APPLIED MECHANICS

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EMBRACING

STRENGTH AND ELASTICITY OF MATERIALS

THEORY AND DESIGN OF STRUCTURES

THEORY OF MACHINES

AND

HYDRAULICS

A TEXT-BOOK FOR ENGINEERING STUDENTS

BY

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WITH 850 ILLUSTRATIONS AND 780 EXERCISES

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PREFACE

THE subject of Applied Mechanics is one which covers a very wide field, and it would not be possible adequately to cover the ground in a single volume. In the present work the author has attempted to compress into one volume of moderate dimensions sufficient material for a two years' course in the subject. To carry out this object the author has endeavoured to be as clear and concise as possible, and he has written the text on the assumption that the student will spend a considerable time in working out the numerous exercises which are given.

The illustrations, which are very numerous, have all been specially prepared for this work, they have been made as small as possible consistent with clearness, and they have been set up with the text in such a manner as to be in close connection with it and to economise space as much as possible.

A special feature has been made of the exercises, which will be found in groups in the various chapters. Of the 780 exercises given, 600 are original, and the author has given as much attention to these as to the text. The remaining 180 exercises have been selected with great care from the examination papers of various examining bodies. Many of the exercises will be found to amplify the text, and thus add to the scope of the book.

The author would here desire to impress upon the student the great importance of working a large number of exercises. A student may imagine, after hearing a lecture, or after reading the text on a part of the subject, that he knows it thoroughly, and that he may therefore leave it, but he will generally find, if he proceeds to apply his knowledge to a practical example, that some important point has escaped his attention or has not been thoroughly understood. This applies to the clever student as well as to the student of ordinary ability. Besides, the working of exercises is essential for thoroughly impressing the subject on his mind. Another matter of very great importance to the student is the cultivation of neatness and accuracy and the systematic arrangement of his work.

The majority of the exercises given involve numerical answers, and these will be found at the end of the book. Some teachers who may use this book in their classes may object to their students having the answers

to the exercises beforehand, but such teachers may, if they choose, make simple alterations in the data of the exercises before giving them to their students, and thus in an easy way have their own set of good exercises. The answers given at the end of the book will, however, be useful to students who may be studying privately, and also to conscientious and industrious students who may desire to get thoroughly familiar with the subject by working examples.

The three chapters on the design of structures have been written and illustrated, on lines suggested by the author, by Mr. E. H. Salmon, B.Sc. (Lond.), A.M.Inst.C.E., and the author feels that these chapters will add very considerably to any merit which the other chapters may give to the book.

To Mr. J. W. Barrett the author is deeply indebted for the great care, intelligence, and skill which he has bestowed on the preparation of the illustrations from the author's pencil drawings and sketches.

A good and enthusiastic teacher interested in his subject does not as a rule follow strictly any particular text-book, not even if he has written it himself, and many of the best teachers seldom refer to any text-book in their lectures. It is, however, very important that a student should form as good a library of his own as he can afford, and the author of this book hopes that it will not be unworthy of a place in such a library, especially in the initial stages of its formation.

D. A. L.

EAST LONDON COLLEGE (UNIVERSITY OF LONDON)
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APPLIED MECHANICS

CHAPTER I

PRELIMINARY

(Mainly for Reference)

1. Definitions Relating to Divisions of Subject.—Newton used the term *mechanics* for "the science of machines and the art of making them," but the term has been used by most writers since Newton's time for the science which treats of the laws of motion and force. This includes (1) *kinematics*, the science of motion without reference to its cause; (2) *statics*, the science of forces which balance one another; and (3) *kinetics*, the science of unbalanced forces, or the relations between motion and force. Many modern writers use the term *dynamics* in the same sense as that of mechanics as just defined, but the most logical writers restrict the term dynamics to statics and kinetics, and consider kinematics as a branch of pure mathematics. Writers who use the term mechanics in place of dynamics generally apply the latter term to what has been defined above as kinetics.

In statics the forces considered may act at a point, or on a solid, a liquid, or a gas. That branch of statics which considers the relations between forces acting on a liquid at rest is called *hydrostatics*, and that branch which considers the equilibrium of a gas is called *pneumatics*. In *hydrodynamics* the relations between motion and force in fluids is considered. *Hydraulics* relates to the application of the principles of hydrostatics and hydrodynamics to engineering.

2. Values of Various Constants.—Except where otherwise given, the values of the more common constants required in working the exercises in this book should be taken as given below. Various useful functions of π are also given.

Ratio of the circumference of a circle to its diameter $= \pi = 3{\cdot}1416$.

$$\pi^2 = 9{\cdot}8696. \quad \pi^3 = 31{\cdot}0063. \quad \sqrt{\pi} = 1{\cdot}7725. \quad \sqrt[3]{\pi} = 1{\cdot}4646.$$

$$\frac{1}{\pi^2} = 0{\cdot}1013. \quad \frac{1}{\pi^3} = 0{\cdot}03225. \quad \frac{1}{\sqrt{\pi}} = 0{\cdot}5642. \quad \frac{1}{\sqrt[3]{\pi}} = 0{\cdot}6828.$$

$$\frac{1}{\pi} = 0{\cdot}3183. \quad \text{Log } \pi = 0{\cdot}49715.$$

Accelerating effect of gravity $= g = 32{\cdot}2$ feet per second per second.
Weight of 1 cubic foot of water $= 62{\cdot}3$ lbs.
1 gallon of water at 62° F. weighs 10 lbs.

3. The C. G. S. System of Units.—This is the system of units recommended, for scientific purposes, by a committee of the British Association. The *centimetre* is the unit of *length*, the *gramme* is the unit of *mass*, and the *second* is the unit of *time*.

The unit of *area* is the *square centimetre.*

The unit of *volume* is the *cubic centimetre.*

The unit of *velocity* is a velocity of a *centimetre per second.*

The unit of *momentum* is the momentum of a *gramme* moving with a velocity of a *centimetre per second.*

The unit of *force* is that force which generates a *unit of momentum in a second*, and is therefore that force which, acting on a gramme for one second, generates a velocity of a centimetre per second. This unit of force is called the *dyne.*

The unit of *work* is the work done by a force of a *dyne* acting through a distance of a *centimetre.* This unit of work is called the *erg.*

4. Equivalents of Ordinary British and C. G. S. Units.

1 foot = 30·479 centimetres.
1 centimetre = 0·0328 foot.
1 square inch = 6·451 square centimetres.
1 square foot = 928·997 square centimetres.
1 square centimetre = 0·155 square inch = 0·001076 square foot.
1 cubic inch = 16·386 cubic centimetres.
1 cubic foot = 28315·3 cubic centimetres.
1 cubic centimetre = 0·061027 cubic inch = 0·00003532 cubic foot.
1 lb. avoirdupois = 453·593 grammes.
1 gramme = 0·0022 lb. avoirdupois.
1 foot per second = 30·479 centimetres per second.
1 mile per hour = 44·703 centimetres per second.
1 centimetre per second = 0·0328 foot per second = 0·02237 mile per hour.
1 lb. per cubic foot = 0·01602 grammes per cubic centimetre.
1 gramme per cubic centimetre = 62·4245 lbs. per cubic foot.
Accelerating effect of gravity = 32·2 feet per second per second = 981·44 centimetres per second per second.
In the equivalents below g is taken = 981 centimetres per second per second.
1 lb. avoirdupois = 444974 dynes.
1 gramme = 981 dynes.
1 foot-pound = 13562570 ergs.
1 kilogrammetre = 98100000 ergs.
1 lb. per square inch = 68974 dynes per square centimetre.
1 lb. per square foot = 478·98 dynes per square centimetre.
1 kilogramme per square centimetre = 981000 dynes per square centimetre.

5. Algebraical Formulæ.—

Quadratic equations.—If $x^2 + ax + b = 0$, then $x = -\frac{a}{2} \pm \frac{\sqrt{a^2 - 4b}}{2}$.

The roots of an equation are the values of x which satisfy the equation. If α and β are the roots of the equation $x^2 + ax + b = 0$, then $\alpha + \beta = -a$, and $\alpha\beta = b$.

Cubic equations.—If $x^3 + ax + b = 0$, then Cardan's solution gives

$$x = \left\{ -\frac{b}{2} + \sqrt{\frac{a^3}{27} + \frac{b^2}{4}} \right\}^{\frac{1}{3}} + \left\{ -\frac{b}{2} - \sqrt{\frac{a^3}{27} + \frac{b^2}{4}} \right\}^{\frac{1}{3}}.$$

The equation $x^3 + px^2 + qx + r = 0$ may be reduced to the form $x^3 + ax + b = 0$ by substituting $x - \frac{p}{3}$ for x in the given equation.

Arithmetical progression.

The terms a, $(a + b)$, $(a + 2b)$, $(a + 3b)$, etc., are in arithmetical progression. The n^{th} term from the beginning is $a + (n - 1)b$.

The sum of n terms $= \frac{n}{2}\left\{2a + (n - 1)b\right\}$.

If M, A, and N are in arithmetical progression, then $A = \frac{M + N}{2}$, and A is the *arithmetical mean* of M and N.

Geometrical progression.

The terms a, ar, ar^2, ar^3, etc., are in geometrical progression.
The n^{th} term from the beginning is ar^{n-1}.
The sum of n term $= \frac{a(r^n - 1)}{r - 1} = \frac{a(1 - r^n)}{1 - r}$.

If r is less than 1, the sum of an infinite number of terms is $\frac{a}{1 - r}$.

If M, G, and N are in geometrical progression, then $G = \sqrt{MN}$, and G is the *geometrical mean* of M and N.

Miscellaneous series. S_n = sum of n terms.

$$S_n = 1 + 2 + 3 + \ldots\ldots + n = \frac{n}{2}(n + 1).$$

$$S_n = 1^2 + 2^2 + 3^2 + \ldots\ldots + n^2 = \frac{n(n + 1)(2n + 1)}{6}$$

$$S_n = 1^3 + 2^3 + 3^3 + \ldots\ldots + n^3 = \left\{\frac{n(n + 1)}{2}\right\}^2.$$

Binomial theorem. $(a + x)^n =$

$$a^n + na^{n-1}x + \frac{n(n - 1)}{1.2}a^{n-2}x^2 + \frac{n(n - 1)(n - 2)}{1.2.3}a^{n-3}x^3 + \ldots\ldots + x^n.$$

The $(r + 1)^{\text{th}}$ term of $(a + x)^n = \frac{n(n - 1)(n - 2) \ldots (n - r + 1)}{\lfloor r} a^{n-r}x^r$,

where $\lfloor r$, read *factorial r*, $= 1.2.3.\ldots\ldots r$.

Exponential and logarithmic series.

$$a^x = 1 + Ax + \frac{A^2x^2}{\lfloor 2} + \frac{A^3x^3}{\lfloor 3} + \frac{A^4x^4}{\lfloor 4} + \ldots\ldots$$

where $A = \log_e a$, and e = base of Napierian system of logarithms.

$$e = 2{\cdot}71828\ldots\ldots$$

$$\log_e(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots$$

Logarithms.—If $a^x = N$, then x is the logarithm of N to the base a. In the *common system* of logarithms the base is 10. In the *Napierian system* of logarithms the base is $e = 2{\cdot}71828 \ldots$ Napierian logarithms are also called *natural* and also *hyperbolic* logarithms.

$$\log (A \times B \times C) = \log A + \log B + \log C.$$

$$\log \frac{A}{B} = \log A - \log B. \qquad \log A^n = n \log A. \qquad \log \sqrt[n]{A} = \frac{1}{n} \log A.$$

If $a^x = b$, then $x \log a = \log b$, and $x = \dfrac{\log b}{\log a}$.

The foregoing rules are true whatever be the system of logarithms used.

If $x = \log_a m$, and $y = \log_b m$, then $y = x \log_b a = \dfrac{x}{\log_a b}$.

If $x = \log_{10} m$, and $y = \log_e m$, then $y = x \log_e 10 = \dfrac{x}{\log_{10} e}$.

$\log_e 10 = 2{\cdot}3026$ nearly, and $\log_{10} e = 0{\cdot}4343$ nearly.

6. Trigonometrical Formulæ.—

$$\operatorname{cosec} A = \frac{1}{\sin A}. \qquad \sec A = \frac{1}{\cos A}.$$

$$\tan A = \frac{\sin A}{\cos A} = \frac{1}{\cot A}. \qquad \cot A = \frac{\cos A}{\sin A} = \frac{1}{\tan A}.$$

$$\sin^2 A + \cos^2 A = 1.$$

$$\sec^2 A = 1 + \tan^2 A. \qquad \operatorname{cosec}^2 A = 1 + \cot^2 A.$$

$$\sin (A + B) = \sin A \cos B + \cos A \sin B.$$
$$\sin (A - B) = \sin A \cos B - \cos A \sin B.$$
$$\cos (A + B) = \cos A \cos B - \sin A \sin B.$$
$$\cos (A - B) = \cos A \cos B + \sin A \sin B.$$

$$\tan (A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

$$\tan (A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}.$$

$$\sin 2A = 2 \sin A \cos A.$$
$$\cos 2A = \cos^2 A - \sin^2 A = 2 \cos^2 A - 1 = 1 - 2 \sin^2 A.$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}.$$

$$\sin 2A = \frac{2 \tan A}{1 + \tan^2 A}. \qquad \cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}.$$

$$\sin 3A = 3 \sin A - 4 \sin^3 A.$$
$$\cos 3A = 4 \cos^3 A - 3 \cos A.$$

$$\tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}.$$

$$\sin(A+B)\sin(A-B)=\sin^2 A-\sin^2 B=\cos^2 B-\cos^2 A.$$
$$\cos(A+B)\cos(A-B)=\cos^2 A-\sin^2 B=\cos^2 B-\sin^2 A.$$

$$\sin\frac{A}{2}=\pm\sqrt{\tfrac{1}{2}(1-\cos A)}.$$

$$\cos\frac{A}{2}=\pm\sqrt{\tfrac{1}{2}(1+\cos A)}.$$

$$\tan\frac{A}{2}=\frac{\sin A}{1+\cos A}.$$

$$\sin\frac{A}{2}+\cos\frac{A}{2}=\pm\sqrt{1+\sin A}.$$

$$\sin\frac{A}{2}-\cos\frac{A}{2}=\pm\sqrt{1-\sin A}.$$

$$2\sin A\cos B=\sin(A+B)+\sin(A-B).$$
$$2\cos A\sin B=\sin(A+B)-\sin(A-B).$$
$$2\cos A\cos B=\cos(A+B)+\cos(A-B).$$
$$-2\sin A\sin B=\cos(A+B)-\cos(A-B).$$

$$\sin A+\sin B=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}.$$

$$\sin A-\sin B=2\cos\frac{A+B}{2}\sin\frac{A-B}{2}.$$

$$\cos A+\cos B=2\cos\frac{A+B}{2}\cos\frac{A-B}{2}.$$

$$\cos A-\cos B=-2\sin\frac{A+B}{2}\sin\frac{A-B}{2}.$$

7. Formulæ for Triangles.—a, b, and c are the sides of a triangle, and A, B, and C are the opposite angles.

$$a+b+c=2s.\qquad A+B+C=180^\circ.$$

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}.\qquad a=b\cos C+c\cos B.$$

$$a^2=b^2+c^2-2bc\cos A.\qquad \cos A=\frac{b^2+c^2-a^2}{2bc}.$$

$$\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}.$$

$$\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}.\qquad \tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

$$\sin A=\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}.$$

$$\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}.$$

$$\text{Area of triangle}=\Delta=\frac{1}{2}bc\sin A=\sqrt{s(s-a)(s-b)(s-c)}$$

R = radius of the circumscribing circle of a triangle.
r = radius of the inscribed circle.

$$R = \frac{a}{2\sin A} = \frac{abc}{4\Delta}. \qquad r = \frac{2\Delta}{a+b+c} = \frac{\Delta}{s} = (s-a)\tan\frac{A}{2}.$$

8. Differential and Integral Calculus.—The curve APB (Fig. 1) is the *graph* of the equation $y = 20x - 2x^3$. Let x and y be the co-ordinates of the point P on the curve. Take another point Q on the curve near to P, and let its co-ordinates be $x + \delta x$ and $y + \delta y$. Then for P, $y = 20x - 2x^3$, and for Q

$$\begin{aligned} y + \delta y &= 20\,(x + \delta x) - 2\,(x + \delta x)^3 \\ &= 20x + 20\delta x - 2x^3 - 6x^2\delta x - 6x(\delta x)^2 - 2(\delta x)^3, \end{aligned}$$

$$\text{therefore } \delta y = 20\delta x - 6x^2\delta x - 6x(\delta x)^2 - 2(\delta x)^3,$$

$$\text{and } \frac{\delta y}{\delta x} = 20 - 6x^2 - 6x\delta x - 2(\delta x)^2.$$

Now let Q approach nearer and nearer to P so that δx and δy get smaller and smaller, then the terms $6x\delta x$ and $2(\delta x)^2$ will get smaller and smaller, and the value of $\frac{\delta y}{\delta x}$ approaches nearer and nearer to $20 - 6x^2$. In the limit when Q is indefinitely near to P the ratio $\frac{\delta y}{\delta x}$ is written $\frac{dy}{dx}$, and is equal to $20 - 6x^2$.

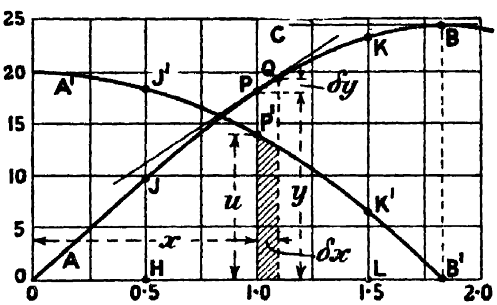

FIG. 1.

The ratio $\frac{dy}{dx}$ is evidently a measure of the slope of the curve or tangent at any point whose co-ordinates are x and y. Also, $\frac{dy}{dx}$ is a measure of the *rate of increase* of y with respect to x.

The ratio $\frac{dy}{dx}$ is called the *differential coefficient* of y with respect to x.

At the highest point B of the curve APB, y has its maximum value, and the slope of the tangent CB is zero. Hence where y is a maximum, $\frac{dy}{dx} = 20 - 6x^2 = 0$, or $x = \sqrt{\frac{20}{6}} = 1{\cdot}826$ nearly, and the maximum value of y is $20 \times 1{\cdot}826 - 2 \times 1{\cdot}826^3 = 24{\cdot}34$ nearly.

The process of finding a differential coefficient is called *differentiation.*

Now let $u = \frac{dy}{dx} = 20 - 6x^2$ be plotted as shown by the curve A'P'B' in Fig. 1. Then, when δx and δy are very small, $u = \frac{\delta y}{\delta x}$ very nearly, and $u\delta x = \delta y$ very nearly. But $u\delta x$ is the area of the shaded strip very nearly when δx is very small, and when δx is indefinitely small $udx = dy$.

Next suppose the figure HJ′P′K′L to be divided into an infinite number of indefinitely narrow vertical strips, each of width dx and variable height u. Let the ordinates of J and K be denoted by y_1 and y_2 respectively; also let the abscissæ of these points be denoted by x_1 and x_2 respectively. The sum of the areas of the strips into which the figure HJ′P′K′L is supposed to be divided is written $\Sigma_{x_1}^{x_2} udx$ or $\int_{x_1}^{x_2} udx$, where Σ is the Greek letter sigma, and $\int$ is the old English letter S. The expression $\int_{x_1}^{x_2} udx$ is read, "the sum of successive values of udx between the limits $x = x_1$ and $x = x_2$."

Since for each strip $udx = dy$, it is obvious that the sum of the areas of the strips is $y_2 - y_1$. But $y_2 = 20x_2 - 2x_2^3$, and $y_1 = 20x_1 - 2x_1^3$, therefore

$$\int_{x_1}^{x_2} udx = 20(x_2 - x_1) - 2(x_2^3 - x_1^3).$$

In Fig. 1 $x_1 = 0{\cdot}5$, and $x_2 = 1{\cdot}5$, and inserting these values in the expression $20(x_2 - x_1) - 2(x_2^3 - x_1^3)$, the area of the figure HJ′P′K′L is found to be 13·5, where the unit of area is a rectangle whose base is 1 inch and height 0·05 inch.

The expression $\int_{x_1}^{x_2} udx$ is called the *definite integral* of udx, and the expression $\int udx$ where no limits are specified is called the *indefinite integral* of udx, or the indefinite integral of u with respect to x.

The process of finding an integral is called *integration*, and $\int$ is the *symbol of integration*.

In the foregoing example $\int udx = \int dy = y = 20x - 2x^3$ is the indefinite integral of udx or $(20 - 6x^2)dx$.

The process of integration is seen to be the reverse of that of differentiation.

If y is a function of x, then $\frac{dy}{dx} = u$, and $\int udx = y$ are equations which follow, the one from the other.

In the process of integration expressed by $\int udx = y$, y has to be found, and u must first be recognised as the differential coefficient of some function of x, and that function of x being known, y is found.

Constant of integration.—Suppose the example already discussed in which $y = 20x - 2x^3$ to be altered so that $y = 20x - 2x^3 + 10$. It is easy to show, by the method already used, that $\frac{dy}{dx} = 20 - 6x^2$, the same as before. Hence in integrating $(20 - 6x^2)dx$ the result, to be quite general, should be written $\int(20 - 6x^2)dx = 20x - 2x^3 + C$, where C is a *constant of integration* which has to be determined from other conditions. For example, it may be known that when $x = 0$, $y = 0$, then C must equal 0.

The following table contains the differential coefficients and integrals

likely to be required in ordinary engineering problems. Those in the first and second lines occur most frequently.

$y=C$	$\dfrac{dy}{dx}=0$	$u=C$	$\int u\,dx=Cx$
$y=ax^n$	$\dfrac{dy}{dx}=nax^{n-1}$	$u=ax^n$	$\int u\,dx=\dfrac{a}{n+1}x^{n+1}$ (except when $n=-1$)
$y=a\log_e x$	$\dfrac{dy}{dx}=ax^{-1}=\dfrac{a}{x}$	$u=ax^{-1}=\dfrac{a}{x}$	$\int u\,dx=a\log_e x$
$y=ae^{bx}$	$\dfrac{dy}{dx}=abe^{bx}$	$u=ae^{bx}$	$\int u\,dx=\dfrac{a}{b}e^{bx}$
$y=a\sin bx$	$\dfrac{dy}{dx}=ab\cos bx$	$u=a\cos bx$	$\int u\,dx=\dfrac{a}{b}\sin bx$
$y=a\cos bx$	$\dfrac{dy}{dx}=-ab\sin bx$	$u=a\sin bx$	$\int u\,dx=-\dfrac{a}{b}\cos bx$
$y=a\tan bx$	$\dfrac{dy}{dx}=ab\sec^2 bx$	$u=a\sec^2 bx$	$\int u\,dx=\dfrac{a}{b}\tan bx$
$y=a\cot bx$	$\dfrac{dy}{dx}=-ab\operatorname{cosec}^2 bx$	$u=a\operatorname{cosec}^2 bx$	$\int u\,dx=-\dfrac{a}{b}\cot bx$

The differential coefficient of a constant is zero.

The differential coefficient of the sum of a number of functions is the sum of the differential coefficients of the functions.

Thus, if $y=u+v+w$, where u, v, and w are functions of x, then

$$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}+\frac{dw}{dx}.$$

The differential coefficient of the product of a number of functions is found by multiplying the differential coefficient of each factor by all the other factors and adding the products thus formed. Thus, if $y=uvw$, where u, v, and w are functions of x, then $\dfrac{dy}{dx}=vw\dfrac{du}{dx}+uw\dfrac{dv}{dx}+uv\dfrac{dw}{dx}$.

The differential coefficient of the quotient of two functions is found as follows: From the product of the denominator and the differential coefficient of the numerator subtract the product of the numerator and the differential coefficient of the denominator, and divide the result by the square of the denominator. Thus, if $y=\dfrac{u}{v}$, where u and v are functions of x, $\dfrac{dy}{dx}=\dfrac{v\dfrac{du}{dx}-u\dfrac{dv}{dx}}{v^2}$.

Function of a function.—If y is a function of u, and u is a function of x, then $\dfrac{dy}{dx}=\dfrac{dy}{du}\cdot\dfrac{du}{dx}$. For example, let $y=\sqrt{a+bx+cx^2}$. Put

$u = a + bx + cx^2$, then $y = \sqrt{u} = u^{\frac{1}{2}}$ and $\frac{dy}{du} = \frac{1}{2}u^{\frac{1}{2}-1} = \frac{1}{2}u^{-\frac{1}{2}} = \frac{1}{2\sqrt{a+bx+cx^2}}$.

Also, $\frac{du}{dx} = b + 2cx$, therefore $\frac{dy}{dx} = \frac{b+2cx}{2\sqrt{a+bx+cx^2}}$.

Successive differentiation.—If y is a function of x, and $\frac{dy}{dx} = u$, where u is also a function of x, then $\frac{du}{dx} = v$, where v is another function of x.

Hence, $\frac{du}{dx} = \frac{d\frac{dy}{dx}}{dx}$, and this is written $\frac{du}{dx} = \frac{d^2y}{dx^2} = v$.

The integral of the sum of a number of functions is equal to the sum of the integrals of the functions. Thus,

$$\int(ax + bx^n)dx = \int axdx + \int bx^n dx + \text{C},$$

where C is the constant of integration.

9. Circle of Curvature.—APB (Fig. 2) is any curve. M and N are two points on this curve, on opposite sides of the point P. A circle may be drawn through the three points M, P, and N. If the points M and N be moved nearer to P, then when M and N are indefinitely near to P, the circle becomes the *circle of curvature* of the curve APB at P. The centre and radius of the circle of curvature are called the *centre of curvature* and *radius of curvature* respectively.

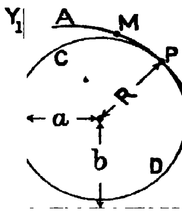

FIG. 2.

Let CPD be the circle of curvature of the curve APB at P. Let X and Y be the co-ordinates of the point P, considered as a point on the circle CPD. Then, $(\text{X} - a)^2 + (\text{Y} - b)^2 = \text{R}^2$.

Differentiating once, $(\text{X} - a) + (\text{Y} - b)\frac{d\text{Y}}{d\text{X}} = 0$.

Differentiating again, $1 + \left(\frac{d\text{Y}}{d\text{X}}\right)^2 + \left(\text{Y} - b\right)\frac{d^2\text{Y}}{d\text{X}^2} = 0$.

Let x and y be the co-ordinates of the point P, considered as a point on the curve APB. Then $\text{X} = x$, and $\text{Y} = y$. Also, since the circle CPD and the curve APB have the same tangent at P, $\frac{d\text{Y}}{d\text{X}}$ and $\frac{dy}{dx}$ denote the slope of this tangent, therefore $\frac{d\text{Y}}{d\text{X}} = \frac{dy}{dx}$. Lastly, since the circle CPD and the curve APB have the same curvature at P, and since curvature is measured by the rate of change of the slope of the tangent $\frac{d^2\text{Y}}{d\text{X}^2} = \frac{d^2y}{dx^2}$; hence, $x - a + (y - b)\frac{dy}{dx} = 0$, and $1 + \left(\frac{dy}{dx}\right)^2 + (y - b)\frac{d^2y}{dx^2} = 0$.

Therefore, $y - b = -\frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}$, and $x - a = \frac{\frac{dy}{dx}\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}{\frac{d^2y}{dx^2}}$.

Substituting these values in the equation $(X-a)^2+(Y-b)^2=R^2$, the result $R=\dfrac{\pm\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}}$ is obtained.

The sign to be taken in the numerator of the right-hand side of this last expression should be the same as that of the denominator, so as to make the value of R positive.

If the curvature of a curve is very small, and the inclination θ of the tangent at any point is small (Fig. 3), then $\frac{dy}{dx}=\tan\theta$ is also small. In this case the expression just found for R becomes $R=\dfrac{1}{\dfrac{d^2y}{dx^2}}$ nearly or $\frac{1}{R}=\frac{d^2y}{dx^2}$, and this is sufficiently accurate for the curves into which beams and struts deflect.

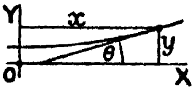

FIG. 3.

10. Construction of Parabola.—A problem of very frequent occurrence is, given the vertex A (Fig. 4), axis AB, and double ordinate CBD of a parabola, to construct the curve. Complete the rectangle CDEAF. Divide AE into any convenient number of equal parts, and divide ED into the same number of equal parts. Join the points of division on ED with A. Lines through the points of division on AE parallel to AB to meet the former lines as shown determine points on one half of the curve. Points on the other half of the curve are found in a similar manner.

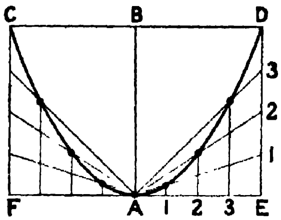

FIG. 4.

11. Equations to Parabola.—OK (Fig. 5) is a fixed straight line, and F is a fixed point. P is a point which moves in the plane of F and OK, so that its distance from F is always equal to its distance from OK. The path of P is a *parabola*, whose axis is the line through F perpendicular to OK. The line OK is called the *directrix*, and the point F the *focus* of the parabola. The curve cuts the axis at A, the *vertex* of the parabola. FA is equal to AO.

Draw PK perpendicular to OK, and PN perpendicular to the axis. Draw the tangent to the parabola at A, and let it meet PK at K′. The tangent at A is obviously perpendicular to the axis of the parabola.

Let $FA=a$, $PN=x$, and $PK'=y$.

Then $PN^2+FN^2=PF^2=ON^2$.

That is, $x^2+(y-a)^2=(y+a)^2$.

Therefore $x^2=4ay$ . . . . (1)

which is the equation to the parabola referred to the axis of the parabola and the tangent at the vertex, the axis being the axis of y, and the tangent the axis of x.

If the axis of x be moved parallel to itself until it is at a distance

b from the vertex (Fig. 6), then y in (1) will become $y+b$, and the new equation will be

$$x^2 = 4a(y+b) = 4ay + 4ab \qquad . \quad . \quad . \quad . \quad (2)$$

If the axis of y be moved parallel to itself until it is at a distance c

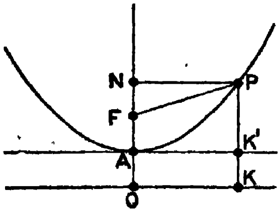

FIG. 5.

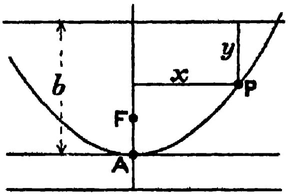

FIG. 6.

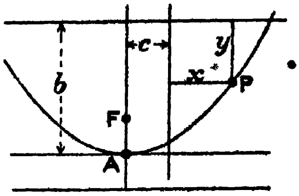

FIG. 7.

from the axis of the parabola (Fig. 7), then x in (2) will become $x+c$, and the new equation will be

$$(x+c)^2 = 4a(y+b) \qquad (3)$$

In the foregoing equations y is positive or negative according as it is measured above or below the axis of x, and x is positive or negative according as it is measured to the right or left of the axis of y.

12. Cycloidal Curves.—If a circle be made to roll along a line, and remain in the same plane with the line, a point on the circumference of the rolling circle will describe a *cycloidal curve*. The line upon which the circle rolls is called a *base line*, a *directing line*, or a *director*. If the base line is a straight line, the curve described is called a *cycloid*. If the base line is a circle, the curve described is called an *epicycloid* or a *hypocycloid*, according as the generating circle rolls on the outside or inside of the directing circle.

The hypocycloid becomes a straight line passing through the centre

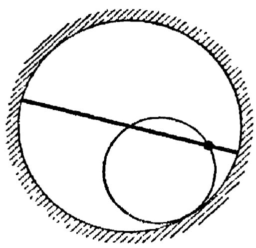

FIG. 8.

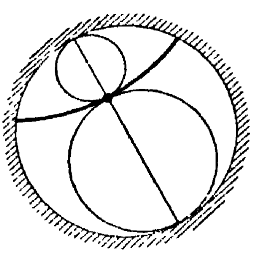

FIG. 9.

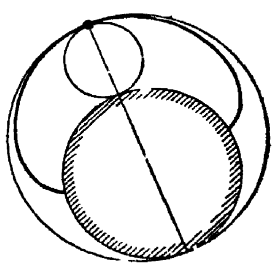

FIG. 10.

of the directing circle (Fig. 8) when the diameter of the rolling circle is equal to the radius of the directing circle.

The same hypocycloid may be described by either of two rolling circles whose diameters are together equal to the diameter of the directing circle (Fig. 9).

The same epicycloid may be described by either of two rolling circles whose diameters differ by an amount equal to the diameter of the directing circle (Fig. 10).

The most convenient method of drawing any of the cycloidal curves is the transparent templet method. Let AB (Fig. 11) be the directing line or circle, and CDP the rolling circle. The directing line or circle is to be drawn on the drawing paper, and the rolling circle is to be drawn on a piece of tracing paper or thin transparent celluloid. Mark the tracing point P by a short radial line cutting the circle, and also by a small needle hole. Place the tracing paper on the drawing paper so that the directing line and the rolling circle touch one another at P. Place a needle through the tracing paper and into the drawing paper at P. Turn the tracing paper round until the rolling circle cuts the directing line at a near point Q_1. Transfer the needle from P to Q_1, and turn the tracing paper until the rolling circle touches the directing line at Q_1. The tracing point will now have moved from P to P_1. Mark the drawing paper at P_1 with a needle-pointed pencil. Again turn the tracing paper until the rolling circle cuts the directing line at another near point Q_2. Transfer the needle from Q_1 to Q_2, and turn the tracing paper until the rolling circle touches the directing line at Q_2. The tracing point will now have moved to P_2. Mark the drawing paper at P_2. Continuing the process, any number of points on the required curve may be obtained, and these points may then be joined by a fair curve.

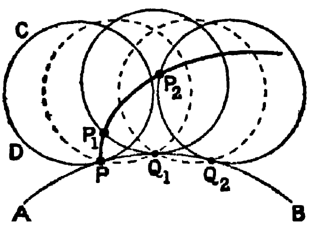

FIG. 11.

13. Scalar and Vector Quantities.— Certain quantities, such as the weight of a body, the volume of a body, a sum of money, the energy stored in a moving body, can be denoted by numbers representing their *magnitudes* in terms of suitable units. For example, a body may weigh 5 lbs., the energy of a moving body may be 205 foot-pounds. All such quantities are called *scalar quantities*.

Other magnitudes, such as velocity, acceleration, force, involve the idea of *direction* as well as magnitude, and they cannot be completely defined by numbers. There must also be descriptions defining their directions. For example, a velocity may be 10 feet per second in a direction from south to north. All such quantities are called *vector quantities*.

A vector quantity may be represented by a straight line, which is called a *vector*. The length of the vector represents the magnitude of the quantity, and the direction of the line represents the direction of the quantity. A line AB (Fig. 12) represents a vector quantity whose magnitude is the length AB, measured with a certain scale, and whose direction is parallel to AB. It is necessary to distinguish between the direction AB and the direction BA, the one being opposite to that of the other. This distinction is the *sense* of the direction, and may be given by the order in which the letters on the line are mentioned in referring to the line. An arrow-head placed on the vector is the best way of showing the sense of the direction. A vector with an arrow-head on it may be referred to by using a single letter, as P.

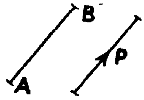

FIG. 12.

14. Addition of Vectors.—A number of vectors, P, Q, R, and S, shown to the left in Fig. 13, are added together as follows: Draw AB parallel and equal to P, BC parallel and equal to Q, CD parallel and equal to R, and DE parallel and equal to S, then the vector AE equal to T is the sum of the vectors P, Q, R, and S. The sum will be the same whatever be the order in which the vectors are taken in performing the addition. The vector T is also called the *resultant* vector. The polygon ABCDEA is called a *vector polygon*.

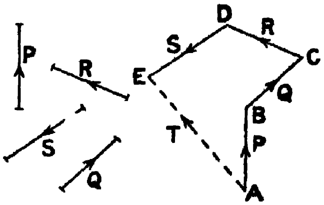

FIG. 13.

15. Bibliography.—The following list gives the titles, authors, publishers, and published prices of some of the more important books on applied mechanics and its subdivisions:—

GENERAL

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Mechanics of Engineering. Church. Wiley, New York. 25s. 6d.
Mechanics Applied to Engineering. Goodman. Longmans, London. 9s.
Applied Mechanics. Lanza. Wiley, New York. 31s. 6d.
Applied Mechanics. Alexander and Thomson. Macmillan, London. 21s.
Applied Mechanics. Perry. Cassell, London. 7s 6d.
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Experimental Engineering. Carpenter. Wiley, New York. 25s. 6d.
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Dynamics of Rotation. Worthington. Longmans, London. 4s. 6d.

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Design of Structures. Anglin. Griffin, London. 16s.
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CHAPTER II

MOTION AND FORCE

16. Rest and Motion.—One point A is said to be *fixed* or to be at *rest* in relation to another point B when the straight line AB does not alter in length or direction. If the straight line AB changes in length or direction, then A is said to *move* or have *motion* in relation to B. If the straight line AB changes in length but not in direction, A has *rectilinear motion* in relation to B, and if AB changes its direction but not its length, A has *angular or rotary motion* in relation to B. If AB changes both in length and direction, then A has both rectilinear and angular motion in relation to B. Motion is therefore change of position, but since position can only be defined in relation to points or bodies which are fixed or whose motions are neglected, *all motion is relative motion.*

A point is said to have *plane motion* when, while it changes its position, it remains in the same plane. Of the many problems on motion which the engineer has to consider, those on plane motion are by far the most common. In an ordinary steam-engine, for example, all the points in the piston, piston-rod, cross-head, connecting-rod, crank, crank shaft, fly-wheel, eccentric, eccentric-rod, and valve have plane motion. The points in the piston, piston-rod, and cross-head have rectilinear motion; the points in the crank, crank shaft, and fly-wheel have angular motion; and the points in the connecting-rod have both rectilinear and angular motions.

17. Velocity.—The rate of motion, or rate of change of position of a point or body, is called the *velocity* of the point or body. When the changes in position are the same in equal intervals of time, however short these intervals may be, the point or body has *uniform velocity.* When the changes in position are not equal in all equal intervals of time, the point or body has *variable velocity.* At any instant the velocity of a moving point is completely known when (1) the direction in which the point is moving, (2) the rate at which it is moving in that direction, and (3) the sense, are known. For example, a point may be moving (1) in a direction perpendicular to the surface of still water, (2) at a rate of so many feet per second, (3) in an upward direction. The statements (1), (2), and (3) are required to completely specify the velocity of a point. The statement (2) is called the *speed* of the point, or the magnitude of the velocity. The term velocity is often used in the same sense as speed, but modern writers incline to using the term speed as defined above.

A velocity may be completely represented by a straight line. The direction of the line is the direction of the velocity, the length of the

line is the magnitude of the velocity or speed, and an arrow-head placed on the line shows the sense of the velocity. The sense may also be given by placing letters, say, A and B, one at each end of the line, and stating that the velocity is AB for one sense, and BA for the opposite sense.

Linear velocity is measured in units of distance per unit of time, as, feet per second, feet per minute, or miles per hour. A *knot* is a linear velocity of one nautical mile (6080 feet) per hour.

If v denote the linear velocity or speed of a point or body, and s the space or distance through which it moves in time t, then $s = vt$. The unit of distance used in measuring v must be the same as that used in measuring s, and the unit of time used in measuring v must be the same as that used in measuring t. For example, if v is in feet per second, s must be in feet, and t in seconds.

Angular velocity is measured in radians * per second, revolutions per second, or revolutions per minute. The Greek letter ω is generally used to denote angular velocity in radians per second. If a point moves in a circle of radius r feet with a linear velocity of v feet per second, and if the point makes n revolutions per second or N revolutions per minute, then

$$\omega = \frac{v}{r} = 2\pi n = \frac{2\pi N}{60}.$$

18. Acceleration.—When a velocity is not uniform, its rate of change is called *acceleration*. Acceleration is positive or negative according as the velocity is increasing or decreasing. Negative acceleration is frequently called *retardation*. *Linear acceleration* is rate of change of linear velocity, and is generally measured in feet per second per second. *Angular acceleration* is rate of change of angular velocity, and is generally measured in radians per second per second. The symbols f and a will be used to denote linear acceleration and angular acceleration respectively. The linear acceleration due to gravity is denoted by the symbol g. The value of g will be taken as 32·2 feet per second per second.

Acceleration, like velocity, is a vector quantity, and may be completely represented by a straight line.

19. Kinematical Equations.—Let v_1 or ω_1 denote the velocity of a point or body at a given instant, and let v or ω denote the velocity after the lapse of t seconds, the acceleration being uniform and denoted by f or a.

Then $$v = v_1 + ft, \text{ and } \omega = \omega_1 + at.$$

During the interval of t seconds the mean velocity is

$$\tfrac{1}{2}(v_1 + v) = v_1 + \tfrac{1}{2}ft \text{ for linear velocity, and}$$
$$\tfrac{1}{2}(\omega_1 + \omega) = \omega_1 + \tfrac{1}{2}at \text{ for angular velocity.}$$

If s is the linear distance moved, or θ the angle described in the interval of t seconds, then

$$s = \tfrac{1}{2}(v_1 + v)t = v_1t + \tfrac{1}{2}ft^2, \text{ and } \theta = \tfrac{1}{2}(\omega_1 + \omega)t = \omega_1t + \tfrac{1}{2}at^2.$$

Eliminating t, it follows that

$$v^2 = v_1^2 + 2fs, \text{ and } \omega^2 = \omega_1^2 + 2a\theta.$$

If $v_1 = o$ and $\omega_1 = o$, then

$$v = ft,\ s = \tfrac{1}{2}ft^2, \text{ and } v^2 = 2fs,$$

also, $$\omega = at,\ \theta = \tfrac{1}{2}at^2, \text{ and } \omega^2 = 2a\theta.$$

* A *radian* is the angle subtended at the centre of a circle by an arc of that circle equal in length to the radius. Hence the number of radians in an angle or the *circular measure* of an angle subtended at the centre of a circle of radius r by an arc of length a is equal to a/r.

20. Composition and Resolution of Velocities and Accelerations.—Two or more velocities may be reduced to a single velocity, and two or more accelerations may be reduced to a single acceleration by composition, exactly as for forces. Also, conversely, a single velocity may be resolved into two or more velocities, and a single acceleration may be resolved into two or more accelerations, exactly as for forces. For the composition and resolution of forces, see Chapter IV.

21. Radial Acceleration of a Point moving in a Circle with Uniform Velocity.—Let a point A (Fig. 14) be moving with a uniform velocity v along the circumference of the circle, whose centre is C and radius $CA = r$. Let A describe the small arc AB in the time t. The velocity of the point when at A is in the direction of the tangent to the circle at A or perpendicular to CA, and the direction of the velocity of the point when at B is in the direction of the tangent to the circle at B or perpendicular to CB. Draw OP perpendicular to CA and equal to v; also draw OQ perpendicular to CB and equal to v, and join PQ. The change in the velocity of the point in moving from A to B is represented by $PQ = u$. If ACB is a very small angle, the difference between the chord AB and the arc AB may be neglected, and ACB and POQ are then similar triangles.

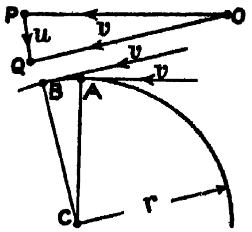


FIG. 14.

Hence $\frac{PQ}{OP} = \frac{AB}{CA}$, that is, $\frac{u}{v} = \frac{vt}{r}$ or $\frac{u}{t} = \frac{v^2}{r}$. But $\frac{u}{t} = f$ is the rate of change of velocity of the point moving in the circle, therefore $f = \frac{v^2}{r}$. Also, when the angle ACB is indefinitely small the direction of u is perpendicular to that of v, and is therefore at any instant in the direction of the radius of the circle from the moving point at that instant. Therefore if a point moves with a uniform velocity v in a circle of radius r, there is a constant acceleration $f = \frac{v^2}{r}$ towards the centre of the circle. If v is in feet per second, and f in feet per second per second, then r must be in feet. If ω is the angular velocity of A about C in radians per second, then $\omega = \frac{v}{r}$ and $f = \omega^2 r$.

22. Instantaneous or Virtual Centre.—Let A and B (Fig. 15) be two definite points in a rigid body which has plane motion, the plane of the paper being the plane of motion of the points A and B. Suppose that at the instant that the body is in the position shown the point A is moving in the direction Aa, and that the point B is moving in the direction Bb. Draw AC and BD perpendicular to Aa and Bb respectively, and let AC and BD meet at O. Just for an instant the point A can be made to revolve about any point in AC without altering the direction of its motion. Also, just for an instant the point B can be made to revolve about any point in BD without altering the

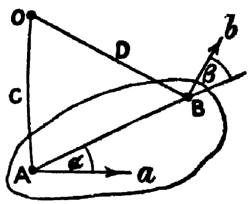


FIG. 15.

direction of its motion. Hence if just for an instant the whole body be made to revolve about O, the directions of the motions of A and B will be unaltered.

Again, since A and B are definite points on a rigid body, they must remain at the fixed distance AB from one another. Hence the components of the velocities of A and B along AB must be equal, that is, $V_A \cos \alpha = V_B \cos \beta$, where V_A and V_B are the velocities of A and B respectively in the directions in which they are actually moving. An inspection of Fig. 15 shows that $\cos \alpha = \sin OAB$, and $\cos \beta = \sin OBA$, therefore $V_A \sin OAB = V_B \sin OBA$, or

$$\frac{V_A}{V_B} = \frac{\sin OBA}{\sin OAB}, \text{ which is equal to } \frac{OA}{OB}.$$

But if the body be made to revolve for an instant about O, then $\frac{V_A}{V_B} = \frac{OA}{OB}$. Hence, for the instant, the motions of A and B are unaltered by making the body revolve about the point O.

The point O is called the *instantaneous centre* or *virtual centre* of the body for the position shown. The instantaneous centre is continually changing, except for a body which has rotary motion only. The locus of the instantaneous centre is called a *centrode*. A line through the instantaneous centre perpendicular to the plane of motion is an *instantaneous axis*, and the locus of the instantaneous axis is a surface called an *axode*.

23. Force.—*Force* is any cause which tends to move a body which is at rest, or which tends to change the motion of a moving body.

A force is completely specified when its magnitude, its direction, and a point in its line of action are given.

A force may be completely specified graphically on paper. Thus, a line *ab* (Fig. 16) has a length which, measured with a certain scale, represents the magnitude of the force, the direction of this line represents the direction of the force, and a point O is given as a point in the line of action of the force. The line along which the force acts will obviously be the line OX drawn through O parallel to *ab*. The force may act from O to X or from X to O, and this may be fixed by an arrow-head placed on the line *ab*. The arrow-head determines the *sense* of the force. The sense may also be fixed by the order in which the letters *a* and *b* at the extremities of the line are stated; thus a force *ab* would be a force acting in the direction *ab* from *a* to *b*, while a force *ba* would be one acting in the direction *ba* from *b* to *a*.

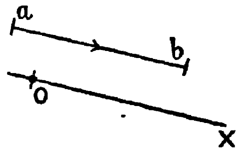


FIG. 16.

24. Mass and Weight.—The *mass* of a body is the quantity of matter which it contains The weight of a body is the force of attraction which the earth exerts on it. The mass of a body is proportional to its weight. The *weight* of a body is, however, slightly different at different parts of the earth's surface.

25. Engineer's Units of Force and Mass.—In engineering calculations the unit of force is the attraction which the earth exerts in the latitude of London on a certain standard piece of platinum, and the unit is called the *pound* or *pound weight*. The unit of mass is taken as the

mass of matter weighing g pounds, so that if W is the weight of a body in pounds, its mass is $M = \frac{W}{g}$.

26. Momentum.—The *quantity of motion* or the *momentum* of a moving body is measured by the product of its mass and velocity. Momentum $= Mv = \frac{W}{g}v$.

27. Newton's First Law of Motion.—*Every body continues in its state of rest or of uniform motion in a straight line, except in so far as it may be compelled by force to change that state.*

28. Newton's Second Law of Motion.—*Rate of change of momentum is proportional to the impressed force, and takes place in the direction of the straight line in which the force acts.*

Rate of change of momentum is equal to mass multiplied by rate of change of velocity or acceleration. Hence if P denotes the impressed force, M the mass of the body, and f the acceleration, $P \propto Mf$. If the unit of force be such as will give unit mass unit acceleration, then $P = Mf$. Using engineer's units, $P = \frac{W}{g}f$ or $\frac{P}{W} = \frac{f}{g}$.

29. Impulse.—If a constant force P acts on a mass M for t seconds, then, since $P = Mf$, it follows that $Pt = Mft = Mv$, where v is the change in the velocity of the body in the time t due to the action of the force P.

The product Pt is called the *impulse* of the force P. The impulse of a force is therefore equal to the change in the momentum which the force produces in the body on which it acts.

If the force is not constant the above equations are only true if t is indefinitely small, or if P is the average value of the force during the time t.

The equation $Pt = Mv$ also shows that equal forces acting on different masses will in the same time produce in these masses equal amounts of momentum. For example, when a projectile is fired from a gun, the forward momentum of the projectile is equal to the backward momentum of the gun.

30. Newton's Third Law of Motion.—*To every action there is always an equal and opposite reaction.* For example, a body is attracted to the earth by the force of gravity, but it is equally true that the earth is attracted to the body by an equal force. Again, a body resting on a table exerts a pressure on the table, but the table exerts an equal pressure on the body.

31. Centrifugal Force.—When a point moves in a circle of radius r feet with a uniform velocity v feet per second, or an angular velocity of ω radians per second, it has been shown (Art. 21) that the point has a radial acceleration f equal to v^2/r or $\omega^2 r$. If the point be replaced by a small body of weight w lbs., then a radial force F must be applied to the body to constrain it to move in the circle, the magnitude of F, in lbs., being $\frac{wf}{g} = \frac{wv^2}{gr} = \frac{w\omega^2 r}{g}$. As here described, the force F is called the *deviating force* or the *centripetal force.* The body in moving in the circle will evidently exert an equal radial force F acting outwards from the centre, and this force is called the *centrifugal force.* Of these two forces, the centripetal and the centrifugal, one may be said to be the reaction of the other.

32. Resultant Centrifugal Force of Two Small Revolving Masses. —*Let A and B (Fig. 17) be two small masses of weight w_1 and w_2* respectively, revolving in the same plane with uniform angular velocity ω about the centre O. The centrifugal force of A is $F_1 = \frac{w_1\omega^2 r_1}{g}$ in the direction OA. The centrifugal force of B is $F_2 = \frac{w_2\omega^2 r_2}{g}$ in the direction OB. Make $OC = F_1$, and draw CD parallel to OB and equal to F_2. Join OD, then OD will be the direction and magnitude of F, the resultant centrifugal force of A and B. Join AB, cutting OD at E. Draw EH perpendicular to OA, EK perpendicular to OB, and EL parallel to CD or OB. Since E is a point in the resultant of the forces F_1 and F_2, the moment of F_1 about E must be equal to the moment of F_2 about E,

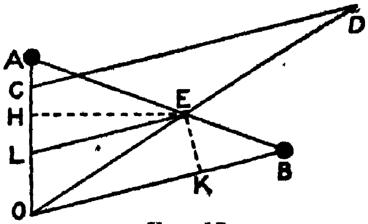


FIG. 17.

therefore $$\frac{w_1\omega^2 r_1}{g} \cdot EH = \frac{w_2\omega^2 r_2}{g} \cdot EK,$$

hence $$\frac{w_1}{w_2} = \frac{r_2 \cdot EK}{r_1 \cdot EH} = \frac{\text{area of triangle OBE}}{\text{area of triangle OAE}} = \frac{BE}{AE}.$$

Therefore E is the centre of gravity of A and B.

Again, $$\frac{OL}{OA} = \frac{BE}{BA} = \frac{w_1}{w_1 + w_2}, \text{ therefore } OL = \frac{w_1 r_1}{w_1 + w_2}.$$

Also, $$\frac{OD}{OE} = \frac{OC}{OL} = \frac{F_1(w_1 + w_2)}{w_1 r_1} = \frac{w_1\omega^2 r_1(w_1 + w_2)}{g w_1 r_1} = \frac{(w_1 + w_2)\omega^2}{g}.$$

But $OD = F$, and if $OE = r$, then $F = \frac{(w_1 + w_2)\omega^2 r}{g}$. That is, the resultant centrifugal force of the two masses A and B is the centrifugal force of the sum of the masses concentrated at their centre of gravity.

33. Centrifugal Force of a Thin Plate revolving about an Axis Perpendicular to its Plane.—If the plate be divided into a large number of small parts of weights w_1, w_2, w_3, etc., then by the preceding Article the resultant centrifugal force of the parts w_1 and w_2 is the same as if these parts were concentrated at their centre of gravity. Again, the resultant centrifugal force of w_1 and w_2 (at their centre of gravity) and w_3 will be the same as if w_1, w_2, and w_3 were concentrated at their centre of gravity. Proceeding in this way until all the masses have been included, it is evident that the centrifugal force of the whole plate will be the same as the centrifugal force of the whole mass concentrated at its centre of gravity.

34. Extension of the Foregoing to Certain Solids.—If a solid can be built up of a number of thin plates, the centres of gravity of which all lie on a line parallel to the axis of revolution, then it is easy to see that the centrifugal force of the whole solid is the same as if the whole mass were concentrated at its centre of gravity.

35. Moment of a Force.—The *moment of a force* about a point, or about an axis perpendicular to it, is the product of the magnitude of the force and its perpendicular distance from the point or axis. This moment is called a *torque*. If the distance is measured in inches and

the force in pounds, the torque is measured in inch-pounds. Other units of torque are—the inch-ton, the foot-pound, and the foot-ton.

36. Rotational Inertia—Moment of Inertia.—*Consider a small body A (Fig. 18), whose mass is m, revolving about an axis O under* the action of a force P, whose line of action is tangential to the path of A. If f is the linear acceleration produced in A by P, then (Art. 28) $P = mf$. If a is the angular acceleration, then $f = ra$; therefore $P = mra$, and $Pr = mr^2a$, or $T = Ia$, where T is the torque causing rotation, and I is called the *moment of inertia* of the body A about the axis O.

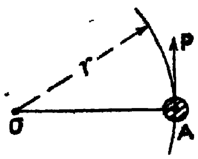


FIG. 18.

If a large body, revolving about an axis, be divided into small parts, whose masses are m_1, m_2, m_3, etc., and whose distances from the axis are r_1, r_2, r_3, etc., respectively, then $T = (m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \text{etc.})a = Ia$, where T is the torque causing the rotation of the body, and $I = m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \text{etc.}$ is the moment of inertia of the body.

If the whole mass M of the body be placed at a distance k from the axis without altering its moment of inertia, then $I = Mk^2$, and k is called the *radius of gyration* of the body.

37. Moment of Momentum—Angular Momentum.—Referring to the small body A of the preceding Article and Fig. 18, if v is its linear velocity and ω its angular velocity, then its linear momentum is $mv = mr\omega$. The moment of this momentum about the axis O is $mr^2\omega = I\omega$. This *moment of momentum* of the body about the axis O is also called the *angular momentum* of the body about that axis.

For a large body made up of small parts, whose masses are m_1, m_2, m_3, etc., and whose distances from the axis about which the body is rotating are r_1, r_2, r_3, etc., respectively, the total *linear momentum* is evidently $(m_1r_1 + m_2r_2 + m_3r_3 + \text{etc.})\omega$, and the sum of the moments of momenta, or the total angular momentum, is

$$(m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \text{etc.})\omega = I\omega,$$

where I is the moment of inertia of the whole body.

Since $T = Ia$, it follows that if the torque T acts on the body for t seconds, $Tt = Iat = I\omega$, where ω is the increase in the angular velocity in the time t, and $I\omega$ is the increase in the angular momentum in that time. Hence, equal torques acting during equal times will produce equal amounts of angular momentum.

Exercises II.

Take 1 metre = 3·281 feet.

1. Express the following velocities in feet per second: 45 miles per hour, 225 feet per minute, 11¼ knots, and 150 metres per minute.

2. Express the following velocities in feet per minute: 3·5 feet per second, 18 miles per hour, 18 knots, and 15 metres per second.

3. Express the following velocities in miles per hour: 33 feet per second, 3080 feet per minute, 16½ knots, and 48 kilometres per hour.

4 Express the following velocities in radians per second: 5 revolutions per second, 270 revolutions per minute.

5. Convert a velocity of 63 radians per second into revolutions per minute.

6. What is the angular velocity, in radians per second, of a train when running round a curve of 18 chains radius at the rate of 35 miles per hour? 1 chain = 22 yards.

7. A traction engine travels at 6 miles per hour; the road wheels are 6 feet in diameter, and are driven through 5 to 1 gearing. Find the angular velocity in radians per second of the fly-wheel on the engine shaft. [B.E.]

8. The speed of a train increases at a uniform rate from 20 to 50 miles per hour in 1 minute 20 seconds. What is the acceleration in feet per second per second?

9. A rotating wheel increases its speed from 100 to 250 revolutions per minute in 35 seconds. What is the mean angular acceleration in radians per second per second?

10. A body starts from rest and its velocity increases at the uniform rate of 8 feet per second per second. How long will it take to travel 144 feet?

11. A moving body has its velocity reduced 2 feet per second in each second while it moves a distance of 45 feet, and its velocity is then 4 feet per second. What was the initial velocity of the body?

12. A stone is dropped into a well, and after the lapse of 2·5 seconds the sound of the splash is heard. Taking the velocity of sound as 1100 feet per second, calculate the depth to the water in the well.

13. At the instant that the brakes are put on, a train has a speed of 40 miles an hour and it is brought to rest, covering a distance of 220 yards during the time of application of the brakes. Assuming the retardation to be uniform, find the time of action of the brakes.

14. The speed of a motor car is determined by observing the times of passing a number of posts placed 500 feet apart. The time of traversing the distance between the first and second posts was 20 seconds, and between the second and third 19 seconds. If the acceleration of the car is constant, find its magnitude in feet per second per second, and also the velocity in miles per hour at the instant it passes the first post. [Inst.C.E.]

15. A rotating wheel has its speed reduced 50 revolutions per minute in each second while it makes 300 revolutions, and its speed is then 60 revolutions per minute. What was the initial speed of the wheel in revolutions per minute? Also, what was the time occupied in making the above reduction in speed?

16. The wheels of a motor car are 30 inches in diameter, what is their angular velocity in revolutions per minute and in radians per second when the speed of the car is 20 miles per hour? If the car is brought to rest in a distance of 60 yards under a uniform retardation, what is the angular retardation of the wheels in radians per second per second?

17. Determine the apparent velocity and direction of rain-drops falling vertically with a velocity of 20 feet per second with reference to a bicyclist moving at the rate of 12 miles an hour. [Inst.C.E.]

18. A cyclist is riding due west at a speed of 12 miles per hour, and the wind is at the time blowing from the south-east with a speed of $5\frac{1}{2}$ miles per hour. If the cyclist carries a small flag, in what direction will this flag fly? At what speed would the cyclist require to ride if the flag is to fly due north? [B.E.]

19. A man standing on a train which is moving with a speed of 36 miles per hour shoots at an object running away from the railway at right angles at a speed of 12 miles per hour. If the bullet, which is supposed to move in a horizontal straight line, has a velocity of 880 feet per second, and if the line connecting man and object makes an angle of 45° with the train when he fires, find at what angle to the train he must aim in order to hit the object. [Inst.C.E.]

20. A and B are two points in a rigid body. A and B are moving in a vertical plane, and when AB is inclined at 30° to the horizontal A is moving in a horizontal direction with a velocity of 10 feet per second. At the same instant the point B is moving in a vertical direction. Find the velocity of the point B.

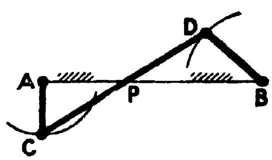


FIG. 19.

21. AC and BD (Fig. 19) are two cranks which oscillate about fixed axes at A and B. These cranks are connected by a link or coupling-rod CD. The axis of the link CD cuts the line AB at P. Show that the ratio of the angular velocity of AC to the angular velocity of BD is equal to the ratio of BP to AP.

22. A rigid body has plane motion. Three points on the body A, B, and C,

in the same plane, are such that AB = 3 feet, BC = 2 feet, and AC = 2·6 feet. At a certain instant it is known that the point A has a velocity of 4 feet per second in the direction from A to C, and that the point B is moving in the direction from C to B. Show how the velocity of any other point on the body may be obtained, graphically or otherwise, and determine the values for the velocities of B and the point midway between A and B.

23. A force P acting on a body weighing 250 lbs. for 10 seconds changes the velocity of the body from 8 to 24 feet per second. Find the magnitude of P.

24. Two men, each exerting a constant force of 50 lbs., set a waggon weighing 5 tons in motion. The frictional resistances amount to 10 lbs. per ton. Find the distance through which the waggon is moved from rest in 1 minute.

25. A locomotive draws a train of 100 tons with a uniform acceleration such that a speed of 60 miles per hour is attained in 4 minutes on the level. If the frictional resistances are 10 lbs. per ton and the resistance of the air, which varies as the square of the speed, is 120 lbs. at 20 miles per hour, find the pull exerted by the locomotive at 30 and at 60 miles per hour. [Inst.C.E.]

26. Assuming that a train may be accelerated by the application of a force equal to one-fortieth of its gross weight, and be braked with a force equal to one-tenth of its gross weight, find the least time in which it may run from one to another of two stopping stations 5000 feet apart. What is the greatest speed during the run? [Inst.C.E.]

27. In an electric railway the average distance between stations is $\frac{1}{2}$ mile, the running time from start to stop $1\frac{1}{2}$ minutes, and the constant speed between the end of acceleration and beginning of retardation 25 miles an hour. If the acceleration and retardation be taken as uniform and numerically equal, find their values; and, if the weight of the train be 150 tons and the frictional resistance 11 lbs. per ton, find the tractive force necessary to start on the level. [Inst.C.E.]

28. A projectile weighing 100 lbs. is fired from a gun weighing 5 tons with a velocity of 1000 feet per second. What is the velocity of free recoil of the gun?

29. A weight of 10 lbs. is suspended from a balloon by a cord. What is the tension in the cord when the balloon is ascending with an acceleration of 3 feet per second per second?

30. A weight of 5 lbs. hangs from the hook of a spring balance suspended within the car of a balloon. What is the vertical acceleration of the balloon when the spring balance indicates 5·5 lbs.?

31. A cage weighing 1000 lbs. is being lowered down a mine by a cable. Find the tension in the cable (1) when the speed is increasing at the rate of 5 feet per second per second, (2) when the speed is uniform, (3) when the speed is diminishing at the rate of 5 feet per second per second. The weight of the cable itself may be neglected. [Inst.C.E.]

32. Two masses weighing W and w lbs. respectively are connected by a fine string passing over a frictionless pulley, as shown in Fig. 20. Show that the tension in the string is $\frac{2Ww}{W+w}$ lbs., and that the acceleration is $\frac{W-w}{W+w}g$. Mass of pulley neglected.

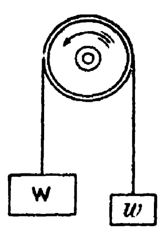


FIG. 20.

33. A locomotive weighing 60 tons runs on a horizontal track of 1000 feet radius at a speed of 15 miles per hour. What is the horizontal thrust exerted on the outer rail?

34. Referring to Fig. 20, W and w are weights of 1000 lbs. and 800 lbs. respectively. The pulley over which the rope passes which connects W and w is 2 feet in diameter, measured to the centre of the rope, its weight is 80 lbs., and its radius of gyration is 10 inches. Assuming that the rope is flexible, and neglecting friction, find the torque which must be applied to the pulley to raise W and lower w with an acceleration of 5 feet per second per second.

CHAPTER III

WORK AND ENERGY

38. Work.—When a force acting on a body causes that body to move, the force is said to do *work*. Also, if a body is moved against a resistance, work is done in overcoming the resistance. The amount of work done depends on the magnitude of the force and also on the distance through which it acts.

In measuring work the unit which is generally used by engineers is the work done when a force of one pound acts through a distance of one foot, this unit being called a *foot-pound*. If the unit taken be the work done when a force of one ton acts through a distance of one foot, it is called a *foot-ton*. The foot-ton is used in measuring large quantities of work. For measuring small quantities of work the *inch-pound*, or the work done when a force of one pound acts through a distance of one inch, is frequently used.

The work done by a force is found by multiplying the magnitude of the force by the distance through which it acts.

39. Work by an Oblique Force.—If a force acting on a body acts in a direction inclined to that of the body's motion the force may be resolved into two components, as explained in Chapter IV., one acting in the direction of the body's motion, and the other perpendicular to that direction. The latter component does no work, and the work done by the former is its magnitude multiplied by the distance through which the body moves. For example, if a body A (Fig. 21) is dragged along a horizontal plane by a force P whose line of action is inclined at an angle θ to the horizontal, the horizontal component of P is P $\cos\theta$ and the work done is $S \times P\cos\theta$, where S is the distance through which A is moved.

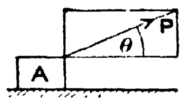


FIG. 21.

40. Work in Raising a System of Weights.—When a number of weights are raised through different heights, or when all the parts of one weight are not raised through the same height, the amount of work done is obtained by multiplying the total weight lifted by the distance through which the centre of gravity of the system is raised. The proof of the foregoing rule is as follows:—Let w_1, w_2, w_3, etc., be the weights of the parts of a system of weights, or the weights of the parts of a single body. Let h_1, h_2, h_3, etc., be the heights of these above a fixed horizontal plane before they are lifted, and let k_1, k_2, k_3, etc., be their heights above the fixed horizontal plane after they are lifted. Also, let H and K be the heights of the centre of gravity of the system above the fixed horizontal plane before and after lifting respectively, and let $W =$ the total weight of the system $= w_1 + w_2 + w_3 +$ etc.

$$\begin{aligned}\text{Work done} &= w_1(k_1 - h_1) + w_2(k_2 - h_2) + w_3(k_3 - h_3) + \text{etc.} \\ &= w_1k_1 + w_2k_2 + w_3k_3 + \text{etc.} \ldots - (w_1h_1 + w_2h_2 + w_3h_3 + \text{etc.} \ldots) \\ &= \text{WK} - \text{WH by a property of the centre of gravity} \\ &= \text{W(K} - \text{H).}\end{aligned}$$

41. Diagram of Work.—If a straight line OC (Fig. 22) represents to scale the distance S through which a body moves under the action of a force, and if OB drawn at right angles to OC represents to scale the magnitude P of the force, then the area of the rectangle BC will represent to scale the work done by P in acting through the distance S. For, let the linear scale be 1 inch to m feet, and the force scale 1 inch to n lbs.; also let OC be l inches, and let OB be h inches long. Then the magnitude P of the force is hn lbs., and the distance S is lm feet. Work done $=$ PS $= hnlm = hlmn$ $= Amn$, where A is the area of the rectangle BC in square inches.

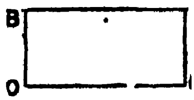


FIG. 22.

If the body moves along the horizontal path represented to scale by OC (Fig. 23) under the action of a force which varies in magnitude, and if the magnitude of the force at each point of the path is represented to scale by the height of the diagram BC at that point, then the area of the diagram will still represent to scale the work done. Consider the work done from E to F, two points near to one another, and let the dimensions of the diagram be in inches, and let the linear and force scales be the same as before. At E the magnitude of the force is $\overline{\text{ED}} \times n$ lbs., and at F the magnitude of the force is $\overline{\text{FH}} \times n$ lbs., and since E and F are near to one another DH may be considered to be a straight line, and the mean magnitude of the force between E and F is $\frac{1}{2}(\overline{\text{ED}} + \overline{\text{FH}}) \times n$ lbs. The work done between E and F is $\frac{1}{2}(\overline{\text{ED}} + \overline{\text{FH}}) \times n \times \overline{\text{EF}} \times m$ foot-pounds. But the area of EDHF is $\frac{1}{2}(\overline{\text{ED}} + \overline{\text{FH}}) \times \overline{\text{EF}}$ square inches, therefore the work done from E to F is equal to the area of the vertical strip DF in square inches multiplied by m and by n. Hence dividing the whole diagram BC into vertical narrow strips, it follows that the work done in moving the body through the distance represented by OC is equal to Amn, where A is the area of the diagram BC in square inches.

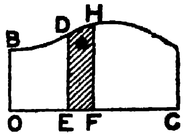


FIG. 23.

42. Turning Moment—Work in Turning.—When a force P acting on a body causes that body to rotate about a fixed axis, the line of action of the force being in a plane perpendicular to that axis, the product of P, the magnitude of the force, and the perpendicular distance R of its line of action from the axis is called the *turning moment* or *torque* of the driving force P. If P is in pounds and R is in feet, the turning moment PR is in *pound-feet* or *foot-pounds*; but if P is in pounds and R is in inches, PR is in *pound-inches* or *inch-pounds*. If the line of action of P is not in a plane perpendicular to the axis of rotation, but makes an angle θ with that plane, then the turning moment is PR $\cos\theta$.

If R, the leverage of P, remains constant during the rotation of the body, and if the magnitude of P is also constant, then if ω is the angle in radians through which the body turns, the distance through which P acts is ωR, and the work done by P is PωR, or Tω, where T is the turning moment. If the leverage R or the force P, or both, should vary, then if T is the *mean turning moment* the work done is Tω.

If the amount of turning is given as n revolutions, then the distance through which P acts is $2\pi Rn$, and the work done is $2\pi PRn$, or $2\pi Tn$.

43. Rate of Work—Horse-power.—The working power of any agent depends on *the amount of work* which it can do in *a given time*. Watt found that a good working horse could do 33,000 foot-pounds of work in one minute, and he introduced this as the unit for measuring the working power of steam-engines. A steam-engine or any working agent is said to be of *one horse-power* when it can do 33,000 foot-pounds of work in one minute, or 550 foot-pounds in one second.

Evidently the simple rule for finding the horse-power of any working agent or the horse-power transmitted by any piece of machinery is to divide the number of foot-pounds of work done or transmitted per minute by 33,000, or horse-power equals work per second divided by 550.

Horse-power is a measure of the *rate* of doing or transmitting work.

44. Electrical Units and their Mechanical Equivalents.—The electromotive force, or electric pressure of an electric current, is measured in volts, and the strength of the current, or the rate of flow of the electricity across a section of the conductor, is measured in *ampères*. The *power* of a current of 1 ampère at an electrical pressure of 1 volt is called a *watt*. Volts × ampères = watts. 1 horse-power = 746 watts. 1 kilowatt = 1000 watts. 1 electrical unit or 1 Board of Trade unit = 1000 watt-hours.

45. Machines.—For the purposes of this Article a machine may be defined as a contrivance for overcoming a force applied at one point by means of another force applied at another point. In books on mechanics it used to be the practice to call the former force the *weight* and the latter force the *power*, but since the force to be overcome is not necessarily that of gravity, it is better to call it the *resistance*, and since the term power is used in connection with rate of work, it is better to use the term *effort* instead of power when referring to the driving force in a machine. In this Article the effort will be denoted by P, and the resistance by W.

The points at which the effort and resistance act may be called the *driving point* and *working point* respectively.

In machines when the driving point moves through a definite distance, say a, the working point moves through another definite distance, say b, and in many machines the ratio of a to b is constant In other machines the ratio of a to b is different for different positions of the driving and working points. In a simple wheel and axle (Fig. 24), for example, the displacement of P will bear a constant ratio to the displacement of W, whereas in a toggle joint (Fig. 25) the ratio of the displacement of P to that of W will be different for different positions of the parts of the machine.

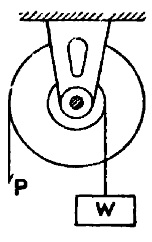


Fig. 24

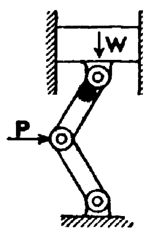


Fig. 25.

In a machine in which the displacement of the driving point bears a constant ratio to the displacement of the working point, this ratio (a/b) is called the *velocity ratio* of the machine. In a machine in which this ratio is variable, the velocity ratio of the machine for any given positions of its parts is the ratio of the displacement of the driving point to the

displacement of the working point when these displacements are indefinitely small.

The ratio of W to P is called the mechanical advantage of the machine.

In all machines a certain amount of work is wasted in overcoming friction, and the result is that Pa, the work done by the effort in a given time, which may be called the *total work* or the work put into the machine, is greater than Wb, the work done on the resistance in the same time, which may be called the *useful work* or the work got out of the machine, and the difference between these two quantities of work is the *lost work.*

The ratio of the useful work to the total work is called the *efficiency* of the machine. The efficiency must evidently be always less than unity. The reciprocal of the efficiency is called the *counter efficiency.* If the machine be reversed so that W becomes the effort and P the resistance, the efficiency under this condition is called the *reversed efficiency.*

Let E = efficiency, M = mechanical advantage, and V = velocity ratio, then $V=\frac{a}{b}$, $M=\frac{W}{P}$, $E=\frac{Wb}{Pa}=\frac{M}{V}$. The lost work is $Pa-Wb$, and assuming that the lost work is the same when the machine is reversed under the action of W as the effort, Wb must be greater than $Pa-Wb$, or $\frac{Wb}{Pa}$ must be greater than $\frac{1}{2}$. A machine will therefore not reverse under the action of the resistance W unless its efficiency is greater than 50 per cent.

46. Usual Relation between the Effort and Resistance in a Machine.—If experiments are made with a machine by varying the useful resistance W and finding the corresponding values of the effort P, it is found that if the results are plotted on squared paper the points thus obtained generally lie very nearly in a straight line, and if the straight line which most nearly contains all the points be drawn, the equation to this line is $P=mW+c$, where m and c are constants for the particular machine.

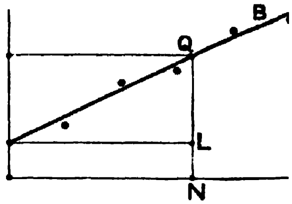


FIG. 26.

In Fig. 26 the dots represent points plotted as described above, the values of W being measured horizontally from the vertical axis OY, and the values of P vertically from the horizontal axis OX. AB is the straight line which most nearly contains the points. Take any point Q in AB, draw QM parallel to OX to meet OY at M, and QN parallel to OY to meet OX at N. Then QM and QN represent corresponding values of W and P respectively. Draw AL parallel to OX to meet QN at L. It is evident that wherever Q may be taken in AB the ratio QL ÷ AL will be the same. Let QL ÷ AL = m, and let OA = c, then $m=\frac{QL}{AL}=\frac{QN-LN}{AL}=\frac{P-c}{W}$, therefore $P=mW+c$.

In plotting it is not necessary that the scale for P be the same as that for W, but in determining the value of m care must be taken to measure QL and OA with the scale for P, and AL with the scale for W.

The relation between P and W is sometimes called the *law* of the

machine. When the law of a machine is known to be a *straight line law* it is evident that since a line is fixed when two points in it are known, the law of the machine can be found from two values of W and the two corresponding values of P.

The quantity c is evidently the magnitude of the effort required to drive the machine unloaded.

The effort to overcome a resistance W when the machine is frictionless is $P' = W \div V$, where V is the velocity ratio of the machine. The efficiency is obviously equal to the ratio of the effort without friction to the effort with friction, hence

$$E = \frac{P'}{P} = \frac{W}{V(mW + c)} = \frac{1}{mV + \frac{cV}{W}}.$$

As W increases the term $\frac{cV}{W}$ diminishes, and by making W large enough $\frac{cV}{W}$ will become so small that it may be neglected; hence the limit to the efficiency of the machine is $\frac{1}{mV}$, assuming of course that the law of the machine, $P = mW + c$, remains true with the increased value of W.

The mechanical advantage

$$M = \frac{W}{P} = \frac{W}{mW + c} = \frac{1}{m + \frac{c}{W}},$$

and reasoning as for the maximum efficiency, it follows that the maximum mechanical advantage is $\frac{1}{m}$.

Since $M = EV$, it follows that if an efficiency curve be drawn this curve will also represent the mechanical advantage, but to a scale whose unit is $1/V$ of the unit of the efficiency scale.

Exercises IIIa.

1. A man raises a weight of 25 kilogrammes to a height of 22 metres. A small steam-engine raises a weight of 98 lbs. to a height of h feet, and does the same amount of work as the man. Find h, having given that 1 kilogramme $= 2{\cdot}2046$ lbs. and 1 metre $= 39{\cdot}371$ inches. *

2. A block of stone is pulled along level ground at a uniform velocity over a distance of 6 yards by a force of 450 lbs. acting on a rope attached to the stone and inclined at 45° to the ground. How many foot-pounds of work have been done?

3. A cylindrical column of granite, 2 feet in diameter and 5 feet long, stands with one end on the ground. If the weight of the stone is 170 lbs. per cubic foot, how many foot-pounds of work are done in tipping it over into the position from which it is about to fall over into a horizontal position?

4. A wire rope 250 feet long, and weighing 1 lb. per foot, hangs from the drum of a winding-engine, and carries a weight of 15 cwt. at its lower end. Find the work done in foot-pounds in winding up the first 200 feet of the rope.

5. A rectangular tank, 7 feet broad, 8 feet long, and 6 feet deep, is half full of water weighing 62·3 lbs. per cubic foot. How many foot-pounds of work will be required to raise all the water over the top of the tank?

6. 250 cubic feet of water are pumped from a rectangular tank, whose base is 5 feet square, into a cylindrical tank, whose base is 6 feet in diameter. The bottom of the cylindrical tank is 20 feet above the bottom of the other. Find the work done in foot-pounds, taking the weight of a cubic foot of water as 62·3 lbs.

7. A weight of 1 lb., hanging at rest on the hook of a spring-balance, has stretched the spring one-tenth of an inch. How many foot-pounds of work must be done in stretching the spring $1\frac{1}{2}$ inches farther?

8. A weight of 50 lbs. is lifted 30 inches. The force which does this work acts through a spring-balance, which at the beginning registers zero and at the end 50 lbs. The stiffness of the spring is such that a weight of 20 lbs. gradually applied stretches the spring 1 inch. Determine the work done by the lifting force.

9. Find the horse-power of a steam-pump which can raise 1100 gallons of water to a height of 90 feet in 5 minutes.

10. What weight will an engine of 8 horse-power raise to a height of 90 feet in $\frac{3}{4}$ minute?

11. Coal is raised from a mine 660 yards deep. The cage and its load weigh 5 tons, and the rope weighs 24 lbs. per fathom. Find the horse-power of the winding-engine if the load is raised from the bottom to the surface in 55 seconds.

12. How many gallons of water may be raised per hour from a depth of 140 feet by an engine of 200 indicated horse-power, the efficiency of the engine and pump being 80 per cent.?

13. An electrically driven overhead crane raises a weight of 5 tons at the rate of 90 feet per minute. What is the horse-power? Convert this into watts. The motor drives the lifting machinery, whose efficiency is 70 per cent. How many ampères of current must be supplied to the motor if the voltage is 220 and the efficiency of the motor is 87 per cent.? If the current is supplied by a company which charges at the rate of $2\frac{1}{2}$d. per Board of Trade unit for power purposes, what is the cost of lifting in pence per foot-ton? [B.E.]

14. Electric current is supplied to a certain motor plant at 220 volts, and 150 ampères are taken. What H.P. does this represent? How much would it cost if used for an average of 6·5 hours per day for a whole year of 313 days? The power is supplied at 2·24d. per H.P. hour (*i.e.* 3d. per B.T.U.). [B.E.]

15. A machine is concealed from sight, except that there are two vertical ropes; when one of these is pulled downwards the other rises. If the falling of a weight A on one causes a weight B on the other to be steadily lifted, first when A is 12 lbs. and B 700 lbs., second when A is 7·6 lbs. and B is 300 lbs., what is A likely to be when B is 520 lbs.? If B rises 1 inch when A falls 70 inches, what is the efficiency of this lifting machine in each of the three cases? [B.E.]

16. In a lifting machine an effort of 26·6 lbs. just raised a load of 2260 lbs.; what is the mechanical advantage? If the efficiency is 0·755, what is the velocity ratio? If on this same machine an effort of 11·8 lbs. raised a load of 580 lbs., what is now the efficiency? What is probably the effort required to raise a load of 1000 lbs., and what would the efficiency be? Explain why, when the efficiency is somewhat less than 0·5, a lifting machine does not overhaul. [B.E.]

17. The law of a machine is $P = 0{\cdot}03W + 1$, and its velocity ratio is 210. What is the mechanical advantage, and what is the efficiency when $W = 300$ lbs.? What is the maximum mechanical advantage, and what is the maximum efficiency? Denoting that part of the effort which is used in overcoming the friction of the machine by F, find F when $W = 300$ lbs., and determine the relation between F and W for any value of W. Plot the effort, friction, and efficiency, on the resistance as a base, from $W = 0$ to $W = 300$ lbs. Scales.—Effort and friction, 1 inch to 2 lbs.; resistance, 1 inch to 50 lbs.; efficiency, 1 inch to 4 per cent.

18. Referring to the machine of the preceding exercise, if Q is the effort to reverse the machine, or lower the load W, find the relation between Q and W.

19. If B be the B.H.P. of a certain gas-engine and I the corresponding I.H.P., and the results of two tests gave $B = 57$ corresponding to $I = 73$, and $B = 117$ corresponding to $I = 139$, what would the B.H.P. be if the I.H.P. were 35? Assume the law $B = xI + y$, where x and y are constants. [Inst.C.E.]

20. Experiments were made with a Weston differential pulley tackle having a velocity ratio of 16, and the results

W	28	56	84	112	140	168	196
P	$7\frac{1}{4}$	$12\frac{1}{4}$	16	21	$25\frac{1}{2}$	$30\frac{1}{4}$	$34\frac{1}{4}$

here tabulated were obtained, W being the load and P the effort, both in lbs. Plot these results. Draw the straight line which most nearly represents the relation between P and W, and find its equation in the form $P = mW + c$. Draw the efficiency curve, and calculate the maximum efficiency.

21. A crane requires the expenditure of 30 foot-tons of work per second to lift 10 tons at the rate of 2 feet per second, and 20 foot-tons per second to lift 3 tons at the rate of 4 feet per second. Assuming that the law connecting the rate at which work is done on the crane (A) with the rate at which the crane does work (B) is of the form $A = pB + q$, where p and q are constants, find the values of p and q, and use the expression to calculate the efficiency of the crane for a load of 5 tons lifted at the rate of 2 feet per second.

47. Energy.—In mechanics the term *energy* means capacity for doing work.

Potential Energy is energy due to the relative position of one body to another, or of one part of a body to another part when the two bodies or the parts of the same body are under the action of a force or forces tending to alter their relative positions. For example, a body which is allowed to fall towards the earth may be made to do work; hence before it begins to fall it possesses potential energy, or energy due to its position in relation to the earth. A compressed spiral spring has potential energy, because if it is allowed to resume its unstrained form it can be made to do work. Likewise compressed air possesses potential energy. The energy stored in a piece of coal is potential energy, and under favourable conditions the atoms of the constituents of the coal and the atoms of the oxygen of the air will rush together and produce heat which may be converted into work.

Kinetic Energy is energy due to the motion of a body. A gallon of water at rest at a height of 100 feet above the level of the sea possesses 1000 ft.-lbs. of potential energy, and if this water is allowed to fall freely to the level of the sea, without doing work on the way, it will in every position of its fall possess 1000 ft.-lbs. of energy, but as it descends its potential energy will diminish, and the remainder of the 1000 ft.-lbs. will be stored in the water as kinetic energy. When the gallon of water has fallen 25 feet its potential energy will be reduced to 750 ft.-lbs., and its kinetic energy will then be 250 ft.-lbs.

If a body of weight W lbs. falls freely from rest through a height of h feet it will then have stored in it Wh foot-lbs. of kinetic energy, and its velocity will then be $v = \sqrt{2gh}$ feet per second. Hence the kinetic energy Wh is equal to $\frac{Wv^2}{2g}$. It is evident that the kinetic energy of a body weighing W lbs., and moving with a velocity of v feet per second, will be the same, namely, $\frac{Wv^2}{2g}$, whatever be the cause of the velocity, whether, for example, the cause be the force of gravity, as in a falling body, or the force of an explosion, as in a gun.

48. Kinetic Energy of a Rotating Body.—If an indefinitely small body of weight w lbs. be moving with a linear velocity v feet per second in a circle of radius r feet, then its angular velocity ω in radians per second is equal to v/r, and its kinetic energy is $\frac{wv^2}{2g} = \frac{w\omega^2 r^2}{2g}$ ft.-lbs.

If a body of weight W, rotating about a fixed axis with an angular velocity ω, be divided into indefinitely small parts of weights w_1, w_2, w_3,

etc., whose distances from the axis of rotation are r_1, r_2, r_3, etc., respectively, then the kinetic energy of the whole body is

$$\frac{\omega^2}{2g}(w_1r_1^2 + w_2r_2^2 + w_3r_3^2 + \text{etc.}) = \frac{W\omega^2k^2}{2g} = \frac{I\omega^2}{2g},$$

where k is the radius of gyration, and I the moment of inertia of the body about the axis of rotation. If these expressions give the kinetic energy in ft.-lbs., then W must be in lbs., k in feet, and I in lb. and foot units.

49. Total Kinetic Energy of a Body.—If a body of weight W rotates about an axis through its centre of gravity with an angular velocity ω, and if the radius of gyration of the body about that axis is k, then its kinetic energy due to its rotary motion is $\frac{W\omega^2k^2}{2g}$. If the centre of gravity of this body has a linear velocity v, then its kinetic energy due to its motion of translation is $\frac{Wv^2}{2g}$. If the body has both kinds of motion simultaneously, then its total kinetic energy is

$$\frac{W\omega^2k^2}{2g} + \frac{Wv^2}{2g}.$$

50. Mechanical Equivalent of Heat.—Heat and work are mutually convertible the one into the other. In a heat engine the heat produced by the combustion of the fuel used is converted into the work done by the engine. When the brakes are applied to the wheels of a moving train, in order to bring it to rest, the kinetic energy of the train is converted into heat at the rubbing surfaces of the brake blocks and wheels, or if the wheels skid the heat is produced at the rubbing surfaces of the wheels and rails.

Careful experiments have shown that 778 ft.-lbs. of work are equivalent to one British thermal unit (B.Th.U.) of heat, or the heat required to raise the temperature of 1 lb. of water 1° Fahrenheit. The number 778 is called the *mechanical equivalent of heat*. In terms of the lb.-degree centigrade unit of heat the mechanical equivalent of heat is 1400 ft.-lbs.

51. Analogies of Linear and Angular Motions.—The student would do well to study the analogies of linear and angular motions exhibited in the following table:—

QUANTITY.	LINEAR.	ANGULAR.
Time	t	t
Distance or displacement	s	θ
Velocity	v	ω
Acceleration	f	a
Inertia	Mass, $M = w/g$	Moment of inertia, I
Effort	Force, $P = Mf$	Torque, $T = Ia$
Momentum	Mv	$I\omega$
Impulse	$Pt = Mv$	$Tt = I\omega$
Work done = U	Ps	$T\theta$
Space average of effort	$U \div s$	$U \div \theta$
Time average of effort	$Mv \div t$	$I\omega \div t$
Kinetic energy	$\frac{1}{2}Mv^2$	$\frac{1}{2}I\omega^2$
Kinematical equations for uniform acceleration from rest in time t	$v = ft$ $s = \frac{1}{2}ft^2$ $v^2 = 2fs$	$\omega = at$ $\theta = \frac{1}{2}at^2$ $\omega^2 = 2a\theta$

In this table $I = Mk^2 = \frac{Wk^2}{g}$.

Exercises IIIb.

1. A railway train has a speed of 30 miles per hour. How far will this train travel, after steam is shut off, under a resistance of 15 lbs. per ton? Also, what additional resistance, in lbs. per ton, must be applied to stop the train in a distance of 100 yards?

2. Through what distance must a force of $7\frac{1}{2}$ lbs. act on a body weighing 20 lbs., in the direction of its motion, in order to change its velocity from 15 to 25 feet per second?

3. What must be the magnitude of a force acting on a body weighing 50 lbs. through a distance of 10 feet, in the direction of its motion, which will double the kinetic energy of the body, if the velocity when the force begins to act is 200 feet per minute?

4. To draw a waggon weighing 10 tons up an incline 50 feet high requires the expenditure of 530 foot-tons of work. If the waggon is liberated at the top of the incline, what speed, in miles per hour, will it have when it reaches the bottom? Assume that the frictional resistances are the same coming down as going up the incline. Further, if the brake is applied during the descent, and the velocity acquired at the foot of the incline is 25 miles per hour, how many foot-tons of work have been absorbed by the brake?

5. How far will a train, moving at the rate of 40 miles per hour, run up an incline of 1 in 150 after steam is shut off? In addition to the resistance of gravity there is a mean resistance of 12 lbs. per ton in the direction opposite to that of the motion.

6. A vehicle weighing 4 tons is proceeding at the rate of 10 miles an hour along a level road; the pull on it is suddenly stopped: supposing the whole resistance equivalent to 500 lbs. applied to the rim of one of the wheels 4 feet in diameter, calculate how far the vehicle will run before stopping. [Inst.C.E.]

7. A fly-wheel alters in speed from 99 to 101 revolutions per minute when its kinetic energy alters by the amount of 500,000 ft.-lbs. What is its moment of inertia? What is its kinetic energy when making 1 revolution per minute? [Inst.C.E.]

8. A fly-wheel of a shearing machine has 150,000 ft.-lbs. of kinetic energy stored in it when its speed is 250 revolutions per minute; what energy does it part with during a reduction of speed to 200 revolutions per minute? If 82 per cent. of this energy given out is imparted to the shears during a stroke of 2 inches, what is the average force due to this on the blade of the shears? [B.E.]

9. A fly-wheel when running at 90 revolutions per minute has a stored energy of 3,000,000 ft.-lbs. By reason of additional load it is slowed down to 86 revolutions per minute in two seconds. By how much will the stored energy be reduced, and what is the average HP. produced by the slowing down of the fly-wheel? [Inst.C.E.]

10. A machine is found to have 300,000 ft.-lbs. stored in it as kinetic energy when its main shaft makes 100 revolutions per minute. A similar machine (that is, made to the same drawings but on a different scale) is made of the same material but with all its dimensions 20 per cent. greater; what will be its store of kinetic energy at 70 revolutions per minute? If when at 70 revolutions per minute energy is being stored for a short time at the rate of 1 horse-power, how does the speed alter during this time? [B.E.]

11. When the fly-wheel of a certain traction engine lessens in speed from 150 to 140 revolutions per minute there is a loss of kinetic energy (on the motion of the whole engine as well as the fly-wheel) of 25,000 ft.-lbs. If the speed is 160 revolutions per minute, how far will the engine travel up an ascent of 1 in 100, before coming to rest, if engine and truck together weigh 30 tons, and there is a constant frictional resistance on a level road of 20 lbs. to the ton? [B.E.]

12. An electric tram-car has a total weight of 10 tons. The driving axle and driving wheels weigh 1050 lbs., and the other axle and its wheels weigh 650 lbs. Each axle with its wheels has a radius of gyration of 11 inches. The diameter of the wheels at the tread is 30 inches. What is the total kinetic energy of this car in foot-tons when it is travelling at 15 miles per hour? What fraction of the total energy of the car is due to the rotation of the wheels and axles?

13. A projectile weighing 12 lbs. has a linear velocity of 2500 feet per second and an angular velocity about its axis of 500 revolutions per second. If its radius of gyration is 0·75 inch, what is the total kinetic energy of the projectile?

14. Apply the principle of the conservation of energy to find the velocity of a thin hollow circular cylinder after rolling a distance of 12 feet down a plane inclined at a slope of 1 vertical in 5 horizontal. [Inst.C.E.]

15. A fly-wheel (Fig. 27) mounted on a horizontal spindle in bearings is rotated by winding a cord on the spindle, attaching a weight to the cord, and allowing the weight to fall to the ground. In an actual experiment the falling weight was 21 lbs., the total height of fall, 5 feet; the height of fall of the weight for one revolution of the spindle was 5·05 inches; the time taken by the weight from starting from rest to reach the floor was 7·6 seconds, the whole time of rotation of the fly-wheel starting from rest was 70·25 seconds, and the total number of rotations of the fly-wheel was 109·9. Find—(*a*) The energy in inch-pounds in the falling weight at the instant of striking the floor; (*b*) the energy in inch-pounds per revolution lost in friction in the bearings of the spindle; (*c*) the moment of inertia of the fly-wheel. [B.E.]

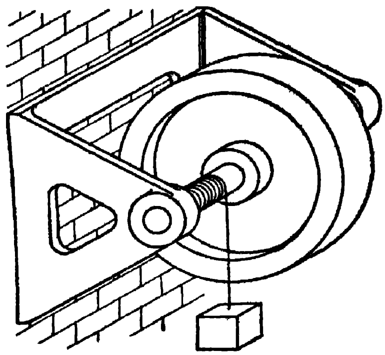


Fig. 27.

16. Through what height must a weight of 10 lbs. fall freely from rest so that its energy at the end of the fall is equivalent to the heat required to warm 10 lbs. of water 1° F.?

17. Two cylindrical tanks, A and B, of, respectively, 4 square yards and 2 square yards horizontal cross section, stand on the same floor and are connected near the bottom by a narrow pipe. A at first contains 8 cubic yards of water, and B is empty. The water flows slowly into B. Find the amount of heat which will have been generated when the water has ceased to flow and it has all come to rest. [Inst.C.E.]

18. Find the combined efficiency of a steam-engine and its boiler when the coal used is $1\frac{1}{2}$ lbs. per horse-power per hour, the calorific value of the coal being 12,500 B.Th.U.

19. The balls on the arms of a fly-press weigh 112 lbs. each, and they are moving with a velocity of 12 feet per second. How many ft.-lbs. of work must be expended in bringing them to rest? If the die sinks $\frac{1}{24}$ inch into the metal while the balls are being brought to rest, find the mean resistance to the motion of the die into the metal.

20. A hammer used for driving a nail weighs 2 lbs., and at the instant of striking the blow it is moving with a velocity of 16 feet per second; the blow causes the nail to penetrate into the wood $\frac{3}{16}$ inch. Find the mean pressure on the head of the nail.

CHAPTER IV

THE POLYGON OF FORCES

52. Composition and Resolution of Forces.—The single force which would produce the same effect as a number of forces acting together is called the *resultant* of these forces, and the forces are called *components* of their resultant

The single force which will balance a number of forces acting together is called the *equilibrant* of these forces. The equilibrant has the same magnitude as the resultant, and acts along the same line, but in the opposite direction.

The process of finding the resultant of a number of forces is called the *composition of forces*, and the converse process of replacing a force by two or more components is called the *resolution of a force*.

The resultant of a number of forces acting in the same straight line is equal to the algebraical sum of the forces. If forces acting in one direction along a straight line are positive (+), those acting in the opposite direction are negative (−).

53. Parallelogram of Forces and Triangle of Forces.—If OP and OQ (Fig. 28) represent in magnitude and direction two forces acting at the point O, then the diagonal OR of the parallelogram OPRQ will represent the magnitude and the direction of their resultant. Conversely, if a parallelogram OPRQ be described on OR as diagonal, OP and OQ will represent components of the force represented by OR.

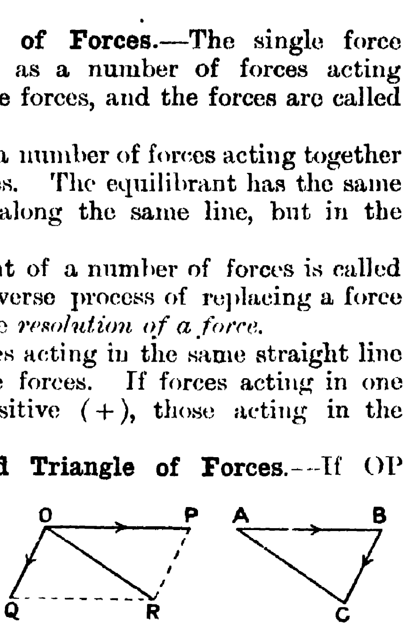


FIG. 28.

In applying the *parallelogram of forces* OPRQ it should be noticed that the forces OP and OQ must both act either from O towards P and Q respectively, or towards O from P and Q respectively. Since the only difference between the resultant and the equilibrant of two forces is that the sense of the one is opposite to the sense of the other, the parallelogram of forces may be applied to find the equilibrant of two forces.

If AB and BC be drawn parallel and equal to OP and OQ respectively, and CA be joined, then it is obvious that the triangle ABC is equal in all respects to the triangle OPR, and therefore AC is equal and parallel to OR. Hence the resultant or the equilibrant of two forces OP and OQ may be determined by drawing the *triangle of forces* ABC. If arrow-heads be placed on the sides of the triangle of forces to show the sense of the forces, then, when two forces and their resultant are represented, the arrow-head for the resultant will point in the *opposite direction* round the triangle to that of the other two arrow-heads. But

when two forces and their equilibrant are represented, the three arrow-heads will point in the *same direction* round the triangle.

Referring to Fig. 28, $OR^2 = OP^2 + OQ^2 + 2OP \cdot OQ \cos POQ$, also $AC^2 = AB^2 + BC^2 - 2AB \cdot BC \cos ABC$.

54. Polygon of Forces.—If a number of forces OP, OQ, OR, and OS act at a point O (Fig. 29), their resultant or their equilibrant may be found by repeated application of the parallelogram or triangle of forces. The parallelogram POQT determines OT, the resultant of OP and OQ. The parallelogram TORU determines OU, the resultant of OT and OR, and therefore the resultant of OP, OQ, and OR. The parallelogram UOSV determines OV, the resultant of OU and OS, and therefore the resultant of OP, OQ, OR, and OS.

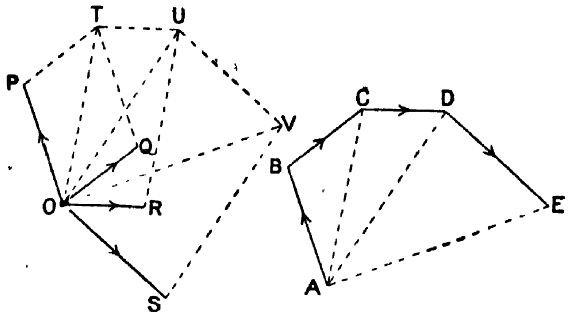


FIG. 29.

If AB be equal and parallel to OP, and BC be equal and parallel to OQ, then AC will be equal and parallel to OT; if CD be equal and parallel to OR, then AD will be equal and parallel to OU; and if DE be equal and parallel to OS, then AE will be equal and parallel to OV Hence if a polygon ABCDE be drawn, having its sides respectively parallel and equal to the given forces, the closing side AE will represent in magnitude and direction the resultant of the given forces, and EA will represent their equilibrant.

If arrow-heads be placed on the sides of the polygon of forces to show the sense of the forces, then when the sides of the polygon represent the given forces and their resultant, the arrow-head for the resultant will point in the opposite direction round the polygon to that of the other arrow-heads. But when the sides of the polygon represent the given forces and their equilibrant, all the arrow-heads will point in the same direction round the polygon.

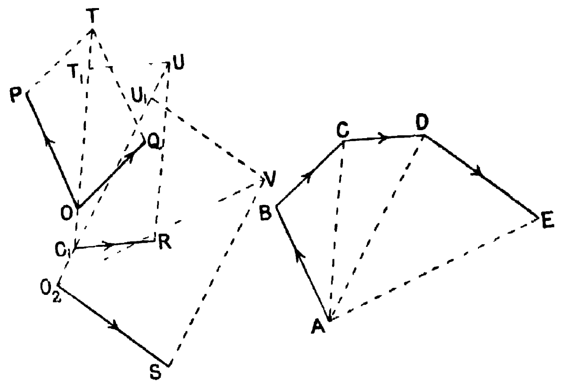


FIG. 30.

When the given forces do not all act at the same point, or when their lines of action are not concurrent and not all parallel, their resultant may still be determined by repeated application of the parallelogram of forces, as shown in Fig. 30.

The lines of action of OP and OQ intersect at O, and the parallelogram POQT determines OT, their resultant. The lines of action of OT and O_1R intersect at O_1, and O_1T_1 being made equal to OT, the parallelogram T_1O_1RU determines O_1U, the resultant of OT and O_1R. The lines of action of O_1U and O_2S intersect at O_2, and O_2U_1 being made equal to O_1U, the parallelogram U_1O_2SV determines O_2V, the resultant of O_1U and O_2S, and therefore also the resultant of OP, OQ, O_1R, and O_2S. By this method the resultant of the given forces is completely determined.

The polygon of forces ABCDE may be drawn as before, and the closing line AE will represent in magnitude and direction the resultant of the given forces, but the line of action is undetermined. A point in the line of action of the resultant may however be found by drawing another polygon, called the *funicular polygon*, which is discussed in the next Article but one.

When the given forces are all parallel, the polygon of forces becomes a straight line.

55. Lettering of Forces—Bow's Notation.—In Fig. 31 the diagram (m) shows the lines of action of a number of forces which act at a point and which are in equilibrium. The diagram (n) is the corresponding polygon of forces. In one system of lettering, each force is denoted by a single letter, as P. In *Bow's notation*, each force is denoted by two letters, which are placed on opposite sides of the line of action of the force in diagram (m), and at the angular points of the polygon in diagram (n). In Bow's notation the force P is referred to as the force AB. In like manner the force Q is referred to as the force BC. The diagram (m), which shows the lines of action of the forces, is called the *space diagram*, and the diagram (n), which shows the polygon of forces, is called the *force diagram*.

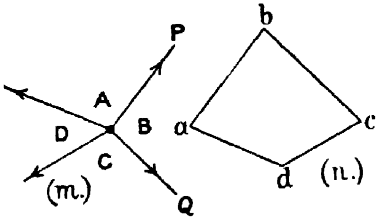


FIG. 31.

In this work, when Bow's notation is used, capital letters will be placed on the space diagram, and the corresponding small letters on the force diagram.

56. The Funicular Polygon.— Let P, Q, R, and S (Fig. 32) be four forces which act on a rigid body, and which are balanced by a fifth force T, which is at present unknown. Draw the polygon of forces *abcde*, then from what has already been shown the line *ea* which closes the polygon will represent the magnitude and direction of the fifth force T. Take any point *o* and join it to *a*, *b*, *c*, *d*, and *e*. Take any point 2 in the line of action of P and draw the line 2B3 parallel to *ob* to meet the line of action of Q at 3. Draw 3C4 parallel to *oc* to meet the line of action of R at 4. Draw 4D5 parallel to *od* to meet the line of action of S at 5. Draw 5E1 parallel to *oe* and 2A1 parallel to *oa*. The latter two lines will meet at a point 1 on the line of action of T.

Conceive that the lines A, B, C, D, and E represent bars jointed to one another at the points 1, 2, 3, 4, and 5. Then these bars may be supposed to take the place of the rigid body upon which the five forces

P, Q, R, S, and T are supposed to act. In the case under consideration (Fig. 32) it is obvious that the bars A, B, C, D, and E are subjected to tension. Consider the point 2. Here there are three forces acting which balance one another, viz. the force P and the tensions in the bars A and B, and these three forces are represented in magnitude and direction by the three sides of the triangle *abo*. Again, the three forces acting at the point 3 are represented by the sides of the triangle *bco*, also the three forces acting at the point 4 are represented by the sides of the triangle *cdo*, and the three forces at 5 by the sides of the triangle *deo*. Now in order that the tensions in the bars E and A may be balanced by the force T, the force T must act at the point of intersection of the bars E and A. The point 1 is therefore a point in the line of action of T.

The polygon 12345 is called the *funicular polygon*, the *link polygon* or the *equilibrium polygon* of the forces P, Q, R, S, and T with reference to the point *o*, which is called the *pole*.

Since the pole *o* may have an infinite number of positions, there are an infinite number of funicular polygons to any system of balanced forces.

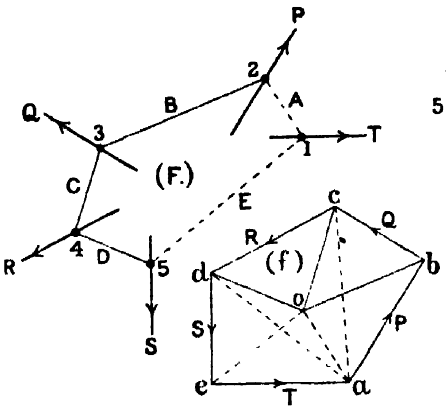


Fig. 32.

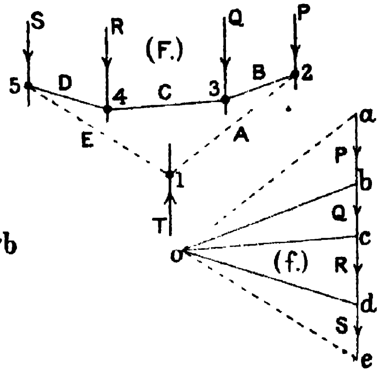


Fig. 33.

If the diagrams (F) and (*f*) (Fig. 32) be compared it will be seen that each line on the one is parallel to a corresponding line on the other. Also, if a system of lines on the one meet at a point, the corresponding lines on the other form a closed polygon. From these properties the diagrams (F) and (*f*) are called *reciprocal figures*.

No reference has yet been made to Fig. 33, but all that has been said with reference to Fig. 32 will also apply to Fig. 33, where the given forces are parallel to one another, except that the bars E and A are in compression, the remaining bars B, C, and D being in tension.

An examination of Figs. 32 and 33 will show that the simple rule to be remembered in drawing the funicular polygon is, that any side of that polygon has its extremities on the lines of action of two of the forces, and that that side is parallel to the line which joins the pole to the point of intersection of the lines which represent these two forces on the polygon of forces.

Referring to Figs. 32 and 33, it may be noted that the equilibrant of

P and Q is represented in magnitude and direction by *ca*, and that the point of intersection of the sides A and C of the funicular polygon is a point in the line of action of this equilibrant. Also the equilibrant of P, Q, and R is represented in magnitude and direction by *da*, and the point of intersection of the sides A and D of the funicular polygon is a point in the line of action of this equilibrant.

Having shown that the funicular polygon together with the polygon of forces may be used to determine the equilibrant of a system of non-concurrent forces, it is obvious that the same construction will also determine the resultant of that system of forces, since the resultant acts along the same line and has the same magnitude as the equilibrant, but acts in the opposite direction.

57. Examples of the use of the Funicular Polygon.—Two examples will now be worked out to further illustrate the use of the funicular polygon.

(1) Three vertical forces, AB, BC, and CD, act on a horizontal beam, as shown in Fig. 34. The beam rests on supports at its ends where there are vertical reactions DE and EA. It is required to determine the magnitudes of these reactions.

Since the forces are all parallel the polygon of forces will be a straight

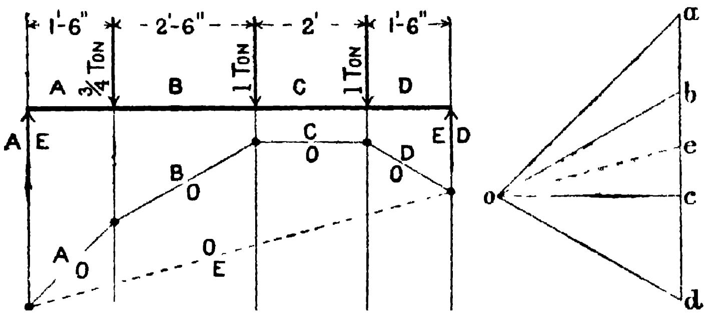


FIG. 34.

line *abcdea*, and the reactions will be represented by *de* and *ea*, the position of the point *e* being as yet unknown.

Choose a pole *o*. Join *oa*, *ob*, *oc*, and *od*. Draw OA, OB, OC, and OD parallel to *oa*, *ob*, *oc*, and *od* respectively as shown. These four lines, OA, OB, OC, and OD, will form four sides of the funicular polygon, of which OE will be the closing side. Draw *oe* parallel to OE to meet *ad* at *e*. This completes the solution. It will be found that DE = 1·48 tons, and EA = 1·27 tons.

(2) A horizontal beam AB (Fig. 35) is acted on by an inclined force P = 200 lbs., a vertical force Q = 150 lbs., and an inclined force R = 500 lbs., as shown. There is also a vertical force T, whose magnitude is unknown, acting at A, and a force S acting at B, whose magnitude and direction are both unknown. These forces being in equilibrium, it is required to determine T and S.

By the polygon of forces (*a*) and the funicular polygon 1234 the line of action 4N of the resultant U of the forces P, Q, and R is found. Replacing P, Q, and R by U, there are now only three forces acting on

the beam AB, viz. U, T, and S, and since these forces are in equilibrium and are not parallel, their lines of action must meet at a point which must be the point N where the lines of action of U and T intersect. This determines the line of action of S, and the polygon of forces being completed, the magnitudes of T and S are found to be, T = 312·6 lbs., and

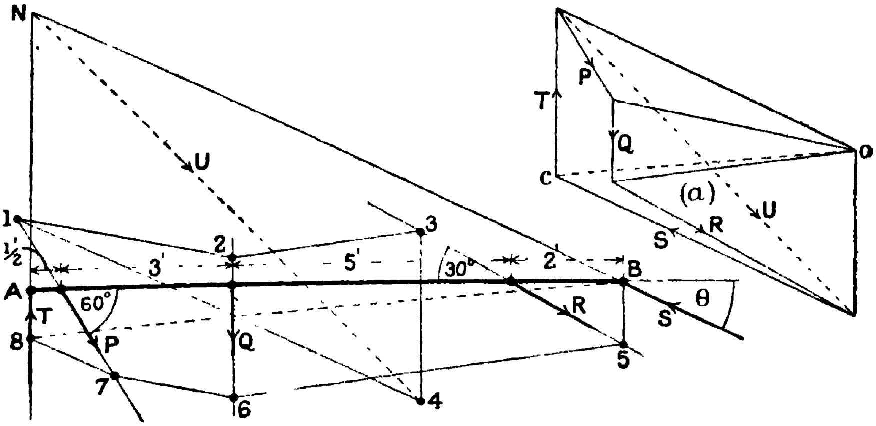


FIG. 35.

S = 593·3 lbs., and θ, the angle which the line of action of S makes with the beam, is 26° 3′.

The forces T and S may, however, be found without the use of the point N, as follows. Draw as much of the polygon of forces as the data of the problem will permit. Choose a pole o and draw the funicular polygon B5678, *starting at the point* B, which is the only point in the line of action of S which is as yet known. 8B is the closing side of this funicular polygon, and a line oc drawn parallel to 8B to meet that side of the polygon of forces which is parallel to the line of action of T will determine the remaining angular point of the polygon of forces, and will therefore fix the magnitude of T and also the direction and magnitude of S.

58. **Analytical Methods.**—The following examples illustrate the methods of solving, by calculation, problems on forces acting in a plane and at a point.

(1) Two forces, P = 20 lbs., and Q = 10 lbs., act as shown at (a), Fig. 36, the angle between their lines of action being 100°. It is required to find R, the resultant of P and Q.

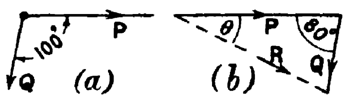


FIG. 36.

Drawing the triangle of forces shown at (b), the angle opposite to R is

$$180° - 100° = 80°.$$

Then, $R^2 = P^2 + Q^2 - 2PQ \cos 80°$

$$= 20^2 + 10^2 - 2 \times 20 \times 10 \times 0·17365 = 430·54.$$

Therefore, $R = \sqrt{430·54} = 20·75$ lbs.

The angle θ which R makes with P is found from the equation

$$\frac{\sin \theta}{\sin 80°} = \frac{Q}{R}.$$

Therefore, $\sin\theta = \dfrac{10}{20\cdot75} \times 0\cdot98481 = 0\cdot4746,$

and $\theta = 28^\circ\, 20'.$

(2) The lines of action of four forces, P, Q, S, and T, are as shown at (*a*), Fig. 37. The magnitudes of P and Q are 10 and 20 units respectively, and the four forces are in equilibrium. It is required to find the magnitudes and senses of S and T.

Drawing the polygon of forces shown at (*b*), the senses of S and T are seen at once.

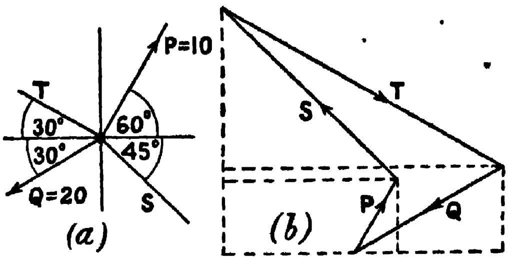


FIG. 37

Projecting the sides of the polygon on to the horizontal, it is evident that the projection of T is equal to the projection of S minus the projection of P plus the projection of Q,

or $$T\cos 30^\circ = S\cos 45^\circ - 10\cos 60^\circ + 20\cos 30^\circ \quad . \quad . \quad . \quad \text{(i)}$$

Projecting the sides of the polygon on to the vertical, it is evident that the projection of T plus the projection of Q is equal to the projection of S plus the projection of P,

or $$T\sin 30^\circ + 20\sin 30^\circ = S\sin 45^\circ + 10\sin 60^\circ \quad . \quad . \quad . \quad \text{(ii)}$$

Solving the equations (i) and (ii),
$T = 10\,(2 + \sqrt{3}) = 37\cdot32$, and $S = 20\sqrt{2} = 28\cdot28$.

(3) P_1, P_2, P_3, etc., are forces acting at a point O (Fig. 38), and their lines of action are inclined to a horizontal axis OX at angles θ_1, θ_2, θ_3, etc., respectively. Produce XO to X_1, and draw the vertical axis YOY_1. Resolve each force into two components, one along the horizontal axis and the other along the vertical axis. The horizontal components are, $P_1 \cos\theta_1$, $P_2\cos\theta_2$, $P_3\cos\theta_3$, etc.; and the vertical components are, $P_1\sin\theta_1$, $P_2\sin\theta_2$, $P_3\sin\theta_3$, etc.

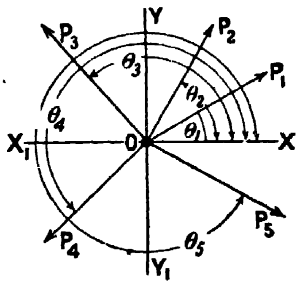


FIG. 38.

The resultant of the horizontal components is,

$$P_1\cos\theta_1 + P_2\cos\theta_2 + P_3\cos\theta_3 + \text{etc.} \;.\;.\;. = \Sigma(P\cos\theta).$$

The resultant of the vertical components is,

$$P_1\sin\theta_1 + P_2\sin\theta_2 + P_3\sin\theta_3 + \text{etc.} \;.\;.\;. = \Sigma(P\sin\theta).$$

If R is the resultant of all the forces, then
$R^2 = \{\Sigma(P\cos\theta)\}^2 + \{\Sigma(P\sin\theta)\}^2$, and the line of action of R makes an angle θ with OX such that $\tan\theta = \dfrac{\Sigma(P\sin\theta)}{\Sigma(P\cos\theta)}$.

If the forces P_1, P_2, P_3, etc., are in equilibrium, then $R = 0$, $\Sigma(P\cos\theta) = 0$, and $\Sigma(P\sin\theta) = 0$.

In applying the foregoing equations to numerical examples care must be taken to give the proper algebraical sign (+ or −) to each quantity.

Exercises IV.

It is intended that all the following exercises should be worked graphically, but in addition the student will find it advantageous to also calculate the results.

1. Determine the resultant of the forces shown at Ex. 1, Fig. 39, (1) by means of the parallelogram of forces; (2) by means of the polygon of forces, taking

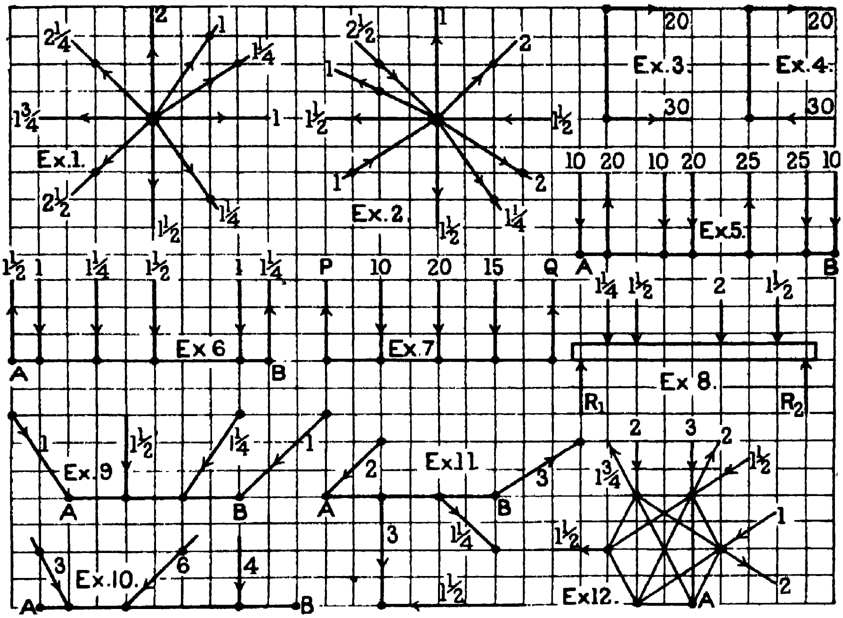


FIG. 39.

In reproducing the above diagrams the sides of the small squares are to be taken equal to half an inch.

the forces in the order given; and (3) by means of the polygon of forces, taking the forces alternately, instead of in the order given.

2. By means of the polygon of forces determine the resultant of the forces shown at Ex. 2, Fig. 39. Also, find the magnitudes of two forces, one acting horizontally and the other vertically, which will balance the given forces.

3. Using a funicular polygon, determine the resultant of the forces shown at Ex. 3, Fig. 39.

4. Using a funicular polygon, determine the resultant of the forces shown at Ex. 4, Fig. 39.

5. Find the resultant of the forces shown at Ex. 5, Fig. 39.

6. Find the resultant of the forces shown at Ex. 6, Fig. 39.

7. The forces shown at Ex. 7, Fig. 39, are in equilibrium. Find the magnitudes of P and Q.

8. A beam, loaded as shown at Ex. 8, Fig. 39, rests on supports at its ends. Determine the magnitudes of the reactions R_1 and R_2 at the supports.

9. Determine the resultant of the forces shown at Ex. 9, Fig. 39.

10. The forces shown at Ex. 10, Fig. 39, are balanced by parallel forces acting at A and B. Determine the forces at A and B.

11. Find the resultant of the forces shown at Ex. 11, Fig. 39.

12. The forces shown at Ex. 12, Fig. 39, are balanced by two forces, one of which (P) acts along the vertical line through A, and the other (Q) acts in a horizontal direction. Determine the forces P and Q.

CHAPTER V

MOMENTS AND CENTROIDS

59. Moment of a Force.—The moment of a force about a point or axis, perpendicular to its line of action, is the measure of its turning power round that point or axis. The magnitude of the moment (generally called the moment) is the product of the magnitude of the force and the perpendicular distance of its line of action from the point or axis. For example, the moment of the force AB (Fig. 40) about the point M is equal to the magnitude of the force AB multiplied by MN, the perpendicular distance of M from the line AB. If the unit of force is the pound, and the unit of distance is the inch, then the unit of moment is the *inch-pound* or *pound-inch*. Other units of moment in common use are the *foot-pound* or *pound-foot*, the *foot-ton* or *ton-foot*, and the *inch-ton* or *ton-inch*.

The construction shown in Fig. 40 is a very convenient one for determining graphically the moment of a force about a point. AB is the line of action of the force, and M is the point. The construction is as follows. Draw ab parallel to AB, and make the length of ab to represent the magnitude of the force. Through M draw $a'Mb'$ parallel to AB. Choose a pole o. Join oa and ob. Take any point o' in AB. Draw $o'a'$ parallel to oa to meet $a'Mb'$ at a', and draw $o'b'$ parallel to ob to meet $a'Mb'$ at b'. Then $a'b'$ measured with a suitable scale will be the magnitude of the moment of the force AB about the point M.

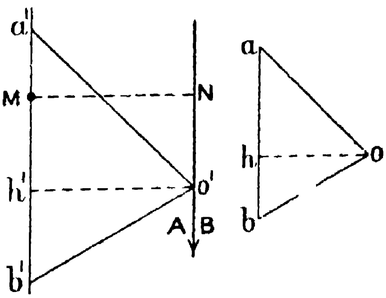


FIG. 40.

Draw oh perpendicular to ab, and $o'h'$ perpendicular to $a'b'$. The triangles oab and $o'a'b'$ are obviously similar, and $ab : a'b' :: oh : o'h'$. Hence $ab \times o'h' = a'b' \times oh$. But ab is the magnitude of the force AB, and $o'h'$, which is equal to MN, is the perpendicular distance of M from AB. Therefore $ab \times o'h'$ is equal to the moment of AB about M, and therefore $a'b' \times oh$ is equal to the moment of AB about M.

If oh is made equal to the linear unit, then $a'b'$ measured with the force scale will give the moment required. For example, if oh is 1 inch and $a'b'$ measures 20 lbs. on the force scale, then the required moment is 20 *inch-pounds*. It is not always convenient to make oh equal to the unit of distance, but it should be made a simple multiple or sub-multiple of it.

The following is the simple rule for determining the moment scale. Let oh be m times the linear unit, and let the force scale be n units of

force per inch. Then the moment scale will be $m \times n$ units of moment per inch. For example, let the linear unit be *one foot*, and suppose that *oh, measured with the linear scale*, is 4 feet. Let the force scale be 100 lbs. per inch, then the moment scale will be $100 \times 4 = 400$ *foot-pounds* per inch.

It may be pointed out that the figure $a'o'b'$ is the funicular polygon of the force AB with reference to the pole o.

60. Resultant Moment of a System of Forces.—The resultant moment of a system of forces about a point is equal to the algebraical sum of the moments of the separate forces about that point, and it is obvious that this sum must be equal to the moment of the resultant of the system about the same point. Hence the graphical determination of the resultant moment of a system of forces about a point resolves into constructing the resultant of the system, and the determination of the

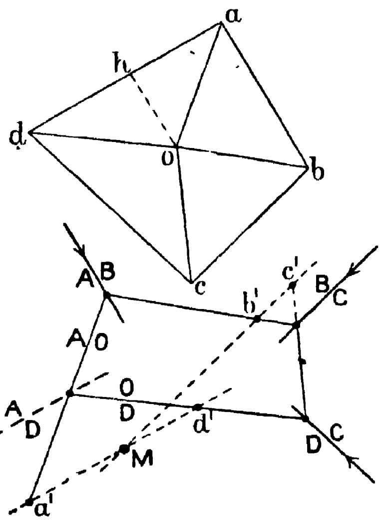


FIG. 41.

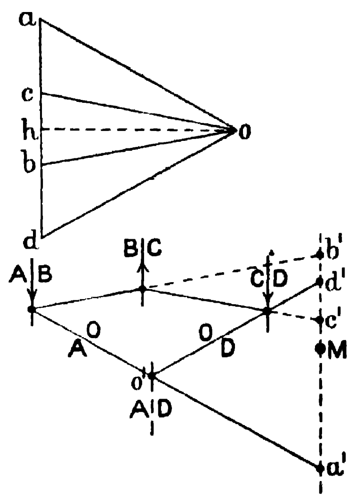


FIG. 42.

moment of this resultant about the given point by the construction of the preceding Article. The two constructions may, however, be combined in one, as shown in Figs. 41 and 42. AB, BC, and CD are three given forces, and M is a given point. It is required to determine the resultant moment of the given forces about the given point.

abcda is the force polygon; *ad*, the closing line, gives the magnitude and direction of the resultant of the three given forces. A pole o is taken at a perpendicular distance oh from ad, which is a simple multiple or sub multiple of the linear unit. The funicular polygon of the forces with reference to the pole o is next drawn, and the intersection of the closing sides OA and OD determines a point on the line of action of the resultant force AD. A line through M parallel to ad intersects the closing sides OA and OD of the funicular polygon at a' and d'. The moment required is equal to $a'd' \times oh$. The triangle $o'a'd'$ is obviously similar to the triangle oad, and therefore, as shown in the preceding Article, the moment of AD about M is equal to $a'd' \times oh$.

It may be observed that the moment of any one of the forces, say BC, is obtained by drawing through M a parallel to BC to intersect the sides of the funicular polygon which meet on BC at b' and c'; $b'c' \times oh$ is the moment of BC about M.

61. Principle of Moments.—When a number of forces acting on a rigid body are in equilibrium, then the moments of all the forces about any given axis being taken, the sum of the moments of those forces which tend to turn the body in one direction about the axis is equal to the sum of the moments of those forces which tend to turn the body in the opposite direction about the same axis.

62. Couples.—A *couple* consists of two equal parallel forces acting in opposite directions. The *arm* of a couple is the perpendicular distance between the lines of action of the two forces. The *moment* of a couple is the product of the magnitude of one of the forces and the arm of the couple. A couple tends to cause a body to rotate.

Two couples will balance one another when (1) they are in the same plane or in parallel planes, (2) they have equal moments, and (3) their directions of rotation are opposite.

63. The Centre of Parallel Forces.—If a system of parallel forces acts at fixed points, the resultant will act through another fixed point, called the centre of the system. This centre is independent of the

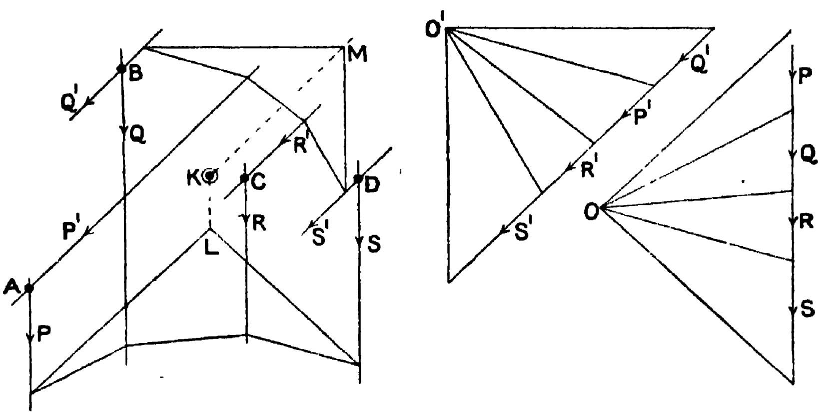


FIG. 43.

direction of the forces so long as the sense of each in relation to the sense of one of the forces is unaltered.

In Fig. 43, P, Q, R, and S are parallel forces acting at the fixed points A, B, C, and D respectively in a plane. By means of the force and funicular polygons the line of action LK of the resultant is determined. Let the direction of the forces be changed so that they act as shown by P′, Q′, R′, and S′. The line of action MK of the resultant is determined as before. The point K, where LK and MK intersect, is the centre of the parallel forces P, Q, R, and S acting at the points A, B, C, and D respectively. If the construction be repeated with the forces acting in any other direction, it will be found that the new resultant will act through the same point K.

In Fig. 43 the forces P, Q, R, and S have all the same sense, and

therefore P′, Q′, R′, and S′ must have the same sense. But if the sense of Q were opposite to that of P, then the sense of Q′ would be opposite to that of P′.

In applying the above method to the determination of the centre of a system of parallel forces, it is usually most convenient to take the two directions of the forces at right angles to one another.

In determining the centre of a system of parallel forces by calculation, it is most convenient to apply the principle of moments. Thus, let P_1, P_2, P_3, etc., be parallel forces in a plane acting at fixed points 1, 2, 3, etc., in a rigid body; choose a point X in the plane of the forces, and let the perpendicular distances of X from P_1, P_2, P_3, etc., be x_1, x_2, x_3, etc., respectively. Let R be the resultant of the forces, and x the perpendicular distance of its line of action from X, then

$$Rx = P_1x_1 + P_2x_2 + P_3x_3 + \text{etc., and } x = \frac{P_1x_1 + P_2x_2 + P_3x_3 + \text{etc.}}{R}.$$

Care must be taken to give the proper signs to the products P_1x_1, P_2x_2, P_3x_3, etc. If one force tending to turn the body in one direction about X be considered as having a positive moment, then another force tending to turn the body in the opposite direction about the point X is to be considered as having a negative moment. A line parallel to the directions of the forces and at a distance x from X will be the line of action of R. Turning all the lines of action of all forces through the same angle in the original plane, and repeating the calculation with reference to the same point X, or any other point in the plane of the forces, a new line of action of R is determined which intersects the first at the centre of the given system of forces.

If the fixed points, and therefore the lines of action of the forces, are not in the same plane, the procedure may be as follows. Select three axes, X, Y, and Z, perpendicular to one another. Take the lines of action of the forces in turn parallel to the axes Y, Z, and X, and in turn take moments about the axes X, Y, and Z to determine z, x, and y the distances from X, Y, and Z respectively of three planes parallel to XY, YZ, and ZX respectively. The point of intersection of these three planes is the centre required.

64. Centres of Gravity or Centroids.—The particles of which any body is made up are attracted to the earth by forces which are proportional to the masses of these particles. For all practical purposes these forces may be considered to be parallel, and their resultant will act through the centre of these parallel forces. In this case the centre of the parallel forces is called the *centre of gravity* or *centroid* of the body, and the determination of a centroid resolves into finding the centre of a system of parallel forces.

The centre of gravity of a body may also be defined as that point from which if the body is suspended it will balance in any position.

When the term centre of gravity is applied to a line, the line is supposed to be made of indefinitely thin wire; and when the centre of gravity of a surface is spoken of, the surface is supposed to be made of indefinitely thin substance.

The following results, which are not difficult to prove, should be noted :—

The centroid of a straight line is at its middle point.

The centroid of a triangle is at the intersection of its medians.

The centroid of a parallelogram is at the intersection of its diagonals.

If a plane figure is symmetrical about a straight line, the centroid of the figure is in that straight line.

65. Examples on the Determination of Centroids.—(1) ABC is a triangle AB = BC = $2\frac{1}{2}$ inches, AC = $3\frac{1}{2}$ inches. D is a point within the triangle ABC $2\frac{1}{4}$ inches from A and $1\frac{1}{2}$ inches from B. Small bodies are placed at the points A, B, C, and D, their masses being proportional to the numbers 3, 3·5, 2·5, and 5 respectively. It is required to find the centre of gravity of the four bodies.

The graphic method of working this example is fully explained in Art. 63, and is illustrated by Fig. 43. The dimensions in Fig. 43 are, however, not the same as given above. (If G be the required centre of gravity, then AG = 1·94 inches, and BG = 1·21 inches.)

(2) A piece of wire of uniform thickness is bent to the form ABCD

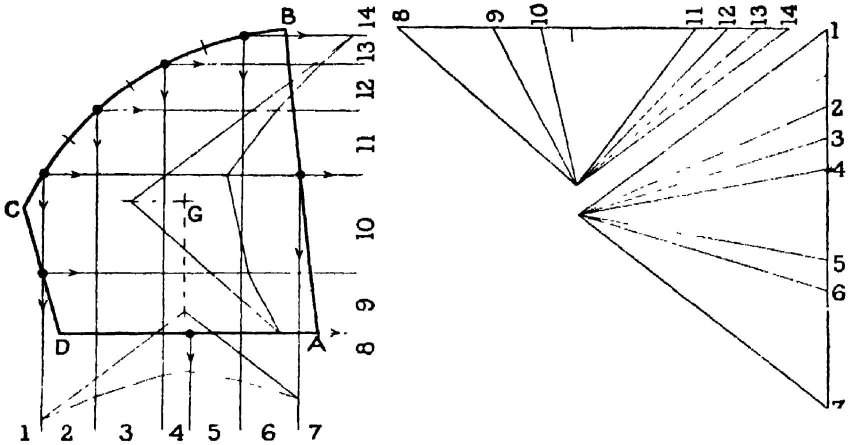


FIG. 44.

(Fig. 44). The parts AB, CD, and DA are straight, and the part BC is an arc of a circle whose centre is A. AB = $2\frac{1}{2}$ inches, CD = 1 inch, DA = 2 inches, and the arc BC subtends an angle of 60° at A. It is required to find the centre of gravity of the frame ABCD.

The arc BC is divided into four equal parts, and the centre of gravity of each of these is assumed to be at its middle point. This assumption only involves a small error, because the arcs are small compared with the radius of the circle. It may also be assumed that the weights of these small arcs of wire are proportional to the length of their chords. The weights of the straight sides are proportional to their lengths, and their centres of gravity are at their middle points. The weight of each part into which the frame is divided may be supposed to act at its centre of gravity, and the problem becomes similar to the preceding one. G is the required centre of gravity.

Scales to be used.—For the frame, full size. For the forces, 1 inch equal to the weight of 2 inches of wire.

(3) A plane-figure is formed by removing from a triangle ABC (Fig. 45) triangles ADE and FHK. It is required to find the centroid of the figure.

The centroids of the triangles ABC, ADE, and FHK are first determined by the intersections of their medians. Conceive that the triangle ABC is made of very thin sheet metal, and that it is suspended from its centre of gravity by a string. The tension R in the string would be equal to the weight (or area) of the triangle. The upward force R would be balanced by the downward forces W, P, and Q, where W is the weight (or area) of the shaded figure acting at its centre of gravity G (as yet unknown), P is the weight (or area) of the triangle FHK acting at its centre of gravity, which is known, and Q is the weight (or area) of the triangle ADE acting at its centre of gravity, which is known. The parallel forces R, P, and Q are completely known, and G, their centre, is the centroid required. The force and funicular polygons for finding G are not shown.

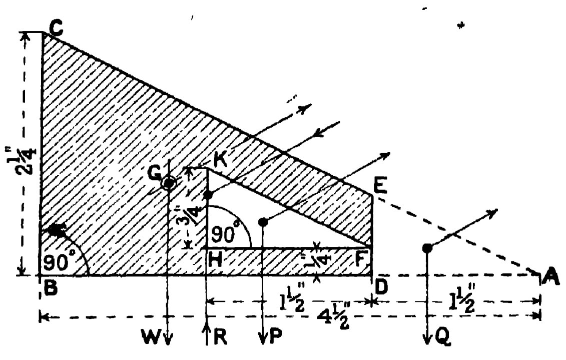


FIG. 45.

To work this example by calculation proceed as follows:—

$R =$ area of ABC $= \frac{1}{2} \times 4\frac{1}{2} \times 2\frac{1}{4} = \frac{81}{16}$ square inches.

$P =$ area of FHK $= \frac{1}{2} \times 1\frac{1}{2} \times \frac{3}{4} = \frac{9}{16}$ square inch.

$Q =$ area of ADE $= \left(\frac{1\frac{1}{2}}{4\frac{1}{2}}\right)^2 \times \frac{81}{16} = \frac{9}{16}$ square inch.

$W =$ shaded area $= R - P - Q = \frac{81}{16} - \frac{9}{16} - \frac{9}{16} = \frac{63}{16}$ square inches.

Distance of centroid of ABC from BC $= \frac{1}{3} \times 4\frac{1}{2} = 1\frac{1}{2}$ inches.

Distance of centroid of FHK from BC $= 4\frac{1}{2} - 1\frac{1}{2} - \frac{2}{3} \times 1\frac{1}{2} = 2$ inches.

Distance of centroid of ADE from BC $= 4\frac{1}{2} - \frac{2}{3} \times 1\frac{1}{2} = 3\frac{1}{2}$ inches.

Distance of centroid G from BC $= x$.

Take forces parallel to BC, and take moments about B, then $\frac{81}{16} \times 1\frac{1}{2} = \frac{9}{16} \times 2 + \frac{9}{16} \times 3\frac{1}{2} + \frac{63}{16}x$. Hence $x = 1\frac{1}{7}$ inches.

Taking the forces parallel to AB, and taking moments about B, the distance y of the centroid G from AB is found to be $\frac{6}{7}$ inch.

Further examples on the determination of centroids will be found later on in this chapter in connection with moments of inertia.

66. Centre of Pressure and Centre of Stress.—If a plane figure be subjected to fluid pressure, the point in the plane of the figure at which the resultant of the pressure acts is called the *centre of pressure*. If a plane figure be a section of a bar which is subjected to stress, the point in the plane of the section at which the resultant of the stress acts is called the *centre of stress*.

If the pressure or stress be uniform over the figure, then the centre of pressure or centre of stress coincides with the centroid of the figure.

A general construction for determining the centre of pressure or centre of stress of any plane figure when the pressure or stress varies

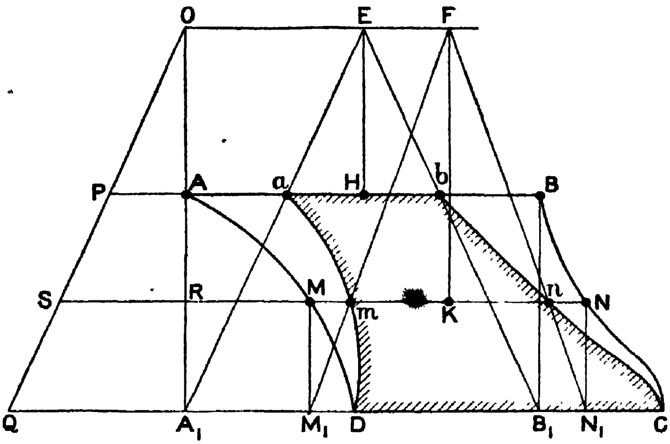


FIG. 46.

uniformly in one direction is illustrated by Fig. 46. ABNCDMA is a plane figure supposed to be vertical, and AB and CD are horizontal, AA_1 is the altitude of the figure, and the pressure or stress is supposed to vary uniformly from an amount represented by AP at the level AB to an amount represented by A_1Q at the level CD. AP and A_1Q are horizontal. Join QP and produce it to meet A_1A produced at O. Draw any horizontal SRMN to cut the given figure. Draw the horizontal OF, and the verticals MM_1 and NN_1. Through K, the middle point of MN, draw the vertical KF. Join FM_1 and FN_1 cutting MN at m and n. If this construction be repeated at a sufficient number of levels, and all points corresponding to m be joined, also all points corresponding to n, a figure abnCDma is obtained, and the centroid of this figure will be the centre of pressure or centre of stress of the original figure.

The proof is as follows. Suppose that the line MN is the centre line of a *very narrow* horizontal strip of the original figure, and let the width of this strip be denoted by w. The magnitude of the resultant pressure or stress on this strip is equal to $MN \times w \times RS$, and it will act at K, the middle point of MN.

Since	SRMN is parallel to QA_1DC,
	$M_1N_1 : mn :: OA_1 : OR$,
and,	$MN : mn :: A_1Q : RS$,
therefore	$MN \times RS = mn \times A_1Q$,
and	$MN \times w \times RS = mn \times w \times A_1Q$,

that is, the resultant of the pressure or stress on the strip of length mn when subjected to a pressure or stress A_1Q will have the same magnitude as the resultant of the pressure or stress on the strip of length MN when subjected to a pressure or stress RS, and it will act at the same point K, which is also the middle point of mn.

It follows that the resultant of the pressure or stress on the figure

*abn*CD*ma* when subjected to a uniform pressure or stress A_1Q will be the same as the resultant of the varying pressure or stress on the original figure. But when the pressure or stress on a plane figure is uniform, the centre of pressure or centre of stress is at its centroid. Therefore the centroid of the figure *abn*CD*ma* is the centre of pressure or centre of stress of the original figure.

Exercises Va.

The following exercises may be worked graphically, or by calculation, or by a combination of these methods.

1. ABCD is a square of $6\frac{1}{2}$ feet side. A force $P=5$ tons acts from A to D, and a force $Q=3\frac{1}{2}$ tons acts from B to C. Determine the moment (in foot-tons) of the resultant of P and Q about a point within the square and 4 feet from AD.

1a. Same as preceding exercise, except that the force P acts from D to A instead of from A to D.

1b. ABC is a triangle, $AB=1\frac{1}{2}$ inches, $BC=2\frac{1}{2}$ inches, and $CA=2$ inches. A force P has a moment of -12 inch-lbs. about A, a moment of -30 inch-lbs. about B, and a moment of $+20$ inch-lbs. about C. Determine the magnitude and line of action of the force P.

2. Six parallel forces, having the same sense, act at the angular points A, B, C, D, E, and F of a regular hexagon of 2 inches side. The magnitudes of the forces, taking them in the order A, B, C, etc., are 2, $1\frac{1}{2}$, $2\frac{1}{2}$, 3, 1, and $1\frac{1}{2}$. Find the centre of these parallel forces.

3. ABC is a right-angled triangle having the right angle at C. $AB=2\frac{1}{2}$ inches, $AC=1\frac{1}{2}$ inches. Determine the centroid of the three squares described on the three sides of this triangle.

3a. Determine the centroid of the three equilateral triangles described on the sides of the triangle given in the preceding exercise.

4. A wire is bent into the zig-zag form ABCD shown at Ex. 4, Fig. 47, and

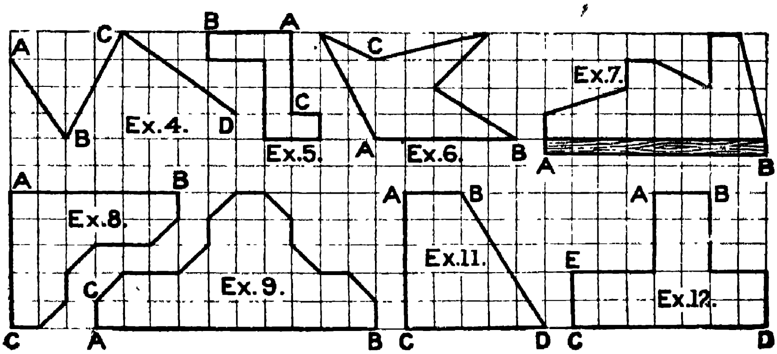


FIG. 47.

In reproducing the above diagrams the sides of the small squares are to be taken equal to half an inch.

is suspended by a string attached to the wire at the point A. Draw the direction of the string.

5. Determine the centroid of the figure shown at Ex. 5, Fig. 47.

6. Determine the centroid of the figure shown at Ex. 6, Fig. 47.

7. The intensity of the load at any point of the beam AB, Ex. 7, Fig. 47, is proportional to the height of the diagram above the beam at that point. The length of AB is 16 feet. Determine the position of the resultant load.

8. Determine the centroid of the figure shown at Ex. 8, Fig. 47.

9. Determine the centroid of the figure shown at Ex. 9, Fig. 47.

10. OAB is a quadrant of a circle, the radii OA and OB being $2\frac{1}{4}$ inches long. CD is a straight line cutting OB at C and OA at D. OC = 2 inches, OD = $1\frac{1}{2}$ inches. Determine the centroid of the figure ABCD.

11. The figure shown at Ex. 11, Fig. 47, is subjected to fluid pressure, which varies uniformly from $\frac{1}{2}$ lb. per square inch at the level AB to $1\frac{1}{2}$ lbs. per square inch at the level CD. Determine the position of the centre of pressure of the figure.

12. The figure shown at Ex. 12, Fig. 47, represents the section of a bar which is subjected to tensile stress. The stress varies uniformly from nothing at AB to 3 tons per square inch at CD. Determine the position of the centre of stress of the section.

13. A vertical wall is 80 yards long and 42 feet high. The adjoining table gives the pressures of the wind on it, p pounds per square foot, at various heights h feet above the ground. Draw a diagram showing the relation between p and h. Find the mean pressure on the wall in lbs. per square foot, and the total wind force on the wall in lbs. Find the line of action of this force. Employ scales of 1 inch to 10 feet, and 1 inch to 10 lbs. per square foot. [B.E.]

h	4	10	18	25	33	42
p	9	12	16·7	20·3	23·5	26

67. Moment of Inertia.—The sum of the products of the mass of each elementary part of a body and the square of its distance from a given axis is called the *moment of inertia* of the body about that axis. Thus, if m_1, m_2, m_3, etc., be the masses of the parts of the body, and r_1, r_2, r_3, etc., be the distances of these parts respectively from the axis, then the moment of inertia $= I = m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \text{etc.} \ldots = \Sigma mr^2$.

The *moment of inertia of an area* and the *moment of inertia of a line* are defined in a similar manner by substituting *area* or *length* for *mass*. But since areas and lines have no inertia, they have, strictly speaking, no moment of inertia.

The *moment of inertia of a force* about an axis perpendicular to the line of action of the force is the product of its magnitude and the square of the distance of its line of action from the axis.

The graphic method of determining the moment of inertia of a plane area, or of a system of parallel forces, will be understood from the two examples worked out in Figs. 48 and 49.

Fig. 48 shows the application of the method to finding the moment of inertia of a force AB about a point M or about an axis through M and perpendicular to the plane of the paper. Through M draw MY parallel to AB. Draw MN perpendicular to AB. Applying the construction explained in Art. 59, $a'b' \times oh = AB \times MN$. Choose a pole o' at a distance $o'h'$ from $a'b'$, which is a simple multiple or submultiple of the linear unit. From a point n'' in AB draw $n''a''$ parallel to $o'a'$ and $n''b''$ parallel to $o'b'$. Since the triangle $a''b''n''$ is similar to the triangle $a'b'o'$, it follows that $a''b'' \times o'h'$ $= a'b' \times MN$, and therefore $a''b'' \times o'h' \times oh = a'b' \times oh \times MN$. But $a'b' \times oh$

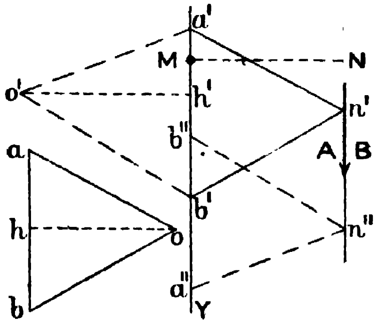


FIG. 48.

$=AB \times MN$. Therefore $a''b'' \times o'h' \times oh = AB \times MN^2 =$ moment of inertia of AB about M. $a'b'n'$ and $a''b''n''$ are funicular polygons, of which the first determines the moment $AB \times MN$, and the second determines the moment of this moment, namely, $(AB \times MN) \times MN$. The lengths $a'b'$ and $a''b''$ must be measured with the force scale, and the lengths oh and $o'h'$ with the linear scale.

Fig. 49 shows the application of the method to the determination of the moment of inertia of the shaded figure about an axis $a'a''$ in the plane of the figure. The area is divided into parallel strips, and parallel forces AB, BC, CD, DE, and EF are supposed to act at the centres of gravity of these strips, the magnitudes of the forces being proportional to the areas of the strips. The sum of the moments of these forces about the given axis is equal to $a'f' \times oh$, and the sum of their moments of inertia is equal to $a''f'' \times o'h' \times oh$. The lengths $a'f$ and $a''f''$ must be measured with the area scale and the lengths oh and $o'h'$ with the linear scale.

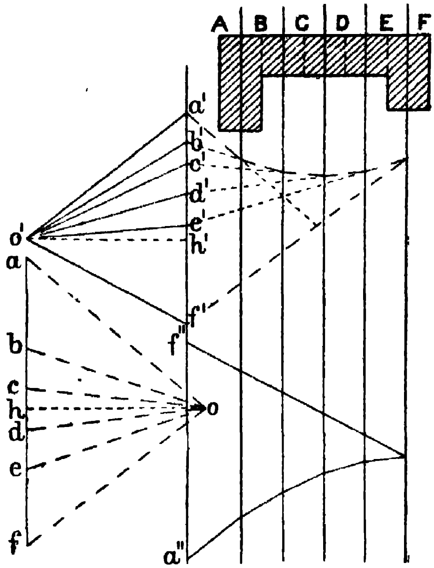


FIG. 49.

68. Moment of Inertia—Theorems.—A knowledge of certain theorems, which will now be proved, will be found of great use in solving problems on moment of inertia.

Theorem I.—If I_x and I_y are the moments of inertia of a plane figure (Fig. 50) about axes OX and OY in its plane and perpendicular to one another, and if I_z is the moment of inertia of the figure about an axis OZ perpendicular to the plane XOY, then $I_z = I_x + I_y$.

Consider a small element P of the figure, whose distance from OY is x, whose distance from OX is y, and whose distance from O is r, and let a denote the area of this small element. Then $r^2 = x^2 + y^2$, $ar^2 = ax^2 + ay^2$,

$\Sigma ar^2 = \Sigma ax^2 + \Sigma ay^2$, therefore $I_z = I_x + I_y$.

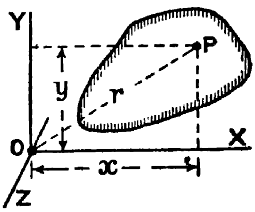


FIG. 50.

Corollary 1.—If OZ is a fixed axis perpendicular to the plane of the figure, and if OX and OY are any two axes in that plane and perpendicular to one another, then $I_x + I_y$ being equal to I_z is constant.

Corollary 2.—Since $I_x + I_y$ is constant, it follows that if I_x is a maximum, I_y is a minimum.

Theorem II.—Let I = the moment of inertia of a surface EF (Figs. 51 and 53) or a body HK (Fig. 52) about an axis XX passing through its centre of gravity G; I_1 = the moment of inertia of the surface or body about an axis X_1X_1 parallel to XX and at a distance r from it; A = area

of surface; W = weight of body; then $I_1 = I + Ar^2$ for the surface, and $I_1 = I + Wr^2$ for the body.

Consider a small element P of the surface or body, and let m denote

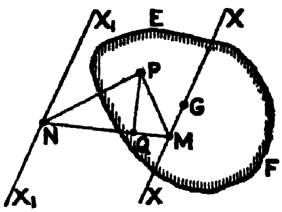


FIG. 51.

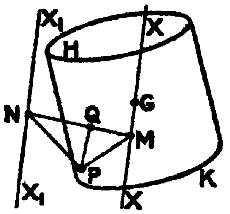


FIG. 52.

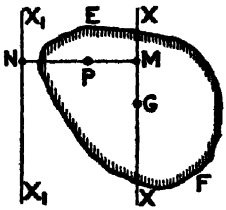


FIG. 53.

its area or weight. Referring now to Figs. 51 and 52, let PM and PN be perpendiculars from P to the axes XX and X_1X_1 respectively, and let PQ be the perpendicular to MN from P. Then,

$$\overline{PN}^2 = \overline{MN}^2 + \overline{PM}^2 - 2\overline{MN} \cdot \overline{MQ}$$

$$\Sigma m\overline{PN}^2 = \Sigma m\overline{MN}^2 + \Sigma m\overline{PM}^2 - 2\overline{MN}\Sigma m\overline{MQ},$$

but $\Sigma m\ \overline{MQ} = 0$, therefore $I_1 = I + Ar^2$ for the surface, and $I_1 = I + Wr^2$ for the body.

The case which is of most importance, on account of its frequent occurrence in practice, is the simple one in which the surface EF is a plane figure (Fig. 53), and the parallel axes XX and X_1X_1 are in the plane of the figure. In this case P and Q coincide.

Corollary 1.—If k and k_1 are the radii of gyration about the axes XX and X_1X_1 respectively, $I = Ak^2$ or Wk^2, and $I_1 = Ak_1^2$ or Wk_1^2. Hence $k_1^2 = k^2 + r^2$.

Corollary 2.—The radius of gyration about a given axis passing through the centre of gravity is less than the radius of gyration about an axis parallel to the given axis, and the axis about which the radius of gyration is least must pass through the centre of gravity.

69. Moment of Inertia—Fundamental Examples.—The graphical method of finding moments of inertia was explained in Article 67, p. 50. The analytical method will now be used, and in practice this is generally the most convenient.

(1) Straight line, or straight and uniform slender rod (Fig. 54) about an axis X_1X_1 perpendicular to it, and passing through one end.

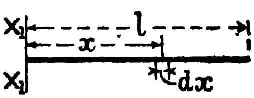


FIG. 54.

Consider an element of length dx at a distance x from the axis. Let w denote the weight of the rod per unit of length. The weight of this element is wdx, its moment of inertia is wx^2dx, and the total moment of inertia

$$I_1 = \int_0^l wx^2dx = w\int_0^l x^2dx = \frac{wl^3}{3} = \frac{Wl^2}{3},$$

where W is the total weight of the rod.

Radius of gyration squared $= k_1^2 = \dfrac{l^2}{3}$.

If the axis passes through the centre of the rod instead of through one end, it follows that $I = \frac{Wl^2}{12}$ and $k^2 = \frac{l^2}{12}$.

(2) Rectangle or parallelogram (Fig. 55) base of length a, and altitude b, about an axis X_1X_1, coinciding with the base.

Consider an element of width dx parallel to the axis, and at a

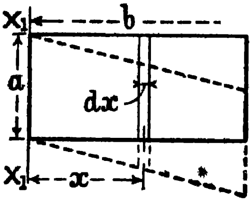


FIG. 55.

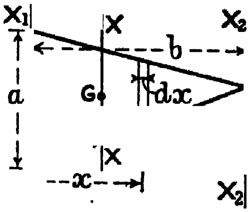


FIG. 56.

distance x from it. The area of this element is adx, its moment of inertia is ax^2dx, and the total moment of inertia

$$I_1 = \int_0^b ax^2dx = a\int_0^b x^2dx = \frac{ab^3}{3}, \text{ and } k_1^2 = \frac{b^2}{3}.$$

If the axis passes through the centre of gravity of the rectangle or parallelogram and is parallel to the base, then it follows that

$$I = \frac{ab^3}{12}, \text{ and } k^2 = \frac{b^2}{12}.$$

(3) Triangle (Fig. 56) base of length a, and altitude b, about an axis X_1X_1 coinciding with the base.

Consider an element of width dx parallel to the axis, and at a distance x from it. The area of this element is $\frac{a(b-x)dx}{b}$, its moment of inertia is $\frac{a(b-x)x^2dx}{b}$, and the total moment of inertia

$$I_1 = \int_0^b \frac{a(b-x)x^2dx}{b} = \frac{a}{b}\int_0^b (bx^2dx - x^3dx) = \frac{a}{b}\left(\frac{b^4}{3} - \frac{b^4}{4}\right) = \frac{ab^3}{12}.$$

If the axis XX passes through the centre of gravity G of the triangle and is parallel to the base, then by Theorem II., Art. 68, p. 51,

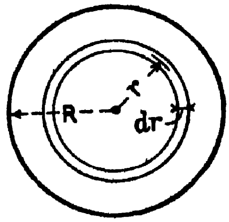
$$I_1 = I + \tfrac{1}{2}ab\left(\frac{b}{3}\right)^2. \text{ Therefore } I = \frac{ab^3}{36}.$$

If the axis X_2X_2 passes through the vertex of the triangle and is parallel to the base, then

$$I_2 = I + \tfrac{1}{2}ab\left(\frac{2b}{3}\right)^2 = \frac{ab^3}{36} + \frac{2ab^3}{9} = \frac{ab^3}{4}.$$

(4) Circle (Fig. 57) of radius R about an axis passing through its centre and perpendicular to its plane.

Consider an element of the form of a ring concentric with the circle and having a width dr and a radius r. The area of this element is $2\pi r dr$, its moment of inertia is $2\pi r^3 dr$, and the total moment of inertia



$$I_0 = \int_0^R 2\pi r^3 dr = 2\pi \int_0^R r^3 dr = \frac{\pi R^4}{2}.$$

FIG. 57.

(5) Circle of radius R about a diameter. If I is the moment of inertia of the circle about a diameter XOX, then I will also be the moment of inertia of the circle about a diameter YOY at right angles to XOX. But the moment of inertia about an axis through the centre O and perpendicular to the plane of the circle is by Theorem 1, Art. 68, p. 51, equal to $I + I = 2I$, and by the preceding example this is equal to $\frac{\pi R^4}{2}$, therefore $I = \frac{\pi R^4}{4}$.

(6) A right prism or right cylinder of any cross-section about an axis X_1X_1 (Fig. 58) in the plane of the base and passing through G_1 the centre of gravity of the base.

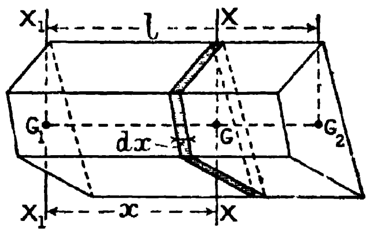


FIG. 58.

Let a = area of base, l = length of solid, I_0 = moment of inertia of base about axis X_1X_1. Consider a thin parallel slice of thickness dx parallel to the base and at a distance x from it. The centre of gravity G of this slice will lie on the line G_1G_2, joining the centres of gravity of the ends. Take an axis XX through G and parallel to X_1X_1. Then, moment of inertia of slice about $XX = I_0 dx$, and moment of inertia of slice about $X_1X_1 = I_0 dx + ax^2 dx$. Hence I_1, the moment of inertia of the whole solid about X_1X_1, is

$$I_1 = \int_0^l I_0 dx + \int_0^l ax^2 dx = I_0 \int_0^l dx + a\int_0^l x^2 dx = I_0 l + \tfrac{1}{3}al^3.$$

(7) A solid of revolution about its axis. Fig. 59 shows the section of a solid wheel or pulley. Take a parallel strip of this section parallel to the axis XX of the solid. The distances of the outside and inside of this strip from XX are R and r respectively, and its mean width is $AB = x$. Consider this strip as the section of a ring whose axis is XX. The moment of inertia of this ring about XX is approximately $\frac{\pi}{2}(R^4 - r^4)x$. If the ends of the ring are parallel its moment of inertia is exactly $\frac{\pi}{2}(R^4 - r^4)x$, and when the ends are

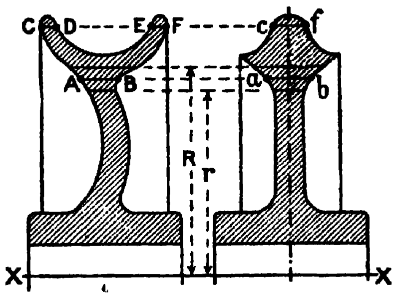


FIG. 59. FIG. 60.

not parallel the error in the above value for its moment of inertia is less the smaller the difference $R - r$. The moment of inertia of the whole solid is the sum of the moments of inertia of all the rings into which it is divided.

Fig. 60 shows how to convert the section of Fig. 59 into an equivalent section symmetrical about an axis perpendicular to XX.

$$ab = AB, \text{ and } cf = CD + EF.$$

70. General Method of finding Moments of Inertia of Irregular Plane Figures.—Taking a standard rail section (Fig. 61) as an example,* let it be required to find the moment of inertia I of the section about an axis XX passing through its centre of gravity and perpendicular to its one axis of symmetry YY. Divide the section into a number of parallel strips, preferably of equal width, perpendicular to YY, and draw the centre lines, shown dotted, of these strips. In Fig. 61, 15 strips, each $\frac{3}{8}$-inch wide, have been taken. Take an axis X_1X_1 perpendicular to YY and touching the lower end of the section. Let x be the length of any one strip measured at the centre of its width, and let y be the distance of its centre line from X_1X_1. To avoid fractions in this example the linear unit is taken in the first instance as one-sixteenth of an inch. The area of any one strip is its width multiplied by x, and is denoted by a. Let $\bar{y}$ denote the distance of the centre of gravity of the section from X_1X_1. Then $\bar{y}\Sigma a = \Sigma ay$, and the moment of inertia about $X_1X_1 = I_1 = \Sigma ay^2$. The work of finding Σa, Σay, and Σay^2 should be tabulated as shown below.

y	x	a	ay	ay^2
87	38	6 × 38	18 × 1102	54 × 31958
81	42	6 × 42	18 × 1134	54 × 30618
75	42	6 × 42	18 × 1050	54 × 26250
69	41	6 × 41	18 × 943	54 × 21689
63	22	6 × 22	18 × 462	54 × 9702
57	11	6 × 11	18 × 209	54 × 3971
51	11	6 × 11	18 × 187	54 × 3179
45	11	6 × 11	18 × 165	54 × 2475
39	11	6 × 11	18 × 143	54 × 1859
33	11	6 × 11	18 × 121	54 × 1331
27	11	6 × 11	18 × 99	54 × 891
21	11	6 × 11	18 × 77	54 × 539
15	23	6 × 23	18 × 115	54 × 575
9	41	6 × 41	18 × 123	54 × 369
3	39	6 × 39	18 × 39	54 × 39
Totals.		6 × 365 $= \Sigma a$	18 × 5969 $= \Sigma ay$	54 × 135445 $= \Sigma ay^2$

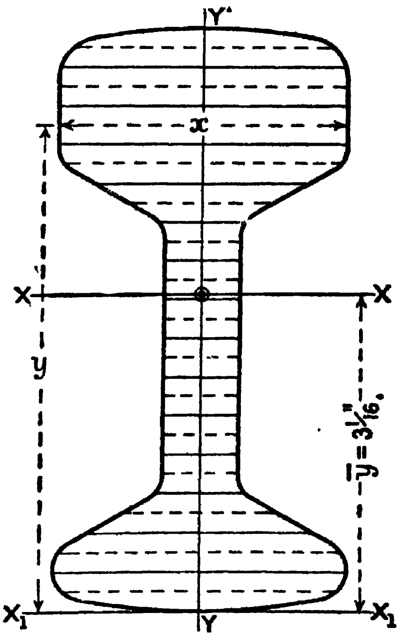


FIG. 61.

$$\bar{y} = \frac{\Sigma ay}{\Sigma a} = \frac{18 \times 5969}{6 \times 365} = 49 \text{ sixteenths of an inch.}$$

$$\bar{y} = \frac{49}{16} = 3\tfrac{1}{16} \text{ inches.} \quad I_1 = \frac{54 \times 135445}{16^4} = 111 \cdot 6 \text{ in inch units.}$$

* Fig. 61 is half full size.

$$\text{Area of section} = A = \frac{6 \times 365}{16^2} = 8{\cdot}55 \text{ square inches.}$$

$$I = I_1 - A\bar{y}^2 = 111{\cdot}6 - 8{\cdot}55 \times \frac{49^2}{16^2} = 31{\cdot}4 \text{ in inch units.}$$

$$(\text{Radius of gyration})^2 = k^2 = \frac{31{\cdot}4}{8{\cdot}55} = 3{\cdot}67, \text{ and } k = \sqrt{3{\cdot}67} = 1{\cdot}92 \text{ inches.}$$

71. Moments of Inertia of Plane Figures made up of Rectangles.—A large number of beam and column sections are made up of rectangles whose sides are either parallel or perpendicular to the axis about which the moment of inertia is required. In such cases the procedure in finding the position of the centre of gravity and moment of inertia is as follows. Referring to Fig 62, let the rectangle of breadth b and depth d be one of the rectangles of which the section is made up, and let X_1X_1 be an axis parallel to the side b. Let y be the distance of the centre of the rectangle from X_1X_1. The area of this rectangle is bd, and the area of the whole section is Σbd. The moment of the area of this rectangle about X_1X_1 is bdy, and $\bar{y} = \dfrac{\Sigma bdy}{\Sigma bd}$, where $\bar{y}$ is the distance of the centre of gravity of the whole section from X_1X_1. The moment of inertia of this rectangle about X_1X_1 is $\dfrac{bd^3}{12} + bdy^2 = bd\left(\dfrac{d^2}{12} + y^2\right)$, and the moment of inertia of the whole section about X_1X_1 is $\Sigma bd\left(\dfrac{d^2}{12} + y^2\right) = I_1$.

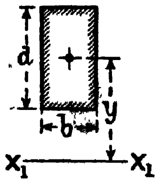


FIG. 62.

The moment of inertia of the whole section about an axis passing through its centre of gravity and parallel to X_1X_1 is $I_1 - \bar{y}^2\Sigma bd$.

The particulars for each rectangle and the results of the calculations should be tabulated in a form such as is shown below.

b	d	bd	y	bdy	$\frac{d^2}{12}$	y^2	$bd\left(\frac{d^2}{12}+y^2\right)$
Totals . .		Σbd		Σbdy			$\Sigma bd\left(\frac{d^2}{12}+y^2\right)$

72. Transformation of Moments of Inertia—Principal Axes of Inertia.—Let EF (Fig. 63) be a plane figure, and OA and OB two axes at right angles to one another in the plane of EF. Let A and B denote the moments of inertia of EF about the axes OA and OB respectively. Let OP and OQ be two other axes in the plane of EF, perpendicular to one another, and inclined at angles θ and $90° + \theta$ respectively to OA. Let P and Q denote the moments of inertia of

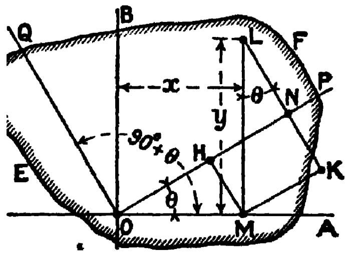


FIG. 63.

EF about the axes OP and OQ respectively. It is required to find the relations between P and Q, and A, B, and θ.

Consider a small element of the figure EF at L, the area of this element being a. Draw LM perpendicular to OA, LNK and MH perpendicular to OP, and MK parallel to OP. Let $OM = x$, and $LM = y$. $LN = LK - KN = LK - MH = y \cos\theta - x \sin\theta$, and

$$\overline{LN}^2 = y^2 \cos^2\theta + x^2 \sin^2\theta - 2xy \sin\theta \cos\theta$$
$$= y^2 \cos^2\theta + x^2 \sin^2\theta - xy \sin 2\theta.$$

The moment of inertia of the element at L about the axis OP is equal to $ay^2 \cos^2\theta + ax^2 \sin^2\theta - axy \sin 2\theta$, and the moment of inertia of the whole figure EF about OP is

$P = \Sigma ay^2 \cos^2\theta + \Sigma ax^2 \sin^2\theta - \Sigma axy \sin 2\theta$, therefore

$P = A \cos^2\theta + B \sin^2\theta - C \sin 2\theta$, where $C = \Sigma axy$.

Changing θ into $90° + \theta$, $Q = A \sin^2\theta + B \cos^2\theta + C \sin 2\theta$.

Hence $P - Q = (A - B) \cos 2\theta - 2C \sin 2\theta$.

If the moment of inertia of the figure is a maximum about the axis OP, then P will be a maximum and Q a minimum, also $P - Q$ will be a maximum.

Differentiating, $\dfrac{d(P-Q)}{d\theta} = -2(A - B) \sin 2\theta - 4C \cos 2\theta$, and when $P - Q$ is a maximum, $-2(A - B) \sin 2\theta - 4C \cos 2\theta = 0$, and

$$2C = -(A - B) \tan 2\theta.$$

Hence when $P - Q$ is a maximum

$P - Q = (A - B) \cos 2\theta + (A - B) \tan 2\theta \sin 2\theta$, therefore

$A - B = (P - Q) \cos 2\theta$. But $A + B = P + Q$, therefore

$$A = P\left(\frac{1 + \cos 2\theta}{2}\right) + Q\left(\frac{1 - \cos 2\theta}{2}\right) = P \cos^2\theta + Q \sin^2\theta, \text{ and}$$

$$B = P\left(\frac{1 - \cos 2\theta}{2}\right) + Q\left(\frac{1 + \cos 2\theta}{2}\right) = P \sin^2\theta + Q \cos^2\theta.$$

The axes OP and OQ, about which the moments of inertia are a maximum and a minimum respectively, are called the *principal axes of inertia* of the figure for the point O. When O is the centre of gravity of the figure the axes OP and OQ are then called *the principal axes of inertia* of the figure.

If a plane figure is symmetrical about an axis in its plane, it is obvious that that axis is one of the principal axes, and if the figure is symmetrical about two perpendicular axes in its plane, these will be the principal axes.

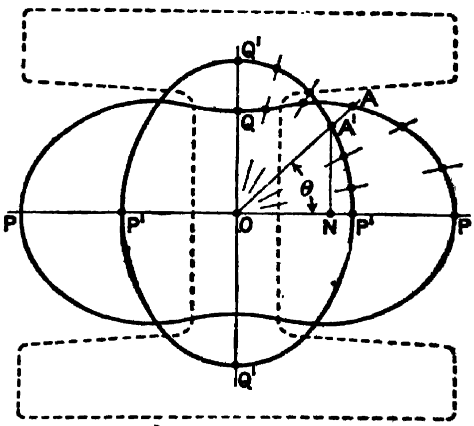


FIG. 64.

73. Inertia Curves and Momental Ellipse.—Let OP and OQ (Fig. 64) be the principal axes of inertia of the plane figure shown by dotted lines.

Let the moments of inertia of the figure about OP and OQ be OP = P and OQ = Q respectively, and let the moment of inertia of the figure about an axis OA making an angle θ with OP be OA = A. Then by the formula proved in the preceding Article, $A = P\cos^2\theta + Q\sin^2\theta$. If P and Q are known, and A be calculated for different values of θ and the results plotted, a curve PAQ, called an *inertia curve*, for the given figure is determined.

Let a denote the area of the given figure. On OP make $OP' = r_1 = \sqrt{\frac{a}{P}}$, on OQ make $OQ' = r_2 = \sqrt{\frac{a}{Q}}$, and on OA make $OA' = r = \sqrt{\frac{a}{A}}$. Draw A′N perpendicular to OP. Let ON = x, and A′N = y.

$$\frac{1}{r^2} = \frac{A}{a} = \frac{P}{a}\cos^2\theta + \frac{Q}{a}\sin^2\theta = \frac{x^2}{r_1^2 r^2} + \frac{y^2}{r_2^2 r^2}, \text{ therefore } \frac{x^2}{r_1^2} + \frac{y^2}{r_2^2} = 1,$$

and therefore the locus of A′ is an ellipse whose principal axes are P′OP′ and Q′OQ′. This ellipse is called the *momental ellipse* of the given figure. It will be noticed that any semi-diameter of the momental ellipse of a given figure is the reciprocal of the radius of gyration of the figure about that diameter.

74. Determination of the Principal Axes of Inertia of an Unsymmetrical Plane Figure.—There are cases in practice in which it is important to know the least moment of inertia, or least radius of gyration, of an unsymmetrical plane figure, a common example being that of the section of an angle-bar used as a strut, and this form of figure will be used to illustrate this problem. Fig. 65 shows an L-section 3 inches × 2 inches × $\frac{1}{2}$ inch, made up of two rectangles. In an actual angle-bar section there is a fillet at the inside angle, and the outer inside corners are rounded, and these modifications of the section shown in Fig. 65 can be allowed for if necessary.

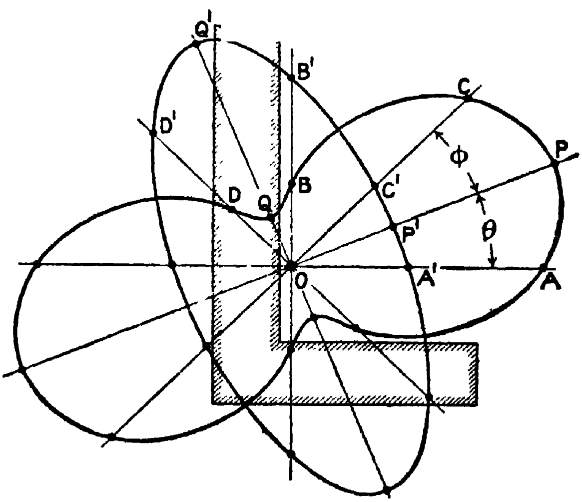


FIG. 65.

Find O the centre of gravity of the section, and draw axes OA and OB parallel to the sides of the section. Determine A and B, the moments of inertia of the section about OA and OB respectively.

Take another axis, OC inclined to OA, preferably at an angle of 45°. Find the moment of inertia C of the section about OC. If D is the moment of inertia of the section about an axis OD perpendicular to OC, then D = A + B − C.

Let OP and OQ be the principal axes of inertia, and let θ denote the

angle POA and ϕ the angle POC. Let P and Q denote the moments of inertia about the axes OP and OQ respectively.

By Art. 72, $A - B = (P - Q)\cos 2\theta$, and $C - D = (P - Q)\cos 2\phi$.

Therefore $\dfrac{\cos 2\theta}{\cos 2\phi} = \dfrac{A - B}{C - D}$. If $\theta + \phi = 45^\circ$, then $\cos 2\phi = \sin 2\theta$.

Hence $\cot 2\theta = \dfrac{A - B}{C - D}$.

Having found θ, $P - Q = \dfrac{A - B}{\cos 2\theta}$, and $P + Q = A + B$. Hence P and Q can be found.

If OA, OB, OC, OD, OP, and OQ be made equal to A, B, C, D, P, and Q respectively, the inertia curve for the section may be drawn. If a is the area of the section, and OP′ be made equal to $\sqrt{\dfrac{a}{P}}$, and OQ′ be made equal to $\sqrt{\dfrac{a}{Q}}$, then OP′ and OQ′ will be the semi-principal axes of the momental ellipse of the section.

In the example illustrated in Fig. 65, $P = 2{\cdot}17$ and $Q = 0{\cdot}42$, in inch units. The student should work out this example, and draw the complete inertia curve and the momental ellipse.

75. Bending Moment and Shearing Force Diagrams for Beams.— When a horizontal beam is acted on by vertical forces or loads, these

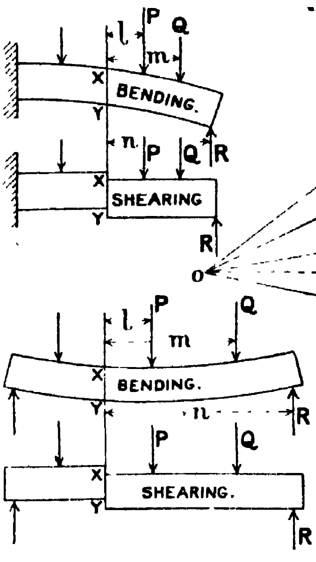


FIG. 66.

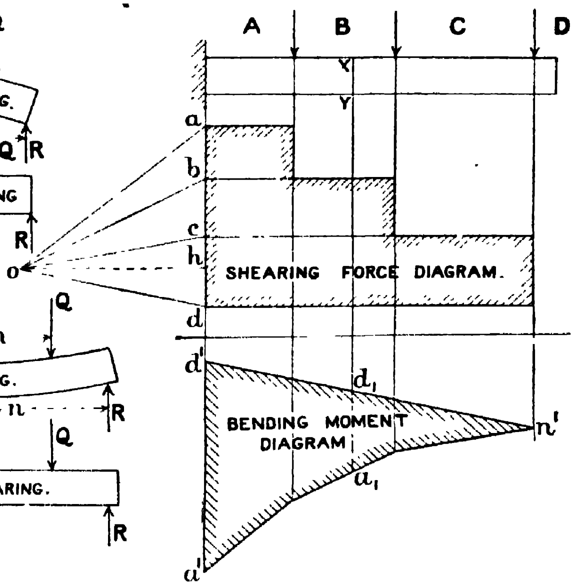


FIG. 67.

forces tend to bend the beam, and the bending action at any transverse section is measured by the algebraical sum of the moments of the forces on one side of the section about a horizontal axis in that section. For

example, the beams shown in Fig. 66 are acted on by forces P, Q, and R to the right of the transverse section XY, and the bending moment at XY is equal to $P \times l + Q \times m - R \times n$.

The loads on a beam also tend to shear the beam transversely, and the shearing action at any transverse section is equal to the resultant of the transverse forces on one side of the section. For example, the shearing action at the section XY of the beams shown in Fig. 66 is equal to the resultant of the forces P, Q, and R which act to the right of the section, and this resultant is equal to $P + Q - R$.

The drawing of the bending moment diagram for a beam is simply the application of the construction explained in Article 59. In Fig. 67 is shown a horizontal cantilever carrying vertical loads AB, BC, and CD. *abcd* is the line of loads, or polygon of forces. A pole *o* is chosen so that the pole distance *oh* is a simple multiple or sub-multiple of the linear unit. The funicular polygon *a'd'n'* is then drawn. It is easy to show, as in Article 60, that the bending moment at any section XY is equal to $a_1d_1 \times oh$, *i.e.* the depth of the funicular polygon under the section multiplied by the pole distance. The depth of the funicular polygon is measured by the force scale, and the pole distance by the linear scale. It follows that, since the pole distance is the same for all parts of the funicular polygon, the depth of the funicular polygon under any section of the beam is a measure of the bending moment on the beam at that section, the scale for measuring the bending moment being found as explained in Article 59.

The shearing force diagram is constructed by drawing horizontals across the spaces A, B, C, and D at the levels *a*, *b*, *c*, and *d* respectively. The depth of this diagram under any section of the beam, measured with the force scale, gives the vertical shearing force on the beam at that section. For example, at the section XY the shearing force is the resultant of the forces to the right of XY, and is equal to $BC + CD = bc + cd = bd$.

Another example is illustrated in Fig. 68. The beam in this case is supposed to rest on supports at its ends. There are three forces AB, CD, and DE acting downwards, a force BC acting upwards, and the reactions EF and FA at the supports acting upwards. The bending moment and shearing force diagrams are drawn as already explained. It will be noticed that the forces DE and EF are equal, and therefore there is no shearing force on that part of the beam in the space D; also, the bending moment on that part of the beam is uniform. The thick line HKLM shows roughly how the beam will bend; the points K and L where the bending moment changes its sign are points of inflexion.

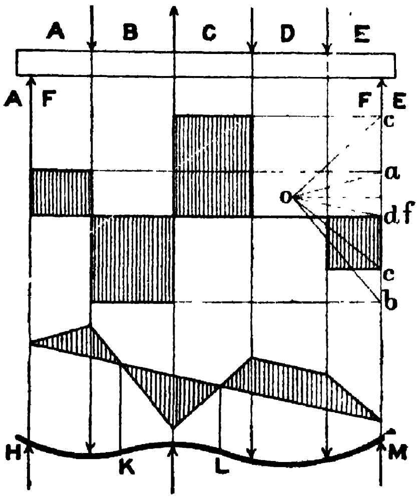


FIG. 68.

In the examples illustrated by Figs. 67 and 68 the loads acting on

the beam are supposed to be concentrated loads, *i.e.* loads acting at definite points. When a load is distributed over the whole length or a part of the length of a beam the bending moment and shearing force diagrams are determined graphically by dividing the part of the beam carrying the distributed load into a number of parts, and assuming the loads on these parts to be concentrated loads acting at the middle points of these parts, and then proceeding as for concentrated loads.

Bending moment and shearing force diagrams are further considered in Chapter VII.

Exercises Vb.

1. Referring to Ex. 5, Fig. 39, p. 41, determine the moment of inertia of the given forces about a point 1 inch to the right of A. The forces are in lbs.

2. Determine the moment of inertia of the figure shown at Ex. 5, Fig. 47, p. 49, about AB as an axis.

3. Determine the moment of inertia of the figure shown at Ex. 9, Fig. 47, p. 49, (*a*) about AC as an axis, (*b*) about an axis parallel to AC and passing through the centroid of the figure.

4. Find the greatest and least moments of inertia of a section of a stanchion built up of an **I** and two channel joists, as shown in Fig. 69. For the **I** section the over-all depth is 8 inches, the greatest moment of inertia is 111·6 inch units, and the least is 22 inch units. For the channel the over-all width of base is 12 inches, the flange width is $3\frac{1}{2}$ inches, and the thickness throughout is $\frac{1}{2}$ inch. Neglect the rivets. [U.L.]

5. A cast-iron beam section is shown in Fig. 70. Find $\bar{y}$, the distance of the centre of gravity of this section from the bottom, and determine I, the moment of inertia of the section about an axis passing through the centre of gravity and perpendicular to the web.

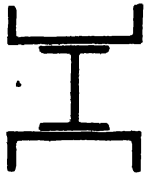


FIG. 69.

6. Fig. 71 shows the section of a Carnegie **Z**-bar column. The web plate and the **Z**-bars are $\frac{1}{2}$ inch thick throughout. Find the square of the least radius of gyration of this section.

7. The cross section of a built up column is shown in Fig. 72. The angles

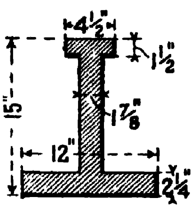


FIG. 70.

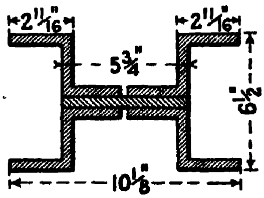


FIG. 71.

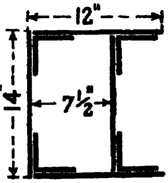


FIG. 72.

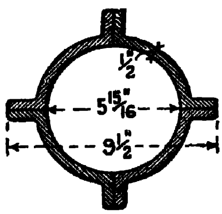


FIG. 73.

are $3\frac{1}{2}$ inches × $3\frac{1}{2}$ inches × $\frac{3}{8}$ inch, and the plates are $\frac{3}{8}$ inch thick. Find the square of the least radius of gyration of this section.

8. The cross section of a Phœnix column is shown in Fig. 73. Find the square of the least radius of gyration of this section.

9. Calculate the greatest and least moments of inertia of a **T**-iron section 5 inches wide, 4 inches deep, and $\frac{1}{2}$ inch thick. Construct the inertia curve and momental ellipse for this section. Linear scale, full size. Inertia scale, $\frac{1}{2}$ inch to 1 unit of moment of inertia, the moment of inertia being in inch units.

10. A **Z**-bar section has a total depth of 5 inches, each flange is 3 inches wide over-all, and the thickness throughout is $\frac{3}{8}$ inch. Find the principal axes of inertia, and construct the inertia curve and momental ellipse for this section. State the value of the square of the least radius of gyration.

11. Calculate the square of the least radius of gyration of an angle-iron section 5 inches × $3\frac{1}{2}$ inches × $\frac{5}{8}$ inch, and construct the inertia curve and momental ellipse for this section.

12. Show that the momental ellipse for any regular polygon is a circle.

13. A column is built up of two channel irons 12 inches × $3\frac{1}{2}$ inches × $\frac{1}{2}$ inch, and two plates each $\frac{1}{2}$ inch thick, as shown in the section Fig. 74. Determine the dimension x in order that the greatest and least moments of inertia about axes through the centre of gravity of the section may be equal.

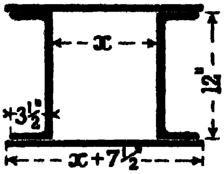


FIG. 74.

14. Fig. 75 shows the British standard section for No. 1 standard rail for tramways, a rail which weighs 90 lbs. per yard. Determine (1) the area of the section, (2) the distance of the centre of gravity of the section from the bottom, and (3) the moment of inertia of the section about an axis through the centre of gravity and parallel to the underside of the bottom flange.

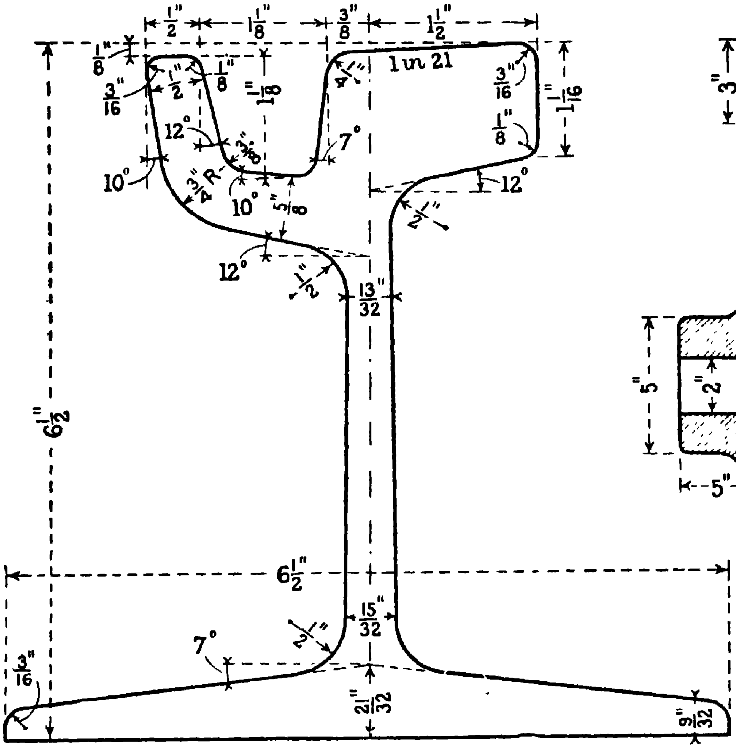


FIG. 75.

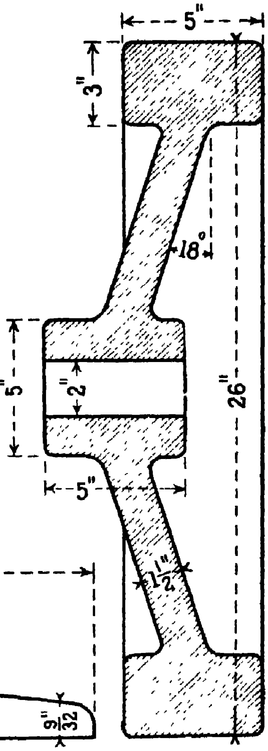


FIG. 76.

15. The section of a small cast-iron fly-wheel is shown in Fig. 76. Find (1) the weight of the wheel in lbs., taking the weight of 1 cubic inch of cast-iron $=0{\cdot}26$ lb., and (2) the radius of gyration of the wheel about its axis.

16. A beam of 20 feet span, supported at the ends, is loaded at points 4, 9, and 17 feet from one end, the loads being $2\frac{1}{2}$, $3\frac{1}{2}$, and $4\frac{1}{2}$ tons respectively. Construct, graphically, the bending moment and shearing force diagrams, and measure the bending moment and shearing force at the middle of the beam. Linear scale, $\frac{1}{4}$ inch to 1 foot. Force scale $\frac{1}{2}$ inch to 1 ton. Pole distance, 10 feet.

17. A beam of 20 feet span, supported at the ends, carries a load of 20 tons uniformly distributed over its length. Determine, graphically, the bending

moment and shearing force diagrams, and state the values of the bending moment and shearing force at a section 5 feet from one end. Linear scale, $\frac{1}{4}$ inch to 1 foot. Force scale, $\frac{1}{4}$ inch to 1 ton.

18. A horizontal lever 10 feet long is hinged at one end and supported by a vertical chain at a point 7 feet from the hinge. The lever carries a load of 1120 lbs. at a point 4 feet from the hinge and a load of 1400 lbs. at the free end. Determine, graphically, the tension in the chain, and construct the bending moment and shearing force diagrams.

19. A beam AB, 30 feet long, rests on two intermediate supports at points C and D, which are 9 feet and 19 feet respectively from the end A. The beam carries a load of 20 tons uniformly distributed over its length, besides concentrated loads of 2, 5, and 3 tons at points 1, 12, and 28 feet respectively from the end A. Determine, graphically, the reactions of the supports at C and D, and construct the bending moment and shearing force diagrams. State the values of the bending moment and shearing force at the centre of the beam.

CHAPTER VI

SIMPLE STRAINS AND STRESSES

76. Load.—The combination of external forces acting on any piece of construction is called the *load* on that piece. The following are examples of forces which may constitute the load on a piece :—(1) Forces arising directly from the purpose for which the piece is designed, and which constitute the *useful load.* For instance, the useful load on the chain or rope of a crane or hoisting engine is the weight to be lifted. (2) Forces due to the weight of the piece, or of pieces connected with it; thus, in the chain or rope mentioned above the load partly consists of the weight of the chain or rope, and in winding engines for deep mines the weight of the wire rope used forms a considerable part of the load which the rope has to carry. (3) Forces due to the inertia of heavy moving parts when their velocities vary; thus, the thrust or pull on the piston-rod of a steam-engine is not simply that due to the pressure of the steam on the piston. When the velocity is increasing the effect of the inertia of the piston is to diminish the thrust or pull due to the steam pressure, and *vice versa.* (4) Centrifugal forces, as in the arms and rim of a rotating wheel or pulley. (5) Forces due to friction. (6) Forces due to the unequal expansion or contraction of parts following variations of temperature.

77. Strain and Stress.—The effect of a load acting on any piece of construction is a change of form or dimensions of the piece, and this change of form or dimensions is called *strain.* The combination of internal forces which are called into play in the material of any piece of construction to resist or balance the load is called *stress.*

There are three kinds of simple strain and stress :—(1) *Tensile strain and tensile stress.* (2) *Compressive strain and compressive stress.* (3) *Shearing strain and shearing stress.*

78. Tensile Strain and Tensile Stress.—If a bar AB (Fig. 77) be pulled in opposite directions by forces PP acting at its ends the bar becomes longer, and a *tensile strain* or *elongation* is produced. If l is the length of the unstrained bar, and x the increase in length produced by the action of the load, then the tensile strain is measured by the fraction x/l. If any imaginary section of the bar be taken at right angles to its length, say at C, the internal forces Q at this section will balance the force P at B, and the internal forces R will balance P at A. These internal forces, which are distributed over the whole of the section at C, resist the tendency of the forces

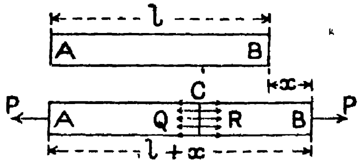


FIG. 77.

PP to pull the bar asunder at C. This system of internal forces is called *tensile stress*.

Since stress is a distributed force, its intensity is measured in the same way as that of fluid pressure, viz. by the number of units of force on a unit of area, such as pounds per square inch, tons per square inch, or tons per square foot.

If the stress at the section C be uniformly distributed over the section, and its intensity be denoted by f (say in lbs. per square inch), and if a denote the area of the section (say in square inches), and P denote the load (say in lbs.), then it is obvious that $P = af$.

79. Compressive Strain and Compressive Stress.—If the external forces acting on the bar AB (Fig. 77) be reversed in direction, the bar becomes shorter by an amount x, and a *compressive strain* is produced whose amount is x/l. At any cross section C there is *compressive stress* which resists the tendency of the forces PP to crush the bar at C.

As in the case of tension, if the stress is uniformly distributed over the cross section, $P = af$, but f now denotes compressive stress.

It should be mentioned here that unless a bar which is subjected to compression by a load acting in the direction of its length is short compared with its transverse dimensions, it has a tendency to bend, and the compressive stress at a transverse section is not uniform, hence the formula $P = af$ only applies to short pieces, or to long pieces if special means are adopted to prevent the bending of the latter. Long pieces in compression are considered in Chapter X.

80. Shearing Strain and Shearing Stress.—Suppose a rectangular block of india-rubber ABCD (Fig. 78) to have its face BC cemented to a vertical wall, and that it has a rigid plate cemented to its opposite face AD. The face ABCD being vertical, let a vertical force P be applied at the middle point of the lower edge of the plate. The force P will evidently tend to make the plate slide on the face AD of the rubber. The force P will also tend to make the face BC of the rubber slide on the wall, and it is also evident that if any vertical transverse section XX be taken dividing the block into two parts, the force P will tend to make the part AXXD slide on the part BXXC along the interface XX, as shown to the right of Fig. 78. In each case the tendency to slide is resisted by a *tangential* or *shearing stress* acting along the face.

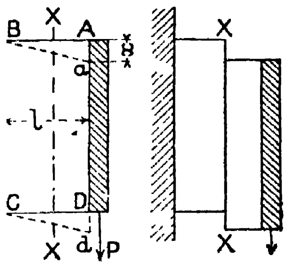


FIG. 78.

The load P will cause the block ABCD to become distorted so that the rectangle ABCD will become a parallelogram aBCd, and the *shearing strain* produced is measured by the fraction Aa/AB or x/l.

If the area of a transverse section XX is denoted by a, and if the shearing stress is uniformly distributed over the section and is denoted by f, then as in the case of tension $P = af$.

81. Volume Strain.—If a body be subjected to pressure all over its surface, as when immersed in water under pressure, it will suffer a change of volume, and if V is the original volume of the body, and v the alteration of volume due to the pressure, then v/V is called the volume strain.

If l is the length of the edge of a cube, which, when placed under pressure all over its surface, becomes $l-x$, then the new volume becomes $l^3-3l^2x+3lx^2-x^3$, and the change in volume is $l^3-(l^3-3l^2x+3lx^2-x^3)$ or $3l^2x-3lx^2+x^3$, but since x is always a very small quantity, the second and third terms of this latter expression may usually be neglected; hence the change in volume is very approximately $3l^2x$, and the volume strain is $3l^2x/l^3$ or $3x/l$, which is three times the linear strain.

82. Elasticity.—Strain is produced in a body by the action of a load on it, and if, when the load is removed, the strain disappears, the body is said to be perfectly elastic up to that particular load, or up to the particular stress corresponding to that load. If when the load is removed the strain does not entirely disappear a *permanent set* has been produced, and the *elastic limit* is reached when the load is the largest which will not cause a permanent set.

It was discovered by Robert Hooke that so long as the elastic limit is not passed the strain produced is directly proportional to the load producing it, and since the stress is directly proportional to the load causing it, it follows that stress ÷ strain is a constant ratio up to the elastic limit for a given material, or more correctly for a given piece of material. This is known as *Hooke's law.*

The value of the constant ratio stress ÷ strain is called the *modulus of elasticity* or the *coefficient of elasticity.*

When a body is subjected to simple tension or simple compression, there being no external forces acting to prevent the exceedingly small lateral contraction or lateral expansion of the body, the coefficient of elasticity is the *coefficient of direct elasticity*, and is called *Young's modulus.* The letter E is generally used to denote Young's modulus.

When the strain is a shearing strain, and the stress of course a shearing stress, the ratio stress ÷ strain is called the *coefficient of transverse elasticity* or the *coefficient of rigidity.* In this work the coefficient of rigidity will be denoted by the letter C.

When the strain is a volume strain the ratio stress ÷ strain is called the *coefficient of elasticity of volume* or the *coefficient of cubical elasticity.* In this work the coefficient of elasticity of volume will be denoted by the letter K.

83. Applications of Young's Modulus.—If a bar of length l, whose area of cross section is a, suffers an alteration of length amounting to x, under the application of a load W acting in the line of the axis of the bar, and if f is the stress produced, then by Art. 78 or Art. 79 $W=af$. Also by Art. 82, $E=\frac{\text{stress}}{\text{strain}}=\frac{fl}{x}$. From these two equations the following results are easily obtained:—

$$x=\frac{fl}{E}=\frac{Wl}{aE}, \quad W=\frac{aEx}{l}, \text{ and } f=\frac{Ex}{l}.$$

If the bar mentioned above be heated or cooled so that if free to expand or contract it would expand or contract by an amount x, then the forces which must be applied at each end of the bar, to prevent the expansion or contraction, will be each equal to W, and the equations above will apply to this case if $l+x$ be substituted for l. But since x is very small compared with l, the error introduced by using l instead of

$l+x$ may be neglected. The quantity x will of course be determined from the change of temperature and the coefficient of expansion of the bar.

If a compound bar be made up of two bars of different materials, firmly united at their ends, so that the component bars must suffer the same alteration of length when the compound bar is placed in tension or compression by a load W, then if a_1 and a_2 be the areas of the cross sections of the component bars, f_1 and f_2 the stresses produced in them by the load W, E_1 and E_2, their coefficients of elasticity, l the original length of the compound bar, and x the alteration in length produced by the load W, then the following equations will obviously apply :—

$$E_1 = \frac{f_1 l}{x}, \quad E_2 = \frac{f_2 l}{x}, \quad \frac{f_1}{f_2} = \frac{E_1}{E_2}, \quad \text{and } W = a_1 f_1 + a_2 f_2.$$

If the foregoing is understood, the case of a compound bar made up of more than two bars of different materials presents no difficulty.

Considering further the compound bar made up of two bars of different materials; suppose that the compound bar is heated or cooled, so that the component bars, if entirely free, would expand or contract by amounts x_1 and x_2 respectively. Assuming that the two materials have different coefficients of expansion, then x_1 and x_2 would not be equal. Let x_1 be the greater. The first bar will tend to lengthen or shorten by an amount x_1, but will be prevented by the other bar, which tends to alter by the amount x_2 by the change of temperature. The first bar will therefore drag the other in one direction, while the second will drag the first in the opposite direction. The result will be that the alteration in length of the compound bar will be an amount x, which will lie between x_1 and x_2. Also the stress produced in one bar will be tensile, while in the other it will be compressive. Using the same notation as before—

$$\text{Strain on first bar} = \frac{x_1 - x}{l + x_1} \qquad E_1 = \frac{f_1(l + x_1)}{x_1 - x}$$

$$\text{Strain on second bar} = \frac{x - x_2}{l + x_2} \qquad E_2 = \frac{f_2(l + x_2)}{x - x_2},$$

and since the pull on the one bar must balance the thrust on the other $a_1 f_1 = a_2 f_2$.

Since x_1 and x_2 are very small compared with l, the error introduced by putting l instead of $l + x_1$ and $l + x_2$ in the above equations may be neglected.

The method indicated above may easily be extended to determine the relations between the various quantities when the compound bar is made up of more than two bars of different materials.

84. Bars of Varying Cross Section.—If at any point in the length of a bar which is in tension the cross section suddenly changes, then the stress at that section will not be uniformly distributed over the section; and at sections for some distance on each side of that section the stress will not be uniformly distributed, and the rules already demonstrated in this chapter will not apply. But if the several parts of a bar between the points where sudden changes of section occur be long compared with their cross sections, the elongations of these several parts of the bar may

be determined without great error by assuming that the stress is everywhere uniformly distributed and given by the formula $f = P/a$, where f is the stress at a cross section whose area is a, and P is the load.

The effect of sudden changes of section on the behaviour of a loaded bar is further considered in Article 165, p. 174.

85. Strength and Factor of Safety.—If the load on a piece which is in tension or shear be continuously increased, the piece will ultimately fracture or break in two, and the smallest load which will do this is called the *breaking load*, and the corresponding stress or breaking load per unit of original section is called the *ultimate stress* or *ultimate strength* of the piece. All solid materials have an *ultimate tensile* and *ultimate shearing strength*, and many have an *ultimate crushing strength*, but for certain ductile materials, such as wrought-iron and mild steel, there is no definite load which will cause complete fracture when they are subjected to compression.

When a piece is loaded up to the elastic limit, the stress produced is the *elastic strength* of the piece.

The largest load, *repeatedly applied*, which a piece will carry without taking a permanent set is called the *proof load*, and the corresponding stress is the *proof stress* or *proof strength*.

The proof strength, as above defined, is less than the elastic strength, because experiment has shown that a load less than that required to produce permanent set may, if repeated a sufficient number of times, cause permanent set, and a load just under the elastic load will, after one or two applications, generally cause permanent set. This proof strength is difficult to determine, and in practice the term proof strength is often taken to mean elastic strength. Also, the elastic strength is frequently taken to mean the stress when the first decided set has taken place, as in mild steel, when the yield point is reached.

The load put upon a piece in actual use is the *working load*, and the corresponding stress is the *working stress* or *working strength*. For safety the working stress must be less than the proof stress. The working stress is usually determined by dividing the ultimate stress by a number called a *factor of safety*, but it may also be fixed by dividing the proof stress by another factor of safety.

The value of the factor of safety to be used in any particular case must be determined by experience and judgment. Some of the considerations which influence the value of the factor of safety are—(1) the degree of certainty as to the magnitude of the greatest load which is likely to act on the piece; (2) the character of the load, *i.e.* whether it is a fixed or constant load, or a constantly changing load; (3) the consequences of a breakdown; (4) the reliability of the material used; (5) the amount of deterioration or wear which may take place in the piece when in use.

86. Stress-strain Diagrams.—If the strains and corresponding stresses on a loaded bar be plotted in the usual way (Fig. 79), then since stress ÷ strain is a constant up to the elastic limit, the diagram up to this point will be a sloping straight line OA. After the elastic limit is reached the strains increase more rapidly than the stresses, and the results are represented by a more or less irregularly curved line AB.

So long as the cross section of a loaded bar does not sensibly alter,

the stress is sensibly proportional to the load, but ductile materials, such as wrought-iron and mild steel, alter considerably in cross section when loaded in tension or compression beyond the elastic limit, and the stresses are therefore no longer proportional to the load. For example, if a bar in tension has an area of cross section at the fracture equal to half its original area, the actual stress at fracture will be twice the nominal stress, the nominal stress being equal to load ÷ original area. If, therefore, the diagram (Fig. 79) is a true stress-strain diagram, it will not be a true *load-strain* diagram. In practice it is the load-strain diagram which is actually drawn by the autographic apparatus on a testing machine. The diagrams are, however, often spoken of as stress-strain diagrams when they should be called load-strain diagrams.

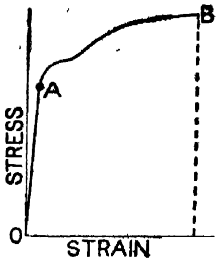


FIG. 79.

The actual form of the stress-strain diagram or load-strain diagram varies greatly for different materials. Different forms of the diagram are considered in Chapter XI.

87. Work done in Producing Strain.—The load-strain diagram is also a diagram representing the work done in producing the strain. In previous Articles of this chapter strain has been denoted by x/l. Hence referring to Fig. 80, it follows that if OX represents a particular amount of strain it will by altering the scale also represent the quantity x. Lengths along ON then represent distances through which the load acts, and the heights of the line OAB above ON represent the variation in the load as the bar is deformed. It follows that the work done in deforming the bar, say by the amount OX, is represented by the area of the figure OAYX, where XY is perpendicular to ON. (See Art. 41, p. 25.)

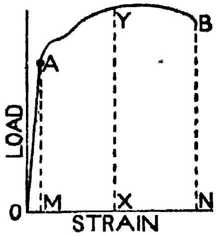


FIG. 80.

88. Resilience and Shock.—The work done in straining a bar up to the elastic limit is called the *resilience* of the bar. Referring to Fig. 80, the area of the triangle OAM represents the work done in straining the bar up to the elastic limit. If the bar is in tension or compression, OM = x, the amount of extension or compression, and AM is the load W at the elastic limit. Hence the resilience $= \frac{1}{2}Wx = \frac{1}{2}afx$, where a is the area of the cross section of the bar, and f the stress at the elastic limit. But it has been shown (Art. 83) that $x = \frac{fl}{E}$, therefore resilience $= \frac{alf^2}{2E}$, but al is the volume of the bar, therefore, putting $V = al$,

$$\text{resilience} = \frac{Vf^2}{2E}$$

If the bar strained to some point below the elastic limit the expression for the work done will still be $\frac{Vf^2}{2E}$, but the stress f will not now be the stress at the elastic limit, but will correspond to the strain produced.

If w is the weight of a unit volume of the bar, then the weight of the bar will be Vw. Let h equal the height through which this weight must fall in order to accumulate an amount of energy equal to the resilience of the bar, then

$$Vwh = \frac{Vf^2}{2E}, \text{ and } h = \frac{f^2}{2wE}.$$

The resilience of a bar is a measure of its power to resist a blow or shock without taking a permanent set.

Suppose a bar AB (Fig. 81) of length l and area of cross section a to be suspended from one end, and let it have a weight W threaded on it as shown. If the weight is allowed to fall freely through a height h before striking the head formed on the lower end of the bar, the bar will lengthen an amount x, and the total fall of the weight will be $h+x$. At the end of the fall the resistance offered by the bar to further stretching will be af, where f is the maximum stress. The diagram of work done on the bar, assuming that it is not strained beyond the elastic limit, will be a triangle whose area $\frac{1}{2}afx$ will equal the work done in stretching the bar, and this must equal the work done by the falling weight.

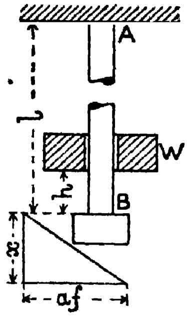


FIG. 81.

Therefore $$W(h+x) = \frac{afx}{2}, \text{ but } x = \frac{fl}{E}.$$

Hence $$W\left(h + \frac{fl}{E}\right) = \frac{af^2l}{2E}.$$

Solving this quadratic equation, $$f = \frac{W}{a} + \frac{\sqrt{W^2l^2 + 2WEalh}}{al}.$$

If $h = 0$, then $f = \frac{2W}{a}$.

When the load W is applied gradually, as when the bar is stretched in a testing machine, the maximum stress, when the load is all on, becomes $\frac{W}{a}$, but if the full load is put on at once the maximum stress, as shown above, is $\frac{2W}{a}$. The effect of a suddenly applied load is therefore to produce a stress double that produced when the load is applied gradually.

Exercises VIa.

1. A steel wire 0·08 inch diameter and 50 feet long is subjected to tension by a load of 112 lbs. Determine (1) the stress in lbs. per square inch, (2) the elongation in inches, (3) the strain, and (4) the work done, in inch-lbs., in producing the strain. $E = 30,000,000$ lbs. per square inch.

2. A steel piston-rod 2 inches diameter is subjected to a pull and thrust alternately. The tensile and compressive stresses are each 8000 lbs. per square inch. Two points A and B on the axis of the rod are 4 feet apart when the rod is unloaded. Determine (1) the effective load on the piston, (2) the difference between the greatest and least distances between A and B. $E = 30,000,000$ lbs. per square inch.

3. A ferro-concrete column is 12 inches square. The principal reinforcement consists of four longitudinal steel bars placed near the angles of the column, and having an aggregate cross sectional area of 11 square inches. The load carried by the column is 50 tons. Determine the compressive stresses in the concrete and steel, in lbs. per square inch, assuming that the modulus of elasticity of the concrete is one-tenth that of the steel.

4. A steel tube, 1·25 inches internal diameter, 0·104 inch thick, and 12 feet long, is covered and lined throughout with copper tubes 0·08 inch thick. The three tubes are firmly united at their ends. This compound tube is subjected to tension, and the stress produced in the steel tube is 9000 lbs. per square inch. Determine (1) the elongation of the tube, (2) the stress in the copper tubes, and (3) the load carried by the compound tube. E = 30,000,000 lbs. per square inch for steel, and 16,000,000 lbs. per square inch for copper.

5. The compound tube in the preceding exercise is raised in temperature 200° F. Find the stresses in the steel and copper, and the increase in length of the tube. Also, what must be the magnitude of the forces, which, applied to the ends of the tube, will prevent its expansion? Coefficients of expansion of steel and copper 0·000006 and 0·0000095 respectively per degree F.

6. If a thin circular hoop is strained and remains circular, prove that the circumferential strain is equal to the diametrical strain.

7. A cylindrical steel hoop has an internal diameter 20 inches, thickness 1 inch, and breadth 1 inch. A second steel hoop has an internal diameter 21·97 inches, thickness 0·7 inch, and breadth 1 inch. The second hoop is expanded by heating and is then shrunk on to the first hoop. Determine (1) the new internal diameter of the first hoop, (2) the tensile stress in the second hoop, and (3) the compressive stress in the first hoop. E = 30,000,000 lbs. per square inch.

8. Referring to the hoops of the preceding exercise. Find what must be the internal diameter of the second hoop so that the stress in it when it is shrunk on to the first will be 10,000 lbs. per square inch. Then determine the stress in the first hoop and its new internal diameter.

9. Calculate the length of a bar of uniform section whose density is 0·28 lb. per cubic inch, and whose coefficient of elasticity is 28,000,000 lbs. per square inch, which when hung from one end causes a maximum tensile stress in it of ¾ ton per square inch. Find also the increase in its length due to the tension.

10. A wrought-iron bar 25 feet long is 2 inches diameter for 6 feet of its length, 1¾ inches diameter for 7 feet of its length, and 1½ inches diameter for the remainder of its length. This bar is in tension, and the stress on the smallest sections is 12,000 lbs. per square inch. Taking E = 28,000,000 lbs. per square inch, find the total elongation of the bar.

11. In testing to destruction a piece of mild steel, 0·937 inch diameter, in tension, a load-strain diagram was taken. The diagram showed the elongations full size, and the loads to a scale of 5 tons to 1 inch. The length of bar under observation was 10 inches. The total elongation after fracture was 2·46 inches. The area of the diagram, measured with a planimeter, was 7·47 square inches. Determine (*a*) the amount of work represented by the diagram, (*b*) the work done in straining the bar up to the elastic limit, taking the length as 10 inches, having given, load at elastic limit 9 tons, and modulus of elasticity 29,900,000 lbs. per square inch. Also (*c*) express (*a*) as a multiple of (*b*).

12. Calculate the resilience, in ft.-lbs., of a cubic inch of steel, in tension, taking the elastic limit at 20,000 lbs. per square inch, and the modulus of elasticity at 30,000,000 lbs. per square inch.

13. A steel bar 1 inch diameter and 6 feet long is put in tension by a force of 3 tons applied suddenly. Determine the maximum stress and the maximum elongation produced. E = 30,000,000 lbs. per square inch.

14. If a bar ½ inch in diameter stretched ⅛ of an inch under a steady load of 1 ton, what stress would be produced in the rod by a weight of 150 lbs. falling through 3 inches before commencing to stretch the rod. The rod is initially unstressed. [U.L.]

15. A steel rod, 2 inches diameter and 10 feet long when unloaded, is suspended from one end, and has a weight of 1000 lbs. threaded on to it. The weight is allowed to fall freely from a height $h = 1$ inch on to a head formed on the lower end of the rod. Find the maximum stress produced in the rod. Also, find h so that the maximum stress may be 10,000 lbs. per square inch. E = 30,000,000 lbs. per square inch.

16. Taking the stress at the elastic limit as 16,000 lbs. per square inch, calculate the resilience of the bar referred to in Exercise 10.

17. Re-work Exercise 15, assuming that there is a steady load of 1 ton hanging from the lower end of the bar before the blow is applied.

18. If the maximum crushing stress of a punch is four times the maximum shearing stress of a plate, show that the smallest hole which can be punched in the plate has a diameter equal to the thickness of the plate.

89. Riveted Joints.—Considering first a simple form of riveted joint, such as the double riveted lap joint shown in Fig. 82. When this joint

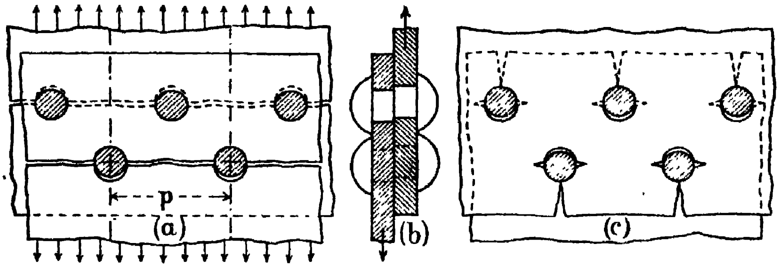


FIG. 82.

is subjected to tension it may give way (1) by the tearing of the plates between the rivets, as shown at (*a*); (2) by the shearing of the rivets, as shown at (*b*); (3) by the crushing of the rivets or of the parts of the plates in contact with them; (4) by the breaking of the plates between their outer edges and the rivet holes, as shown at (*c*).

p = pitch of rivets. f_t = tensile stress in plates.
d = diameter of rivets. f_s = shearing stress in rivets.
t = thickness of plates. f_c = crushing stress in rivets or plates.

Considering a strip of the joint equal in width to the pitch p,

Resistance of this strip to tearing $=(p-d)tf_t$.

„ „ „ shearing $=\frac{\pi}{4}d^2f_s\times 2$.

„ „ „ crushing $=dtf_c\times 2$.

In the last of these expressions the bearing area of the rivet on the plate is taken as its projected area on a plane containing the axis of the rivet and perpendicular to the direction of the pressure.

If the values of the stresses f_t, f_s, and f_c be given, then the three expressions above must be equal to one another. This gives two equations to determine p and d. Solving these equations,

$$d=\frac{4tf_c}{\pi f_s}\quad\text{and}\quad p=\frac{4tf_c}{\pi f_s}\left(1+\frac{2f_c}{f_t}\right).$$

The stresses f_t and f_s can usually be definitely settled, but for materials used in riveted joints the value of the stress f_c is more difficult to decide.

Very often the diameter of the rivets is fixed empirically, and the resistance to tearing is then equated to the resistance to shearing to determine the pitch. In that case the crushing stress should then be calculated

by equating the resistance to crushing to either the resistance to tearing or the resistance to shearing.

The ratio of the strength of a riveted joint to the strength of the solid plate is called the *efficiency of the joint.* The efficiency is called either the tearing, the shearing, or the crushing efficiency, according to the kind of resistance which is taken as the strength of the joint. The resistance of the solid plate being ptf_t, the various efficiencies for a double riveted lap joint are

$$\text{Tearing efficiency} = \frac{(p-d)\,tf_t}{ptf_t} = \frac{p-d}{p}.$$

$$\text{Shearing efficiency} = \frac{\pi d^2 f_s}{2ptf_t}.$$

$$\text{Crushing efficiency} = \frac{2dtf_c}{ptf_t} = \frac{2df_c}{pf_t}.$$

It is usual to express the efficiencies as percentages by multiplying the above by 100.

The lowest efficiency is the real efficiency of the joint.

To resist the rupture of the plate between its edge and the rivets, as shown at (*c*), Fig. 82, it has been found by experiment that if the least distance between the edge of the plate and the nearest rivet is equal to the diameter of the rivet this is sufficient, and this is the rule followed in practice.

For simple lap joints other than the double riveted joint which has been considered, the multiplier 2 which was used in the expressions for the resistance to shearing and the resistance to crushing must be changed to n, where n denotes the number of rows of rivets in the joint.

The determination of the strength and proportions of riveted joints other than simple lap joints presents no particular difficulty, but a few cases will now be considered briefly.

Fig. 83 shows an ordinary double riveted butt joint with two cover straps. Considering a strip of the joint equal in width to the pitch p, and using the same notation as before.

Resistance to tearing $= (p-d)tf_t$ . . . (1)

,, ,, shearing $= \frac{\pi}{4}d^2 f_s \times 2 \times 2$ . . (2)

,, ,, crushing $= dtf_c \times 2$ . . . (3)

Equating (2) to (3) $d = \frac{2tf_c}{\pi f_s}$ . . . . (4)

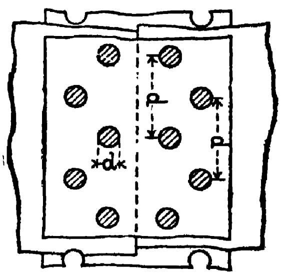


FIG. 83.

Equating (1) to (3) and substituting from (4) $p = \frac{2tf_c}{\pi f_s}\left(1 + \frac{2f_c}{f_t}\right)$.

If d is given, or fixed empirically, equate (1) to (2), then

$$p = \frac{\pi d^2 f_s}{tf_t} + d,$$

and equating (2) to (3) $\qquad f_c = \frac{\pi d f_s}{2t}.$

Of the two factors 2×2 in expression (2), one is for two rivets and the other is for two sections of each rivet, the rivets being in double shear. Experiment has shown that the strength of a rivet in double shear is twice that of the same rivet in single shear. The Board of Trade, however, only allow a load on a rivet in double shear equal to 1·875 times the load allowed on the same rivet in single shear.

If each cover strap carries half the load on the joint, it is obvious that each should have a thickness equal to $\frac{1}{2}t$, but in practice the straps in an ordinary butt joint have a thickness equal to $\frac{5}{8}t$.

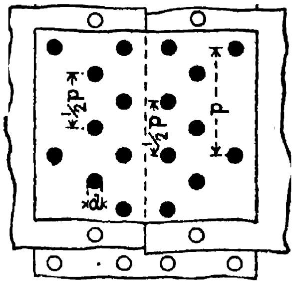


FIG. 84.

Fig. 84 shows a treble riveted butt joint with two cover straps, in which the pitch of the rivets in the outer rows is twice the pitch of the rivets in the other rows. In this form of joint, and in all joints where there are fewer rivets in the outer rows than in the others, there is another way in which the joint may fracture in addition to those already considered, viz. the outer row of rivets may shear, and at the same time the plate may tear between the rivets of the next row.

Considering a strip of the joint equal in width to the greatest pitch p.

Resistance to tearing between rivets of outer rows $=(p-d)tf_t$ . . (1)

" " shearing $=\frac{\pi}{4}d^2 \times 5 \times 2f_s = \frac{5\pi}{2}d^2f_s$ . . . . (2)

Resistance to shearing of rivets in outer row and tearing between rivets of next row $=\frac{\pi}{4}d^2 \times 2f_s + (p-2d)tf_t$ . . . . (3)

Resistance to crushing $= 5dtf_c$ . . . . (4)

These four expressions yield three equations to determine p and d if all the stresses are given, and this is more than sufficient.

Equating (1) to (3) $d = \frac{2tf_t}{\pi f_s}$ . . . . (5)

Equating (1) to (2) and substituting from (5) $p = \frac{12tf_t}{\pi f_s} = 6d$.

Equating (2) to (4) and substituting from (5) $f_c = f_t$.

Since the safe crushing stress is always greater than the safe tensile stress, it is evident that, with the proportions deduced above, the joint will have an excess of crushing strength.

The combined resistance to tearing of the two cover straps is $2(p-2d)t_1f_t$, where t_1 is the thickness of each cover strap. Equating this to $(p-d)tf_t$, the resistance of the plates to tearing, $t_1 = \frac{(p-d)t}{2(p-2d)}$. In practice this would be made $\frac{5(p-d)t}{8(p-2d)}$. Making $p = 6d$, then $t_1 = \frac{25t}{32}$.

Fig. 85 shows a tie-bar joint. Here a bar of width b and thickness t has a lap joint in it containing nine rivets arranged as shown. This joint may give way (1) by tearing at A or E; (2) by shearing of all the rivets; (3) by tearing at B or D, and shearing at A or E; (4) by tearing at C, and shearing at A and B or at D and E. The values of these various resistances are—

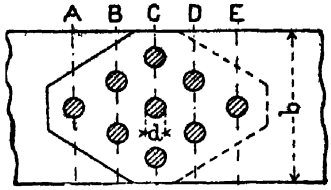


FIG. 85.

$$(b-d)tf_t \quad . \quad . \quad . \quad (1) \qquad (b-2d)tf_t+\frac{\pi}{4}d^2f_s \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$9\frac{\pi}{4}d^2f_s \quad . \quad . \quad . \quad (2) \qquad (b-3d)tf_t+3\frac{\pi}{4}d^2f_s \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The strength of the solid bar is btf_t, and the various efficiencies are

$$\frac{b-d}{b} \quad . \quad . \quad . \quad . \quad . \quad (1) \qquad \frac{(b-2d)}{b}+\frac{\pi d^2f_s}{4btf_t} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\frac{9\pi d^2f_s}{4btf_t} \quad . \quad . \quad . \quad . \quad (2) \qquad \frac{(b-3d)}{b}+\frac{3\pi d^2f_s}{4btf_t} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

There is only one dimension to determine, viz. d, and a value of d may be found by equating any two of the four expressions which give the resistance of the joint. Six possible values of d may be found in this way, but generally there is only one value which will give the highest minimum efficiency of joint.

The most satisfactory way of finding the best value of d is to plot the various efficiencies for different values of d, as shown in Fig. 86. In this case b has been taken equal to 10 inches, $t=1$ inch, and $f_s=0{\cdot}8f_t$. The best diameter of rivet must be under the intersection of a pair of efficiency curves, and an inspection of the figure shows that the diameter which gives the best efficiency is under the intersection of (2) and (3). This diameter is 1·23 inches, and the minimum efficiency is 85·6 per cent. In plotting the efficiency curves it is only necessary to show the parts in the neighbourhood of their intersections.

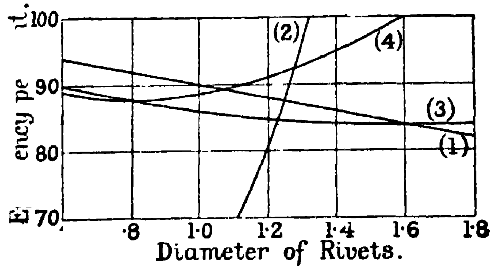


FIG. 86.

A butt joint with two cover straps, such as is shown in Fig. 87, is a more satisfactory joint for a tie-bar than the lap joint, because in the case of the butt joint the pulling forces on opposite sides of the joint are in the same plane, whereas in the case of the lap joint the pulling forces are in different planes, and in consequence there is a bending action on the bar in the neighbourhood of the joint.

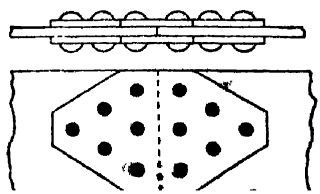


FIG. 87.

90. Thin Cylindrical Shells.—A thin cylindrical shell or pipe (Fig. 88) of internal diameter d, thickness t, and length l is exposed to internal fluid pressure of intensity p. Let the shell be divided into two equal parts by a plane of section containing the axis. The resultant R of the pressure on either of these parts is evidently independent of the shape of the other part. Let the other part be replaced by a flat plate, as shown in Fig. 89. Then the resultant pressure on the flat plate is $S = pdl$. But S must balance R, therefore $R = S = pdl$.

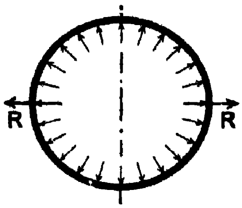


FIG. 88.

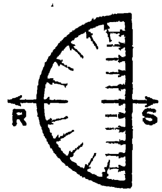


FIG. 89.

If f_t is the stress in the material of the shell at the plane of section, then $R = pdl = 2tlf_t$ or $pd = 2tf_t$.

If the shell has longitudinal riveted joints whose efficiency is e, then $pd = 2tf_te$.

The assumptions made in determining the last two equations are, (1) that the stress f_t is uniformly distributed over the section of the shell, and this is justified if the shell is thin compared with its diameter; (2) that the shell derives no assistance from the ends, and this is justified if the cylinder is not very short compared with its diameter.

The resultant pressure on the ends of the shell is $\frac{\pi}{4}d^2p$, and the resistance of the shell to tearing at a section perpendicular to the axis is πdtf_t, therefore $\frac{\pi}{4}d^2p = \pi dtf_t$ or $pd = 4tf_t$, which shows that the resistance to tearing at a circumferential section is twice the resistance to tearing at a longitudinal section, the effect of the riveted joints being neglected.

91. Thin Spherical Shells.—By the method of the preceding Article, and using the same notation, the resultant pressure on one half of the shell is $\frac{\pi}{4}d^2p$, and the resistance to tearing is πdtf_t, therefore $\frac{\pi}{4}d^2p = \pi dtf_t$ or $dp = 4tf_t$.

92. Centrifugal Tension in a Revolving Hoop.—Each part of a hoop revolving about its axis tends to fly outwards because of centrifugal force, and the effect on the hoop is the same as that of an internal fluid pressure acting on it.

Let a be the area of the cross section of the hoop in square inches; w the weight of a cubic inch of the material in pounds; v the linear velocity of the hoop in feet per second; and d the diameter of the hoop in inches. The hoop is supposed to be thin compared with its diameter.

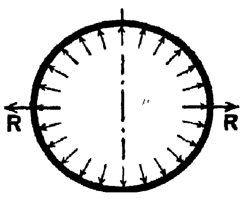


FIG. 90.

The weight of a portion of the hoop 1 inch long is aw lbs., and the centrifugal force q of this portion is $24awv^2/gd$. Each inch of hoop will have the same amount of centrifugal force acting on it, and the result is a uniformly distributed radial force acting on the hoop, as shown by the small arrows in Fig. 90.

From the analogy between this case and that of a thin cylindrical shell under fluid pressure (Art. 90) it may be concluded that R, the resultant of the centrifugal forces qq . . . . on one half of the hoop, is equal to $dq = 24awv^2/g$, and equating this to the resistance of the hoop to bursting, $2af_t = 24awv^2/g$, therefore $f_t = 12wv^2/g$, where f_t is the stress (in lbs. per square inch) due to centrifugal force.

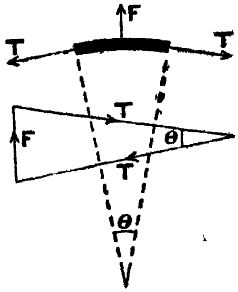


FIG. 91.

The foregoing result may be demonstrated in another way. Consider a small portion of the hoop (Fig. 91) subtending an angle θ at the centre. This is in equilibrium under the action of the centrifugal force F and the tensions TT. The weight of the portion under consideration is $\frac{1}{2}awd\theta$, and $F = 12aw\theta v^2/g$. From the triangle of forces $F = T\theta$, since θ is a very small angle. Also $T = f_t a$, therefore $T\theta = f_t a\theta = 12aw\theta v^2/g$, and $f_t = 12wv^2/g$.

93. Cottered Joints.—Fig. 92 shows two bars of diameter d joined together with their axes in the same straight line. The upper bar is enlarged at its lower end to form a socket, which fits over the enlarged upper end of the lower bar. A cotter passes through the two as shown. It will be assumed in what follows that the two bars are made of the same material, and that they are in tension under a load T.

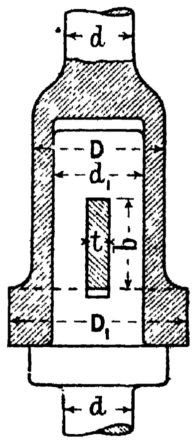


FIG. 92.

For the parts of diameter d,

$$T = \frac{\pi}{4}d^2 f_t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The weakest cross section of the part of diameter d_1 is at the cotter hole, where the area of the cross section is very nearly $\frac{\pi}{4}d_1^2 - d_1 t$, and therefore

$$T = \left(\frac{\pi}{4}d_1^2 - d_1 t\right) f_t \quad . \quad . \quad . \quad . \quad (2)$$

The weakest part of the socket is the cross section at the cotter hole, where the area is

$$\frac{\pi}{4}(D^2 - d_1^2) - (D - d_1)t,$$

therefore

$$T = \left\{\frac{\pi}{4}(D^2 - d_1^2) - (D - d_1)t\right\} f_t \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The cotter will shear at two sections, therefore

$$T = 2btf_s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The bearing area of the cotter on the lower bar is $d_1 t$, therefore

$$T = d_1 t f_c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The bearing area of the cotter on the socket is $(D_1 - d_1)t$, therefore

$$T = (D_1 - d_1)tf_c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Assuming T and the stresses to be known, the foregoing six equations are sufficient for determining the dimensions d, d_1, D, D_1, b, and t.

Exercises VIb.

In the following exercises on riveted joints, t = thickness of plates, d = diameter of rivets, p = pitch (or greatest pitch) of rivets, all in inches. In all cases where p has to be determined the result is to be stated and taken to the nearest eighth of an inch, and where d has to be found, the result is to be stated and taken to the nearest sixteenth of an inch.

Unless otherwise stated the tenacity of the plates is to be taken at 28 tons per square inch, and the resistance of the rivets to shearing at 23 tons per square inch.

1. In a single riveted lap joint $t=\frac{7}{16}$, $d=\frac{3}{4}$, and $p=2$. Calculate the efficiencies, and find the crushing stress on the rivets when the joint gives way by tearing.

2. A treble riveted lap joint has the following dimensions—$t=\frac{13}{16}$, $d=1\frac{1}{16}$, and $p=3\frac{3}{4}$. Calculate the efficiencies, and find the crushing stress on the rivets when the joint gives way by tearing.

3. Find p for a double riveted lap joint in which $t=\frac{9}{16}$, and $d=\frac{15}{16}$. Then calculate the efficiencies.

4. Design a double riveted lap joint for plates $\frac{5}{8}$ inch thick, and find the efficiencies, having given $f_s=0{\cdot}8f_t$ and $f_c=1{\cdot}3f_t$.

5. In the treble riveted lap joint shown in Fig. 93, $t=\frac{15}{16}$, $d=1\frac{1}{4}$, and $p=4\frac{5}{8}$. Calculate the efficiencies of the joint.

6. Having given $t=\frac{13}{16}$, determine d and p for the joint shown in Fig. 93, so that the three principal efficiencies shall be as nearly equal as possible (d being to the nearest sixteenth, and p to the nearest eighth of an inch), then calculate the efficiencies.

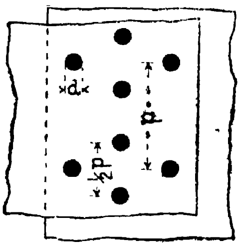


FIG. 93

7. Plates 1 inch thick are connected by a treble riveted butt joint with two cover straps. The pitch of the rivets in the outer rows is twice the pitch of those in the other rows, and the diameter of the rivets is 1 inch. Taking the resistance of rivets in double shear equal to 1·75 times their resistance in single shear, determine p (to nearest eighth of an inch) for equal tearing and shearing resistances. Then determine the efficiencies.

8. Same as preceding exercise, except that the resistance of rivets in double shear is to be taken equal to twice their resistance in single shear

9. In a riveted joint of the form shown in Fig. 94, $t=\frac{9}{16}$, $d=\frac{15}{16}$, and $p=7$. Taking the shearing resistance of rivets in double shear equal to 1·75 times their resistance in single shear, determine the efficiencies of this joint.

10. Determine p and d for the joint shown in Fig. 94 ($t=\frac{9}{16}$), so that the efficiencies may be as nearly as possible equal to one another, taking p to the nearest eighth, and d to the nearest sixteenth of an inch. Take shearing resistance of rivets in double shear equal to 1·75 times their resistance to single shear. Give the efficiencies.

FIG. 94.

11. In a double riveted lap joint $t=\frac{1}{2}$, $d=\frac{7}{8}$, and $p=2\frac{3}{4}$. Find the efficiencies. This joint is strengthened by the addition of a cover strap, as shown in Fig. 95. The rivets at A and B being $\frac{7}{8}$ inch diameter, and $5\frac{1}{4}$ inches pitch. Calculate the efficiencies of the altered joint.

12. The tie-bar lap joint shown in Fig. 96 has rivets $\frac{15}{16}$ inch diameter. The

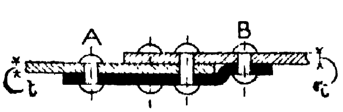


FIG. 95.

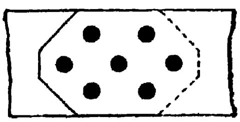


FIG. 96.

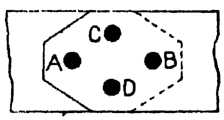


FIG. 97.

bar is 6 inches wide and $\frac{3}{4}$ inch thick. Determine the lowest efficiency of this joint, and describe how it will give way.

13. A tie-bar 5 inches wide and $\frac{1}{2}$ inch thick has a lap joint in it, as shown in Fig. 97. The rivets at A and B have a diameter d, and those at C and D have a diameter d_1. Find the best values of d and d_1 (to the nearest sixteenth of an inch) so that the efficiencies may be as nearly as possible equal to one another. Give the efficiencies.

14. Taking the joint referred to in the preceding exercise, but making all the rivets of the same diameter, plot on squared paper the various efficiencies of the joint for different sizes of rivets up to $1\frac{1}{4}$ inches diameter.

15. Determine the best diameter of rivets (to the nearest $\frac{1}{16}$ inch) for a tie-bar butt joint with double cover straps, and 12 rivets in all, arranged as in Fig. 87, p. 75. Width of bar, 9 inches; thickness, $\frac{5}{8}$ inch. $f_s = 0{\cdot}8f_t$. Resistance of rivets in double shear $= 1{\cdot}75$ times their resistance in single shear. Find the lowest efficiency of the joint.

16. A cylindrical boiler shell is 7 feet in diameter, and the plates are $\frac{5}{8}$ inch thick. The longitudinal riveted joints have a tearing efficiency of 70 per cent. Find the steam pressure which will cause a tensile stress of 5 tons per square inch in the plates between the rivets. Also, what tensile stress will this steam pressure produce in the plates between the rivets of the circumferential joints which have a tearing efficiency of 60 per cent.?

17. The end plates of a boiler shell are $\frac{9}{16}$ inch thick, and are dished to a radius of 6 feet. Find the tensile stress in these plates due to a steam pressure of 150 lbs. per square inch. If the thickness is altered from $\frac{9}{16}$ inch to $\frac{1}{2}$ inch, to what radius must the end plates be curved so that the stress shall be unaltered under the same steam pressure?

18. Find the centrifugal tension (in lbs. per square inch) in the rim of a cast-iron fly-wheel 25 feet in diameter when running at 250 revolutions per minute. Weight of cast-iron $= 0{\cdot}26$ lb. per cubic inch.

19. Determine the speed, in revolutions per minute, of a cast-iron fly-wheel 20 feet in diameter when the centrifugal tension in the rim is 4250 lbs. per square inch. Weight of 1 cubic inch of cast-iron $= 0{\cdot}26$ lb.

20. Fig. 98 shows the lower end of a foundation bolt; the part A is round and of diameter d, the part B is square in cross section, s being the side of the square. The effective width of the cotter C is b, and its thickness t. Taking $f_t = 8$, $f_s = 6$, and $f_c = 15$, all in tons per square inch, express the dimensions s, b, and t in terms of d.

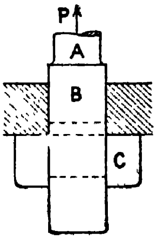


FIG. 98.

21. Referring to the bolt of the preceding exercise, if $f_c = 1{\cdot}6f_t$, $f_s = 0{\cdot}8f_t$, $f_t = 5$ tons per square inch, and P = 10 tons. Find the dimensions d, s, b, and t in inches.

22. Referring to the joint shown in Fig. 92, p. 77, if $f_c = 15$, $f_t = 8$, and $f_s = 6$, all in tons per square inch, determine d_1, b, t, D, and D_1 in terms of d.

94. Simple Torsion.—If two equal and opposite parallel forces P and Q act at opposite ends of a straight lever (Fig. 99) which is fixed to a shaft S, and which lies in a plane at right angles to the axis of the shaft, then, the forces P and Q cannot be balanced by any single force, *i.e.* they have no single force for their resultant, from which it follows that the forces P and Q will only tend to rotate the shaft about its axis.

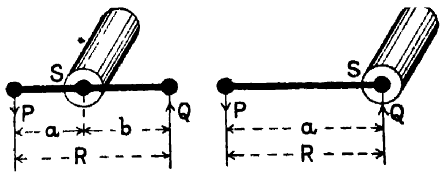


FIG. 99.

If the lines of action of P and Q be at perpendicular distances a and b respectively from the axis of the shaft, then the *turning moment*, *twisting moment*, or *torque* will be measured by $Pa + Qb = T$. But since $P = Q$,

then $T = P(a+b) = PR$. If P is in pounds and R is in inches, then T will be in *inch-pounds*. The inch-pound is generally the most convenient unit for the torque on a shaft, but the *foot-pound*, *foot-ton*, and *inch-ton* are also used.

If T is the torque on a shaft in inch-pounds, N the number of revolutions per minute, and H the horse-power transmitted, then it follows that

$$H = \frac{2\pi RPN}{12 \times 33000} = \frac{\pi TN}{6 \times 33000}.$$

95. Angle of Twist of a Shaft.—Let a shaft of length l, radius or diameter d be subjected to pure torsion by torques each equal to applied at its ends, as shown in Fig. 100. A straight line AM drawn on the surface of the shaft and parallel to the axis when the shaft is unstrained, will become a helix when the shaft is twisted. This follows from the following consideration. If the shaft be divided into a number of parts each of unit length by planes perpendicular to the axis each part will be subjected to the same torque, and the angular movement of one end of each part of unit length relative to the other end will be the same, and therefore the angular movement of one end of the shaft relative to the other end will be the sum of the angular movements due to each part, and therefore the movement of A relative to M, namely, the arc AB, will be proportional to l.

If a small square MN be drawn on the surface of the shaft when the latter is unstrained and having a side on AM, this square, shown enlarged to the left of Fig. 100, will become a parallelogram. If this

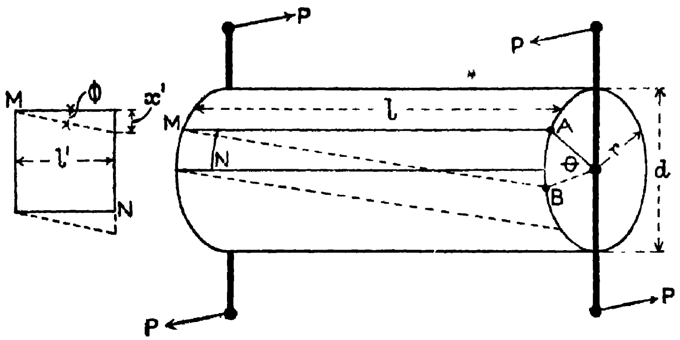


FIG. 100.

square be the outer face of a thin layer of material on the shaft, then the edges or narrow faces of this layer are subjected to shear stress of, say, an intensity f, and the shear strain is $x'/l' = x/l$ where x is the length of the arc AB. But shear strain $= \frac{\text{shear stress}}{\text{modulus of rigidity}}$, therefore $x/l = f/C$ or $x = fl/C$. If θ is the angle of twist in circular measure, then $\theta = x/r = 2x/d$, but $x = fl/C$, therefore $\theta = \frac{2fl}{Cd}$.

If n is the angle of twist in degrees, then since $\theta/\pi = n/180$, $n = \frac{360fl}{\pi Cd}$.

In stating that the shear strain on the small square element or thin layer MN is equal to x'/l', it is assumed that only the edges of that layer are subjected to shear stress, and that there is no stress on the square faces. It is obvious that there cannot be any stress on the outer square face, and since the angular movement of each particle of the shaft about the axis of the shaft is proportional to its distance from that axis, particles which are on the same radial line when the shaft is unstrained will remain on that line as the latter revolves; there can therefore be no relative movement between the layer MN and the layer next it within the shaft, and therefore there cannot be any stress on the inner face of the layer MN.

96. Moment of Resistance of a Shaft to Torsion.—It has been shown that the angle of twist of a circular shaft is $\theta = 2fl/Cd$ or fl/Cr. Hence $f = \theta Cr/l$, and for given values of θ, C, and l, f is proportional to r. Now for all parts of a given shaft subjected to a given torque, θ, C, and l are the same, therefore if a circular shaft be conceived to be made up of a number of thin tubes, the shear stress on any one of them will be proportional to its radius. Let r_1 be the mean radius of one of these tubes, a_1 the area of its cross section, and f_1 the shear stress on it. Then $f_1/f = r_1/r$, or $f_1 = r_1 f/r$. The total shear stress on the cross section of this tube is $f_1 a_1$, and the moment of this about the axis is $f_1 a_1 r_1$. Hence the moment of resistance of this tube to torsion is $\frac{f}{r} a_1 r_1^2 = \frac{f}{r} I_1$, where I_1 is the polar moment of inertia of the cross section of the tube about its axis. In like manner the moment of resistance to torsion of each of the other tubes is the factor f/r multiplied by the polar moment of inertia of its cross section about the axis. Hence M, the moment of resistance of the whole of the tubes, or of the solid shaft, is f/r multiplied by the sum of the polar moments of inertia of the separate annular parts of the cross section, which is equal to f/r multiplied by the polar moment of inertia of the whole cross section. But the polar moment of a circle of radius r about an axis through its centre and perpendicular to its plane is $\frac{\pi r^4}{2}$,

therefore $M = \frac{\pi}{2} r^3 f = \frac{\pi}{16} d^3 f.$

The following is another way of determining the moment of resistance of a circular shaft to torsion. Consider a small sector OAB of the cross

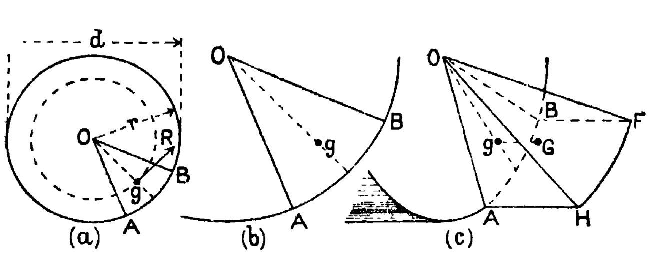


FIG. 101.

section of the shaft (Fig. 101). The full section of the shaft is shown at (a). The part in the neighbourhood of the sector is shown enlarged at (b), and an oblique view of this is shown at (c). Let f denote the

intensity of the shear stress along AB. Let perpendiculars to the sector be erected all over its surface to represent the intensity of the stress at each point, and consider for the moment that the stress is perpendicular to the plane of the sector. These perpendiculars will be enveloped by a pyramid OABFH, in which AH and BF are each equal to f. The volume of this pyramid will be the magnitude of the resultant R of the stress over the sector, and this resultant will act through G, the centre of gravity of the pyramid, and its line of action will be perpendicular to OAB, meeting the latter at g. The real line of action of R is in the plane of OAB and perpendicular to Og, as shown at (a). The moment of resistance of the sector to torsion is $R \times Og$. But R is equal to the volume of the pyramid $OABFH = AB \times f \times \frac{1}{3}r$, and $Og = \frac{3}{4}r$. Therefore $R \times Og = AB \times \frac{1}{4}fr^2$. The moment of resistance of the whole section will be $R \times Og$ multiplied by the number of times that the circle contains the sector OAB, that is, by the number of times that the circumference of the circle contains the arc AB, which is $\frac{2\pi r}{AB}$. Therefore

$$M = AB \times \tfrac{1}{4}fr^2 \times \frac{2\pi r}{AB} = \frac{\pi}{2}r^3f = \frac{\pi}{16}d^3f.$$

If the first method adopted in this Article for finding the moment of resistance of a solid circular shaft to torsion be applied to a hollow circular shaft (Fig. 102), it follows that the moment of resistance of the hollow shaft is $\frac{f}{R}$ multiplied by the polar moment of inertia of the section, *i.e.*

$$\frac{f}{R} \times \frac{\pi}{2}(R^4 - r^4), \text{ or } \frac{\pi}{16}\left(\frac{D^4 - d^4}{D}\right)f.$$

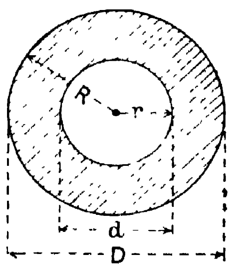


FIG. 102.

The moment of resistance of the hollow shaft may, however, be deduced directly from the moment of resistance of the solid shaft as follows. The moment of resistance of a solid shaft of diameter D is $\frac{\pi}{16}D^3f$. The moment of resistance of a solid shaft of diameter d *when it forms the central portion of the other* is $\frac{\pi}{16}d^3f_1$, where f_1 is the shear stress at radius r (Fig. 102), but $f_1 = f\frac{r}{R} = f\frac{d}{D}$. Hence the moment of resistance of the hollow shaft is $\frac{\pi}{16}D^3f - \frac{\pi}{16}d^3f\frac{d}{D} = \frac{\pi}{16}\left(\frac{D^4 - d^4}{D}\right)f.$

97. Formulæ for Shafts subjected to Simple Torsion.—It will be convenient to collect here the formulæ which have been proved in the three preceding Articles, and give several additional formulæ easily deduced from them.

T = torque or twisting moment on shaft in inch-pounds.
N = number of revolutions of shaft per minute.
H = horse-power transmitted by shaft.
l = length of shaft in inches.

d = diameter of solid shaft *or* internal diameter of hollow shaft in inches.
D = external diameter of hollow shaft in inches.
f = maximum shear stress on shaft in lbs. per square inch.
C = modulus of rigidity of shaft in lbs. per square inch.
θ = angle of twist of shaft in circular measure.
n = angle of twist of shaft in degrees.

For both Solid and Hollow Shafts—

$$\mathrm{H} = \frac{2\pi \mathrm{TN}}{12 \times 33000}. \qquad \mathrm{T} = \frac{12 \times 33000\mathrm{H}}{2\pi \mathrm{N}}.$$

For Solid Shafts—

$$\mathrm{T} = \frac{\pi}{16}d^3 f. \qquad f = \frac{16\mathrm{T}}{\pi d^3}.$$

$$\theta = \frac{2fl}{\mathrm{C}d} = \frac{32\mathrm{T}l}{\pi \mathrm{C}d^4}. \qquad n = \frac{360fl}{\pi \mathrm{C}d} = \frac{360 \times 16\mathrm{T}l}{\pi^2 \mathrm{C}d^4}.$$

For Hollow Shafts—

$$\mathrm{T} = \frac{\pi}{16}\left(\frac{\mathrm{D}^4 - d^4}{\mathrm{D}}\right)f. \qquad f = \frac{16\mathrm{TD}}{\pi(\mathrm{D}^4 - d^4)}.$$

$$\theta = \frac{2fl}{\mathrm{CD}} = \frac{32\mathrm{T}l}{\pi \mathrm{C}(\mathrm{D}^4 - d^4)}. \qquad n = \frac{360fl}{\pi \mathrm{CD}} = \frac{360 \times 16\mathrm{T}l}{\pi^2 \mathrm{C}(\mathrm{D}^4 - d^4)}.$$

98. Helical Springs.— It has already been shown that if a shaft or wire of diameter d and length l be subjected to a torque T, the angle of twist is $\theta = \frac{32\mathrm{T}l}{\pi \mathrm{C}d^4}$. Also, if f is the maximum shear stress, $\mathrm{T} = \frac{\pi}{16}d^3 f$, and $\theta = \frac{2fl}{\mathrm{C}d}$.

Now suppose the wire to be bent round so that its axis forms a semicircle of radius R, and let two radial arms AB and CD be connected to the

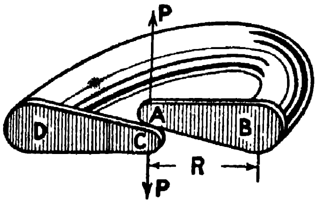


Fig. 103.

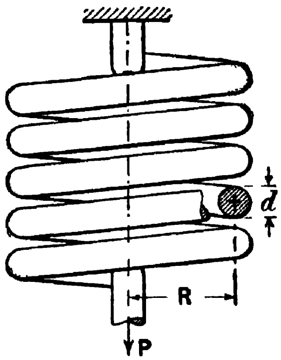


Fig. 104.

free ends of the wire, as shown in Fig. 103. If forces PP be applied to the inner ends of these arms, the forces acting through the centre of the

semicircle and perpendicular to its plane, the wire will be under a torque equal to PR. The length of the wire is πR. Hence the total angular movement of the two arms is $\theta=\frac{32PR^2}{Cd^4}$, and the relative movement of A and C is $\delta=R\theta=\frac{32PR^3}{Cd^4}$.

The half ring shown in Fig. 103 corresponds very approximately with a half coil of a helical spring (Fig. 104), in which the slope of the coils is small.

Extending the above formulæ for the half ring (Fig. 103) to the helical spring (Fig. 104), if n is the number of complete coils, then since for one half coil $\delta=\frac{32PR^3}{Cd^4}$, the extension for n coils is $\delta=\frac{64PR^3n}{Cd^4}$.

Also, $PR=\frac{\pi}{16}d^3f$. Therefore also, $\delta=\frac{4\pi R^2fn}{Cd}$, and $P=\frac{\pi d^3f}{16R}=\frac{Cd^4\delta}{64R^3n}$.

The length of wire, $l=2\pi Rn$, nearly.

The above formulæ also apply when the load is reversed, so that the spring as a whole is in compression.

Helical springs are also called cylindrical spiral springs

Exercises VIc.

1. Find the twisting moment in inch-pounds on a shaft which transmits 50 horse-power at a speed of 120 revolutions per minute, and calculate its diameter, taking the maximum stress at 9000 lbs. per square inch.

2. The turbine shaft of a 5 horse-power De Laval steam turbine is 0·2[illegible]3 inch in diameter, and its speed is 30,000 revolutions per minute. What is the maximum stress on this shaft when transmitting 5 horse-power?

3. A shaft transmits 100 horse-power at 120 revolutions per minute. The maximum torque is 1·4 times the mean torque, and the maximum stress is 9500 lbs. per square inch. Find the diameter of the shaft.

4. The diameter of a shaft is 9¼ inches. Determine the horse-power transmitted when the maximum stress is 9000 lbs. per square inch, and the speed is 130 revolutions per minute. The couplings of this shaft have each six bolts 2⅜ inches diameter, whose centres lie on a circle 14⅛ inches diameter. Find the average shear stress on the bolts.

5. A cast-iron flanged shaft coupling has six bolts 1¼ inches diameter. The axis of each bolt is 7⅜ inches from the axis of the shaft. Diameter of shaft, 5½ inches. When the maximum shearing stress on the shaft, due to the twisting moment, is 10,000 lbs. per square inch, what is the average shearing stress on the bolts?

6. If a shaft 2 inches diameter safely supports a torque of 15,000 inch-pounds, what torque would a shaft of the same material 5½ inches diameter support with the same factor of safety? What horse-power would the former shaft transmit at 150 revolutions per minute, and what should be the speed of the latter to transmit 600 horse-power.

7. Determine the horse-power transmitted by a hollow steel shaft whose external diameter is 18 inches, and internal diameter 12 inches. The speed is 90 revolutions per minute, and the maximum stress 10,000 lbs. per square inch.

8. A hollow steel shaft, external diameter d, internal diameter $\frac{3}{4}d$, is subjected to pure twisting, and transmits 8000 horse-power at a speed of 110 revolutions per minute. Taking the maximum shear stress at 9000 lbs. per square inch, find d.

9. A hollow steel shaft is made to replace a solid wrought-iron one of the same diameter, the material being 35 per cent. stronger than the iron; find what fraction of the outside diameter the internal diameter may be, and,

neglecting the couplings, find the percentage saving of weight by the substitution, assuming that the steel is 2 per cent. heavier than the wrought-iron.

10. A steel shaft $2\frac{1}{2}$ inches diameter, which is subjected to pure twisting, is 50 feet long, and is driven at one end, while the power is taken off at the other. One end of the shaft moves 30° in advance of the other. Find (1) the maximum shear stress, (2) the torque, and (3) the horse-power transmitted at 180 revolutions per minute. $C = 13,000,000$ lbs. per square inch.

11. A wrought-iron shaft 3 inches diameter and 40 feet long is subjected to pure twisting by couples at its ends. The maximum shear stress is 9000 lbs. per square inch, and the speed is 120 revolutions per minute. Determine the horse-power transmitted, and the angle of twist in degrees. $C = 10,500,000$ lbs. per square inch.

12. Find the diameter of a steel shaft which will transmit 15 horse-power at 130 revolutions per minute with an angle of twist amounting to 1° in a length equal to twenty times the diameter. Find also the maximum shear stress. $C = 13,000,000$ lbs. per square inch.

13. Determine the diameter of a solid shaft which shall have the same stiffness, under the same twisting moment, as a hollow shaft of the same material whose external and internal diameters are 9 inches and 6 inches respectively. Find also the ratio of the maximum shear stresses in the two shafts.

14. A steel shaft $2\frac{1}{2}$ inches in diameter is driven by a 20 horse-power gas-engine at 100 revolutions per minute. The shaft is supported by three bearings, spaced 15 feet apart between centres, and the centre of the driving pulley is 6 inches beyond the centre of one of the end bearings. Pulleys are arranged, as shown on the sketch (Fig. 105), to work certain machines, and the horse-

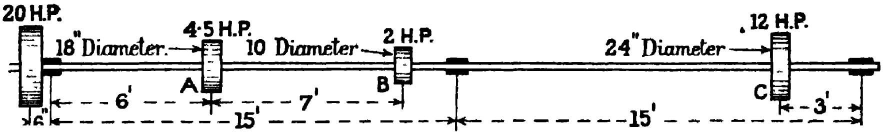


FIG. 105.

power taken off each of these pulleys is shown on the sketch; in addition, each bearing absorbs $\frac{1}{2}$ horse-power. Assuming that all the loads are applied at the centres of the respective pulleys and bearings, calculate the angle of twist in the shaft at each of these points, reckoning from either end of the shaft. The modulus of rigidity is 12,500,000 lbs. per square inch. [B.E.]

15.—Two closely coiled spiral springs were made out of round steel wire $\frac{1}{8}$ inch diameter. The one spring, A, had a mean diameter of coil of 4 inches, and the other, B, had a mean diameter of coil of 5 inches, both springs had 12 complete coils. These two springs were tested by loads extending them axially, and the results of the tests are shown in the table below:—

W	2	4	6	8	10	12	14	16	18	20
x_1	0·26	0·52	0·79	1·06	1·32	1·59	1·86	2·12	2·39	2·66
x_2	0·51	1·02	1·53	2·04	2·55	3·06	3·57	4·09	4·60	5·12

Where W is the axial load in pounds, and x_1 and x_2 are the extensions in inches of the springs A and B respectively.

Plot the results on squared paper.

Given that the law connecting the extension of these springs with their mean diameter of coil is of the form

$$\frac{\text{Extension of B}}{\text{Extension of A}} = \left(\frac{\text{mean diameter of coil of B}}{\text{mean diameter of coil of A}}\right)^n$$

what is the probable value of n?

Determine from these experiments the average value of the modulus of shear elasticity C for this quality of steel wire. [B.E.]

16. A closely coiled helical spring is made out of round steel wire $\frac{1}{4}$ inch in diameter, the coils having a mean diameter of 3 inches. What axial pull will produce a shear stress of 20,000 lbs. per square inch? If the modulus of

rigidity of the wire is 11,000,000 lbs. per square inch and the spring has 20 coils, how much will the spring extend under this pull, and how many ft.-lbs. of work must be done in producing this extension?

17. What is the stiffness, in lbs. per inch of axial deflection, of a close coiled helical spring having 10 coils whose mean diameter is 2 inches, diameter of wire 0·128 inch, and modulus of rigidity 11,000,000 lbs. per square inch?

18. What length of wire 0·232 inch in diameter will be necessary to form a closely coiled helical spring which shall extend 0·05 inch per lb. of axial load if the mean diameter of the coils is 3·5 inches, and the modulus of rigidity of the wire is 11,500,000 lbs. per square inch?

19. It is required to design a close coiled helical spring which shall deflect 1 inch under an axial load of 100 lbs. with a shear stress of 50,000 lbs. per square inch. The spring is to be made out of round steel wire having a modulus of rigidity of 11,000,000 lbs. per square inch, and the mean diameter of the coils is to be 10 times the diameter of the wire. Find the diameter and length of the wire necessary to form the spring.

CHAPTER VII

BEAMS AND BENDING

99. Bending Moments and Shearing Forces on Beams.—In general the lines of action of the external forces acting on a beam are perpendicular to the length of the beam, and they may be considered as being in one plane, and this plane, in what follows, will be taken as the plane of the paper. The external forces acting on a beam must of course balance one another if the beam is at rest.

Consider the case of a horizontal beam ABC (Fig. 106) acted on by any number of vertical forces P_1, P_2, etc. Take any cross section YY dividing the beam into two parts AB and BC. Consider the equilibrium of the part BC. Let the distances of P_1, P_2, etc., from YY be x_1, x_2, etc. Let R (Fig. 107) be the resultant of the external forces P_1, P_2, etc., acting on BC. The magnitude of R will be the algebraical sum of the forces P_1, P_2, etc., acting on BC. (For the forces shown in Fig. 106,

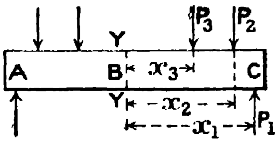


FIG. 106.

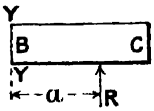


FIG. 107.

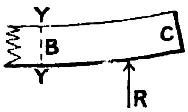


FIG. 108.

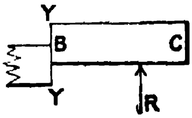


FIG. 109.

$R = P_1 - P_2 - P_3$). The distance a of R from YY must be such that the moment of R about a horizontal axis in YY is equal to the algebraical sum of the moments of P_1, P_2, etc., about the same axis. (For the forces shown in Fig. 106, $Ra = P_1x_1 - P_2x_2 - P_3x_3$). Expressed in another way, $R = \Sigma P$ and $Ra = \Sigma(Px)$.

The force R tends to turn BC about a horizontal axis in YY; in other words, R tends to bend the beam at YY (Fig. 108), and the bending moment is $Ra = \Sigma Px$. R also tends to make BC slide on AB at YY; in other words, R tends to shear the beam at YY (Fig. 109), and the shearing force is $R = \Sigma P$.

The bending and shearing effects of R on BC may perhaps be made more apparent by the artifice of applying at YY two opposite forces R_1 and R_2, each equal and parallel to R, as shown in Fig. 110. The addition of the forces R_1 and R_2 will evidently not affect the equilibrium of BC. The forces R and R_1 being equal and acting in opposite parallel directions, form a couple which can only produce a turning or bending action on BC, while the remaining force R_2 can only make or tend to make BC slide on YY.

To make the existence of the turning or bending and the sliding or shearing actions more evident, consider the case of a block BC (Fig. 111) resting on a horizontal plane YY. Let this block be pushed by a horizontal force R as shown. If the resistance to sliding on YY be not too great the block will slide to the left into, say, the position shown by the dotted lines. But if the resistance to sliding is great enough the force R will cause the block BC to tilt over, as shown by the dotted lines in Fig. 112, provided the resistance to tilting is not too great. R_1 is the resistance to sliding at YY, and R_2 is the effect of R transmitted, by reason of the rigidity of BC, from section to section downwards to YY.

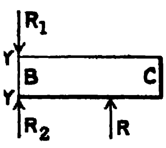


FIG. 110.

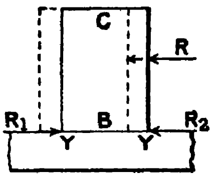


FIG. 111.

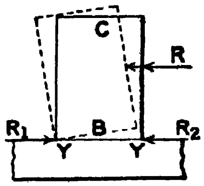
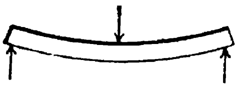


FIG. 112.

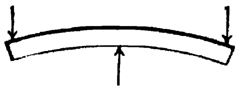
It is evident that the magnitude of the shearing action at YY depends only on the magnitude of R, and not on the distance of R from YY. But the turning or bending action at YY depends on both the magnitude of R and the distance of R from YY.

100. Positive and Negative Bending and Shearing. — When a horizontal beam is bent by the action of the loads on it, it will either "sag" or "hog," that is, it will either become concave or convex on the



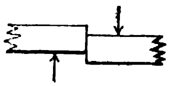
+ *Bending.*

FIG. 113.



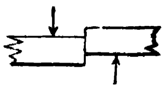
− *Bending.*

FIG. 114.



+ *Shearing.*

FIG. 115.



− *Shearing.*

FIG. 116.

top, and it will be convenient to call one of these, say the first (Fig. 113), *positive* (+) *bending*, and the other (Fig. 114) *negative* (−) *bending*. Again, in considering the shearing action at a section of the beam the loads will tend either to cause the portion to the right of the section to descend and the portion to the left to ascend, or *vice versa*, and it will be convenient to call one of these, say the first (Fig. 115), *positive* (+) *shearing*, and the other (Fig. 116) *negative* (−) *shearing*.

101. Bending Moment and Shearing Force Diagrams.—If the bending moments and shearing forces at a sufficient number of sections of a beam be determined and the results plotted to scale at right angles to a base line representing the length of the beam, diagrams are obtained by joining the points plotted, which are called the *bending moment* and *shearing force diagrams*. In cases where there is both positive and negative bending, or positive and negative shearing on the same beam, it is necessary to distinguish between the positive and negative quantities by measuring them on opposite sides of the base line, and it is desirable in all cases to measure positive bending moments and positive shearing

forces above the base line, and negative bending moments and negative shearing forces below the base line.

102. Examples of Bending Moment and Shearing Force Diagrams.—Two examples of bending moment and shearing force diagrams have already been given in Art. 75, pp. 59–61, where the graphic method of constructing these diagrams was explained. In this chapter the bending moments and shearing forces will be found by calculation.

In what follows M will denote the bending moment and F the shearing force at a section which is at a distance x from some fixed point, generally the free end of a cantilever, and either the centre or one end of a beam. M_m will denote the maximum bending moment, and F_m the maximum shearing force. XX will denote the base line upon which the bending moment diagram or shearing force diagram is plotted.

EXAMPLE I.—Cantilever (Fig. 117) with loads W_1 and W_2.

Between A *and* B. $M = -W_1x$, which is the equation to a straight line. $M_m = -W_1a$ at B. $F = W_1$.

Between B *and* C. $M = -\{W_1x + W_2(x-a)\}$, which is the equation to a straight line. $M_m = -\{W_1(a+b) + W_2b\}$. $F = W_1 + W_2$.

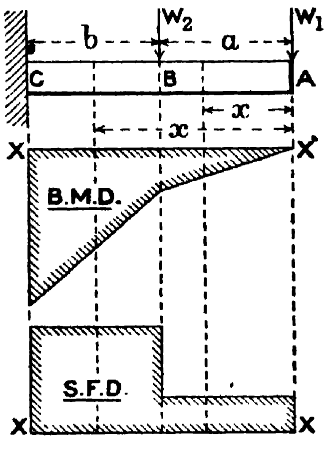


FIG. 117.

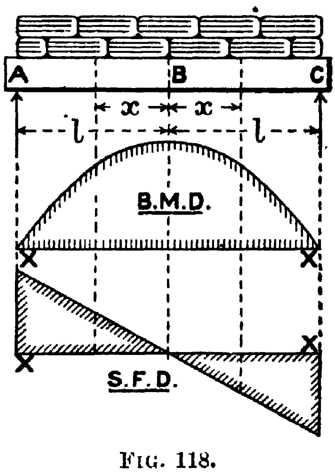


FIG. 118.

EXAMPLE II.—Beam (Fig. 118) supported at the ends and carrying a load w per unit of length uniformly distributed.

Reactions at supports $= wl$.

Between A *and* B. $M = wl(l-x) - w(l-x)\frac{(l-x)}{2} = \frac{w}{2}(l^2 - x^2)$, which is the equation to a parabola whose axis is the vertical through B, $M = 0$ at A where $x = l$. M becomes M_m where $x = 0$. Hence $M_m = \frac{wl^2}{2}$ at B.

$F = wl - w(l-x) = wx$, which is the equation to a straight line. $F = 0$ at B where $x = 0$. F becomes F_m where $x = l$. Hence $F_m = wl$ at A.

Between B *and* C. $M = \frac{w}{2}(l^2 - x^2)$, the same as between A and B.

$F = w(l - x) - wl = -wx$, which is numerically the same as between A and B, but of the opposite sign.

EXAMPLE III.—Beam (Fig. 119) supported at two points equally distant from the ends, and carrying a load w per unit of length uniformly distributed.

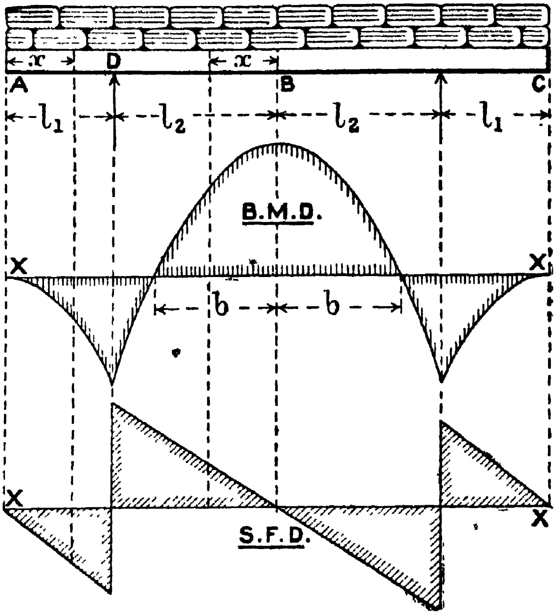


FIG. 119.

Reactions at supports $= w(l_1 + l_2)$.

Between A *and* D.

$M = -\frac{wx^2}{2}$, which is the equation to a parabola, vertex at A and axis vertical. $M = 0$ at A where $x = 0$. $M_m = -\frac{wl_1^2}{2}$ at D.

$F = -wx$, which is the equation to a straight line. $F = 0$ at A where $x = 0$. $F_m = -wl_1$ at D.

Between D *and* B.

$$M = w(l_1 + l_2)(l_2 - x) - \frac{w}{2}(l_1 + l_2 - x)^2 = \frac{w}{2}(l_2^2 - l_1^2 - x^2),$$

which is the equation to a parabola whose axis is the vertical through B.

$$\text{At D where } x = l_2, \ M = -\frac{wl_1^2}{2} \text{ as before.}$$

At B where $x = 0$, $M = \frac{w}{2}(l_2^2 - l_1^2)$. $M = 0$ where $x = b = \pm\sqrt{l_2^2 - l_1^2}$.

$F = w(l_1 + l_2) - w(l_1 + l_2 - x) = wx$, which is the equation to a straight line. $F = 0$ at B where $x = 0$, $F_m = wl_2$ at D where $x = l_2$.

Between B *and* C the bending moment diagram is the same as between B and A, and the only difference in the shearing force diagrams is in sign.

The maximum bending moment on the beam will be least when the bending moment at B is equal to the bending moment at D, that is, when $\frac{w}{2}(l_2^2 - l_1^2) = \frac{wl_1^2}{2}$ or $l_2 = l_1\sqrt{2}$. If $l = l_1 + l_2$, then $l_1 = l(\sqrt{2} - 1) = 0{\cdot}414l$. This shows where the supports should be placed when the beam is uniformly loaded. An example of this is found in the floats of a paddle-wheel.

EXAMPLE IV.—Two equal cantilevers (Fig. 120) carrying a beam with a load W at its centre.

Reactions at supports of beam = loads at free ends of cantilevers $= \frac{1}{2}W$.

For the beam AB.

$M = \frac{1}{2}W(l_2 - x)$.

$M = 0$ at A and B where $x = l_2$.

$M_m = \frac{1}{2}Wl_2$ at the centre C where $x = 0$.

$F = \frac{1}{2}W$ between A and C and $F = -\frac{1}{2}W$ between C and B.

For the cantilever AD. $M = -\frac{1}{2}Wx$, a straight line which is a continuation of the bending moment line for AC.

$M = 0$ at A where $x = 0$. $M_m = -\frac{1}{2}Wl_1$ at D where $x = l_1$. $F = \frac{1}{2}W$.

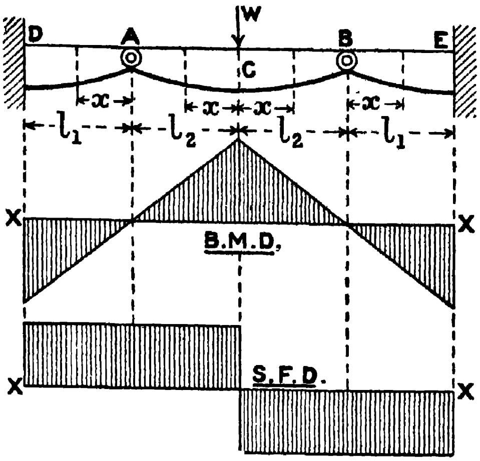


FIG. 120.

For the cantilever BE. $M = -\frac{1}{2}Wx$, a straight line which is a continuation of the bendin moment line for BC.

$M = 0$ at B where $x = 0$. $M_m = -\frac{1}{2}Wl_1$ at E where $x = l_1$. $F = -\frac{1}{2}W$.

If M_m for the beam $= M_m$ for the cantilevers, then $\frac{1}{2}Wl_2 = \frac{1}{2}Wl_1$ or $l_2 = l_1$.

103. Shearing Force at a Section where there is a Concentrated Load.—Let AC and BD (Fig. 121) be cross sections of a beam on opposite sides of a concentrated load Q. Let R_1 be the resultant of all the loads to the right of Q, and R_2 the resultant of all the loads to the left of Q. The shearing force at BD is equal to R_1, and the shearing force at AC is equal to R_2. In moving the section BD towards Q the shearing force remains equal to R_1 so long as the section is to the right of Q and *however near it may be to* Q. In like manner, in moving the section AC towards Q the shearing force remains equal to R_2 so long as the section remains to the left of Q and *however near it may be to* Q. The question then is, what is the shearing force at Q? is it equal to R_1 or is it equal to R_2? The answer is that it is probably near the algebraical mean of the two. In practice there is no such case as a load acting at a point or line. What is called a concentrated load must act over a certain amount of surface, even if it acts through what is called a "knife edge." At (*a*) (*a*) (Fig. 121) are shown examples of shearing force diagrams as usually drawn in the neighbourhood of a concentrated load. At (*b*) (*b*) the diagrams are shown cor-

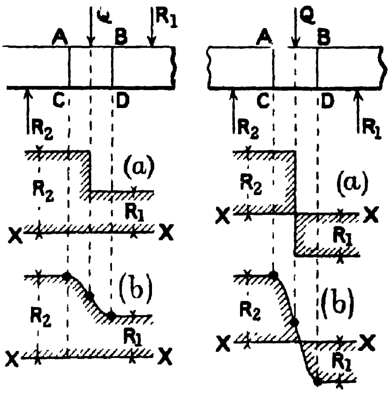


FIG. 121.

rected on the assumption that the load Q is distributed over the surface AB. Generally the distribution of the load on AB would not be uniform. The point just discussed is of little practical importance, but sometimes students find it to be a difficulty.

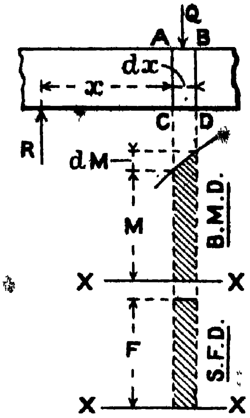


FIG. 122.

104. Relations between Bending Moment and Shearing Force Diagrams.—Let M be the bending moment and F the shearing force at a section AC of a beam (Fig. 122). The section AC may be anywhere except at a point where there is a concentrated load, *but it may be as near to that point as is desired.* Take a section BD at an indefinitely small distance dx from AC, and let $M+dM$ be the bending moment at BD. Let Q be the resultant of any loads which there may be on the beam between the sections AC and BD, and let its distance from BD be ndx where n is a fraction.

Let R be the resultant of all the external forces acting on the beam to the left of the section AC, and let its distance from AC be x. Then, $M = Rx$, $F = R$, $M + dM = R(x + dx) - Qndx = Rx + Rdx - Qndx$, therefore, $dM = Rdx - Qndx$, and $\frac{dM}{dx} = R - Qn = F - Qn$.

If Q is the resultant of a load which is distributed over AB, then, since dx is indefinitely small, Q will be so small that it may be neglected, and then $\frac{dM}{dx} = F$.

Again, if Q is the resultant of loads concentrated at points in AB, these loads may be avoided by taking dx small enough, and then as before $\frac{dM}{dx} = F$.

The shearing force F at the section AC is therefore a measure of the slope of the bending moment line at the point corresponding to AC. In other words, the shearing force at any section is equal to the rate of increase of the bending moment at that section.

Again, $dM = Fdx$, therefore the difference between the bending moments at two sections indefinitely near to one another is equal to the area of the shearing force diagram between these sections, and, in passing from one section to any other section, it is obvious that the sum of all the increments dM will be equal to the sum of all the increments Fdx, and therefore the difference between the bending moments at any two sections is equal to the area of the shearing force diagram between these sections.

These results are very interesting and very useful. For example, referring to a horizontal beam, if the bending moment line is horizontal at any point its slope is nil, and there can therefore be no shearing force at the corresponding section. Again, at the highest point of the bending moment line the slope changes from positive to negative, and therefore where the maximum bending moment occurs the shearing force must change its sign, and the shearing force line will cross the base line. The converse of this is not always true, and all that can be said about the

bending moment at a section where the shearing force is zero or changes from positive to negative is that at that section the bending moment has ceased to increase or ceased to decrease, but in seeking for the section where the bending moment is a maximum, the shearing force diagram is very useful.

105. Travelling Loads.—By a travelling, moving, or rolling load is meant one that comes on to a girder at one end, moves along the girder, and comes off at the other end. It is evidently necessary to know what the maximum bending moment and the maximum shearing force are at any section of the girder, and the bending moment and shearing force diagrams are constructed so as to show the maximum bending moment and maximum shearing force at every section.

The first step is to find the position of the travelling load in relation to any section selected which will make the bending moment a maximum at that section; then an expression is found for that bending moment in terms of the load and the distance of the section from a fixed selected point, and from this expression the bending moment diagram can be constructed. The positive and negative shearing force diagrams are determined in a similar manner.

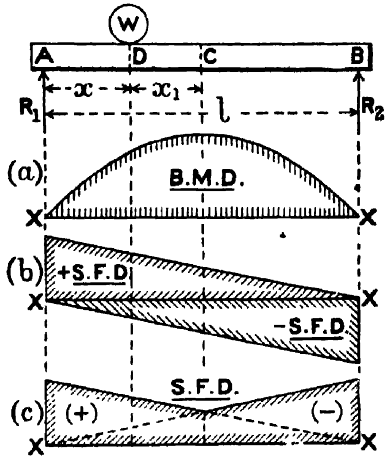


FIG. 123.

EXAMPLE I.—A single load W travelling along a girder AB (Fig. 123) supported at its ends.

When the load W is to the right of a section D which is at a distance x from A the bending moment at D is R_1x, and R_1 is greater the nearer W is to D. Again, when W is to the left of D the bending moment at D is $R_2(l-x)$, and R_2 is greater the nearer W is to D. Hence the bending moment at D is a maximum when W is at D.

Placing W at D, $R_1 = \frac{W}{l}(l-x)$, and the maximum bending moment at $D = M = \frac{Wx}{l}(l-x)$. If D be referred to the vertical through C, the centre of the span, so that $CD = x_1$, then since $x = \frac{1}{2}l - x_1$, $M = \frac{W}{l}\left(\frac{l^2}{4} - x_1^2\right)$, which is the equation to a parabola whose axis is the vertical through C. The height of the vertex above the base line is the maximum value of $\frac{W}{l}\left(\frac{l^2}{4} - x_1^2\right)$, which is obtained by putting $x_1 = 0$, then $M_m = \frac{Wl}{4}$. The bending moment for the travelling load is shown at (*a*). This bending moment diagram is the same as for a uniform dead load of w per unit of length, where $\frac{wl^2}{8} = \frac{Wl}{4}$, or $w = \frac{2W}{l}$. The load w per unit of length is called the *equivalent uniform dead load.*

When W is to the right of D, the positive shearing force at D is equal to R_1 the reaction at the left-hand support, and this increases the nearer W is to D. When W is to the left of D, the shearing force at D is negative and equal to R_2. Hence the maximum positive shearing force at D occurs when W is at D. Placing W at D, $R_1 = \frac{W}{l}(l - x)$, and the maximum positive shearing force at any section $D = F = \frac{W}{l}(l - x)$, which is the equation to a straight line. $F_m = W$ at A where $x = 0$, and $F = 0$ at B where $x = l$.

It is also evident that the maximum negative shearing force at $D = F = \frac{Wx}{l}$, which is the equation to a straight line. $F_m = W$ at B where $x = l$, and $F = 0$ at A where $x = 0$. The complete shearing force diagram for the travelling load is shown at (*b*).

In designing the section of a plate girder to resist the shearing force, it is the maximum numerical value of the shearing force which must be known, its sign being of no consequence. Hence it is convenient to place the (+) and (−) shearing force diagrams on the same side of the base line, as shown at (*c*). If however lattice work takes the place of the web plate, the lattice work must be designed so that it will take *either* the maximum positive shearing force *or* the maximum negative shearing force, *but not both at the same time.*

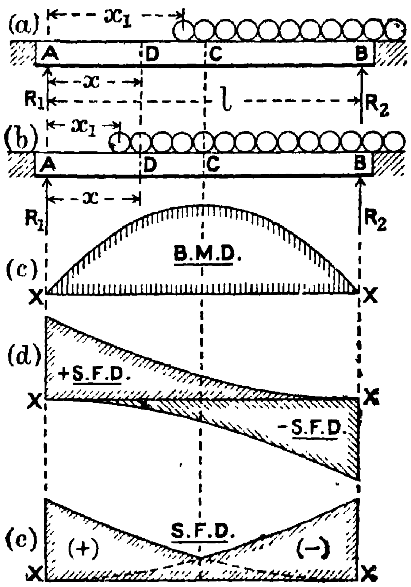


FIG. 124.

EXAMPLE II.—A uniform load of w per unit of length travelling along a girder AB (Fig. 124) supported at its ends, the length of the load being not less than the span l.

As the load advances over the girder it is evident that the bending moment at any section D will continue to increase until the girder is covered by the load, because a load placed anywhere on the girder will add to the bending moment at D. The bending moment diagram for this case is therefore a parabola having for its axis the vertical through C, the centre of the span as shown at (*c*), the height of the vertex above the base line XX being $\frac{wl^2}{8}$.

The load being in the position shown at (*a*) or (*b*), the reaction $R_1 = \frac{w}{2l}(l - x_1)^2$.

When the load is in the position shown at (a), the positive shearing force at D is

$$F = R_1 = \frac{w}{2l}(l - x_1)^2,$$

which shows that F increases as x_1 decreases.

When the load is in the position shown at (b), the positive shearing force at D is $F = R_1 - w(x - x_1)$ $= \frac{w}{2l}(l - x_1)^2 - w(x - x_1)$ $= \frac{w}{2l}(l^2 + x_1^2) - wx$, which shows that F decreases as x_1 decreases.

It is therefore evident that the positive shearing force at D is greatest when $x_1 = x$, that is, when the left-hand end of the load is at D. The shearing force at D is then $F = \frac{w}{2l}(l - x)^2$, which is the equation to a parabola having for its axis the vertical through B, the right-hand end of the span, the vertex being on the base line XX, as shown at (d). The maximum positive shearing force is $\frac{wl}{2}$ at A where $x = 0$.

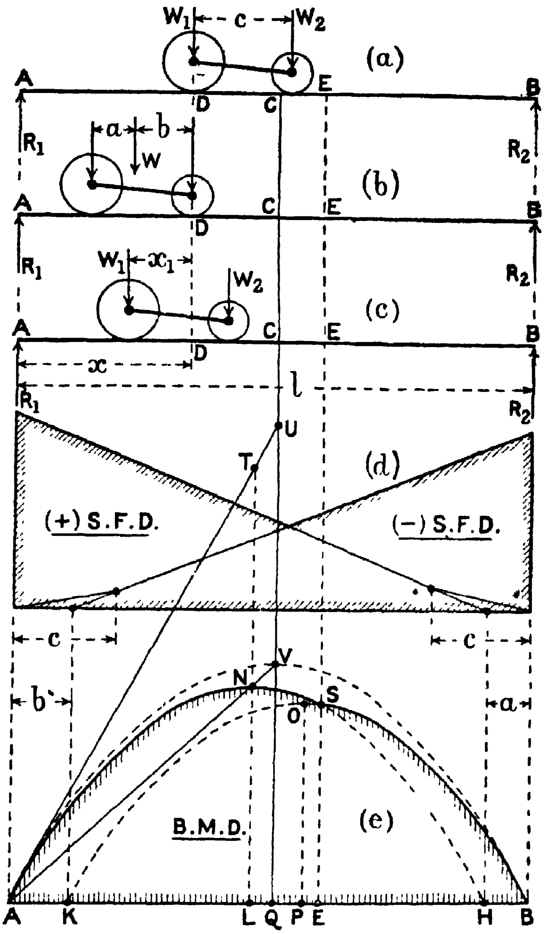


FIG. 125.

In like manner it can be shown that the negative shearing force at D is greatest when the right-hand end of the load is at D. The shearing force at D is then $F = -\frac{wx^2}{2l}$, which is the equation to a parabola having for its axis the vertical through A, the left-hand end of the span, the vertex being on the base line XX, as shown at (d). The maximum negative shearing force is $\frac{wl}{2}$ at B where $x = l$.

The positive and negative shearing force diagrams are shown plotted on the same side of the base line XX at (e).

EXAMPLE III.—Two loads W_1 and W_2 at a fixed distance c apart, moving along a girder AB (Fig. 125) supported at its ends.

Let W, the resultant of W_1 and W_2, act at a distance a from W_1 and

a distance b from W_2, then $c=a+b$, $W=W_1+W_2$, $Wa=W_2c$, and $Wb=W_1c$.

When both loads are to one side of any section D, which is at a distance x from A, the bending moment at D will evidently increase as the loads move towards D. Hence the bending moment at D will be a maximum, either when W_1 is at D, or when W_2 is at D, or when W_1 and W_2 are on opposite sides of D. It will now be shown that when W_1 and W_2 are on opposite sides of D, the bending moment at D has a value which lies between the values of the bending moment at D when W_1 and W_2 are in turn placed at D.

At (*a*) W_1 is placed at D, at (*b*) W_2 is placed at D, and at (*c*) W_1 and W_2 are placed on opposite sides of D, the distance of W_1 from D being x_1 in the latter case. Let M_1, M_2, and M_3 be the bending moments at D, corresponding to the positions of the loads shown at (*a*), (*b*), and (*c*) respectively. It is easy to show that $M_1=\frac{Wx}{l}(l-x-a)$, $M_2=\frac{Wx}{l}(l-x+b)-Wb$, and $M_3=\frac{Wx}{l}(l-x-a+x_1)-W_1x_1$.

$$\text{If}\quad M_3>M_1,\quad\text{then}\quad \frac{Wx}{l}(l-x-a+x_1)-W_1x_1>\frac{Wx}{l}(l-x-a),$$

$$\text{therefore}\qquad \frac{Wx}{l}>W_1.$$

$$M_2-M_3=\frac{Wx}{l}(l-x+b)-Wb-\frac{Wx}{l}(l-x-a+x_1)+W_1x_1$$

$$=\left(\frac{Wx}{l}-W_1\right)(a+b-x_1).$$

Since x_1 is less than $a+b$, the quantity $a+b-x_1$ is positive, and if $M_3>M_1$, $\frac{Wx}{l}>W_1$, therefore the quantity $\frac{Wx}{l}-W_1$ is positive. Hence M_2-M_3 is positive when $M_3>M_1$, that is, $M_2>M_3$ when $M_3>M_1$. In like manner, if $M_3>M_2$, $M_1>M_3$. Therefore M_3 lies between M_1 and M_2, and the bending moment at D is a maximum either when W_1 is at D or when W_2 is at D.

$$\text{If}\quad M_2>M_1,\ \text{then}\ \frac{Wx}{l}(l-x+b)-Wb>\frac{Wx}{l}(l-x-a),$$

therefore $x>\frac{bl}{c}$ or $x>\frac{W_1l}{W}$, and if $M_1>M_2$, then $x<\frac{bl}{c}$ or $x<\frac{W_1l}{W}$. Hence if the span be divided into two parts, the one to the left, AE, having a length $=\frac{W_1l}{W}$, and the other to the right, $EB=\frac{W_2l}{W}$, the maximum bending moment at any section in AE will occur when W_1 is at that section, and the maximum bending moment at any section in EB will occur when W_2 is at that section. AE may be called the field of W_1, and EB the field of W_2.

$M_1=\frac{Wx}{l}(l-x-a)$. $M_1=0$ when $x=0$ or $x=l-a=AH$. The

curve for M_1 is a parabola whose axis LN bisects AH at right angles. $AL = \frac{l-a}{2}$. Putting $x = \frac{l-a}{2}$ in the equation $M_1 = \frac{Wx}{l}(l-x-a)$, $LN = \frac{W(l-a)^2}{4l}$.

In like manner it can be shown that the curve for M_2 is a parabola KOB whose axis PO bisects KB at right angles, AK being equal to b, and $PO = \frac{W(l-b)^2}{4l}$. These two parabolas intersect at a point S on the vertical through E, which divides the field of W_1 from the field of W_2, and the bending moment diagram for the whole girder is ANSB.

A parabola, whose axis is the vertical through C, the middle of the span, and which touches the larger of the two parabolas at the base line, will be the bending moment curve for the equivalent uniform dead load. Let ANH be the larger of the two parabolas ANH and KOB. Produce the axis LN to T, making NT = LN, then AT is the tangent to the parabola ANH at A. AT will also be the tangent to the circumscribing parabola at A. Produce AT to meet the vertical CQ at U. Bisect QU at V, then V is the vertex of the circumscribing parabola.

It is easy to show that QV : LN :: AQ : AL, and therefore that the point V may be found by joining AN and producing to meet CQ at V.

$$QV = \frac{LN \times AQ}{AL} = \frac{W(l-a)^2}{4l} \times \frac{l}{2} \div \frac{l-a}{2} = \frac{W(l-a)}{4}.$$

If the equivalent uniform dead load is w per unit of length, then

$$\frac{wl^2}{8} = \frac{W(l-a)}{4}, \quad \text{therefore} \quad w = \frac{2W(l-a)}{l^2}.$$

When both loads are to the right of D the positive shearing force at $D = R_1$, and as the loads move towards D, R_1 increases, and is greater the nearer W_1 is to D. When W_1 passes to the left of D, the positive shearing force at D is suddenly diminished by the amount W_1, and is then equal to $R_1 - W_1$, but as W_1 moves to the left, W_2 being still to the right of D, R_1 increases, and therefore $R_1 - W_1$ increases until W_2 is at D. When both loads are to the left of D there is no positive shearing force at D. Hence the positive shearing force at D is a maximum, either when W_1 is at D or when W_2 is at D, or, more correctly, when W_1 or W_2 is just to the right of D.

Placing the loads so that W_1 is at D, or just to the right of D, as shown at (*a*), Fig. 125, the positive shearing force at $D = F_1 = R_1 = \frac{W}{l}(l-x-a)$.

Placing the loads so that W_2 is at D, or just to the right of D, as shown at (*b*), the positive shearing force at $D = F_2 = R_1 - W_1 = \frac{W}{l}(l-x+b) - W_1$.

F_2 will be greater than F_1 when $\frac{W}{l}(l-x+b) - W_1 > \frac{W}{l}(l-x-a)$,

that is, when $\frac{cW}{W_1} > l$, and F_2 will be less than F_1 when $\frac{cW}{W_1} < l$.

The equation for the maximum positive shearing force is therefore either

$$F_1 = \frac{W}{l}(l - x - a), \text{ or } F_2 = \frac{W}{l}(l - x + b) - W_1.$$

F_1 is a maximum when $x = 0$, and F_2 is a maximum when $x = 0$,

$$F_1 = 0 \text{ when } x = l - a, \text{ and } F_2 = 0 \text{ when } x = l + b - \frac{W_1 l}{W}.$$

The equations for the positive shearing force at D only apply when both loads are on the girder. The equation $F_1 = \frac{W}{l}(l - x - a)$ applies between $x = 0$ and $x = l - c$, and the equation $F_2 = \frac{W}{l}(l - x + b) - W_1$ applies between $x = c$ and $x = l$. For the remainder of the beam in each case the shearing force is due to one load only.

The positive shearing force diagram shown at (*d*), Fig. 125, is for the case where F_1 is greater than F_2.

The maximum negative shearing force at any section is determined in a similar manner.

106. Travelling Loads—Graphic Method.—The method discussed in connection with Example III. of the preceding Article for determining

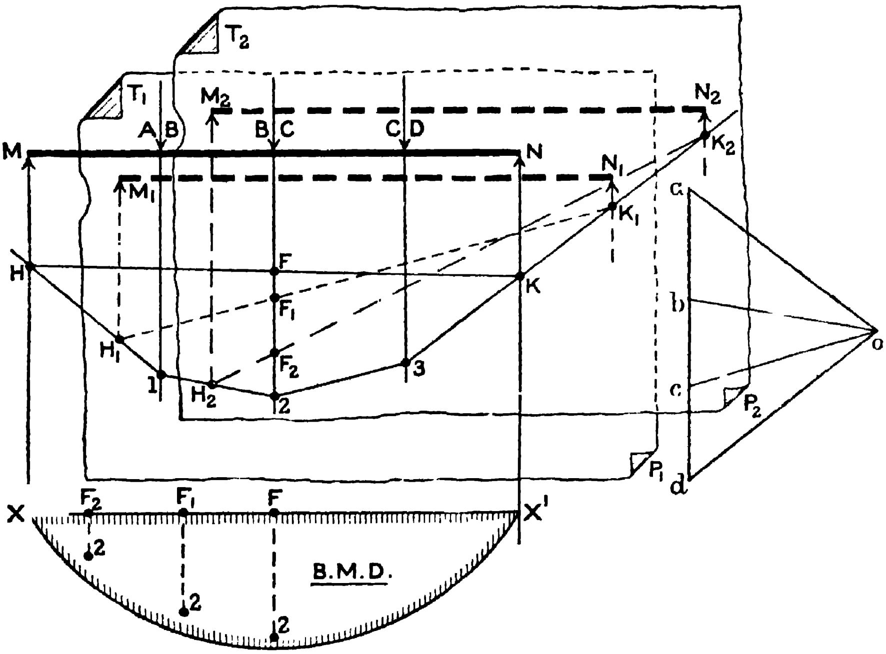


FIG. 126.

the maximum bending moment and maximum shearing force diagrams for two travelling loads may be extended to cases where there are more than two travelling loads, but for such cases the graphic method now to be described is simpler.

MN (Fig. 126) is a beam, supported at the ends, along which three

loads AB, BC, and CD, at fixed distances apart, travel. If instead of the loads travelling over the beam in one direction while the beam is stationary, the beam and its supports are moved under the loads in the opposite direction while the loads are stationary, the resulting maximum bending moment diagram would be the same. On the drawing paper draw the beam MN and show the loads in one position. Still working on the drawing paper draw (Art. 56) the funicular polygon H123K, which will also be the bending moment diagram (Art. 75) for the position of the loads assumed. Draw a horizontal line XX′ under MN to serve as a base line for the required diagram of maximum bending moments. Transfer the ordinates of the points 1, 2, and 3 of the bending moment diagram H123K to the lower diagram; thus F2 in the lower diagram is made equal to F2 in the upper diagram, XF being equal to the horizontal distance of F in HK from MX.

On a sheet of tracing paper TP make a tracing of the beam MN and the lines of the reactions of the supports. Let the tracing paper be moved into another position T_1P_1, the beam MN coming into the position M_1N_1. For clearness in the figure M_1N_1 is not at the same level as MN, but M_1N_1 is parallel to MN. The bending moment diagram for the altered position of the beam in relation to the loads will now be H_1123K_1, and this should be drawn on the tracing paper, or, at all events, the points such as F_1 and 2 should be clearly marked on the tracing paper.

Next transfer the tracing paper so that F_1 is on XX′, and M_1H_1 is on the vertical through X. If the point 2 on the tracing paper be now pricked through on to the drawing paper, the point 2 under F_1 on the lower diagram is obtained, and this gives the bending moment under the load BC when the latter is at F_1 in XX′. The points 1 and 3 are to be similarly dealt with.

The tracing paper is next moved into another position T_2P_2, and the corresponding bending moment diagram is H_223K_2. Points such as F_2 and 2 are to be clearly marked on the tracing paper, and transferred to

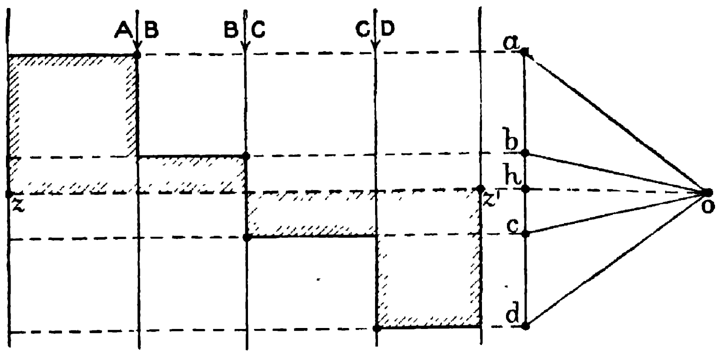


FIG. 127.

the lower diagram as before. Proceeding in this way until a sufficient number of points have been fixed under XX′, a fair curve drawn to enclose them all and pass through the outer ones is the required diagram of maximum bending moments on the base XX′.

The shearing force diagram for each position of the beam in relation to the loads will be a stepped diagram, the levels of the steps being the levels of the points *a*, *b*, *c*, and *d* (Fig. 127) on the line of loads, and the zero line zz' being level with a point *h* obtained by drawing through *o* a line *oh* parallel to the closing line of the funicular polygon.

Horizontal lines are drawn on the tracing paper through the points *a*, *b*, *c*, and *d*, and the right-hand top corners of the steps above the zero line, and the left-hand bottom corners of the steps below the zero line, are marked on the tracing paper, as shown by the prominent dots in Fig. 127. *z* and z', the extremities of the zero line, are also marked on the tracing paper. The tracing paper is then transferred so that zz' coincides with a base line ZZ' (Fig 128) on the drawing paper, and the points marked on the tracing paper are pricked through on to the drawing paper. This is repeated for each position into which the tracing paper was placed in determining the bending moments. It will be found that the outside points lie on a series of straight lines.

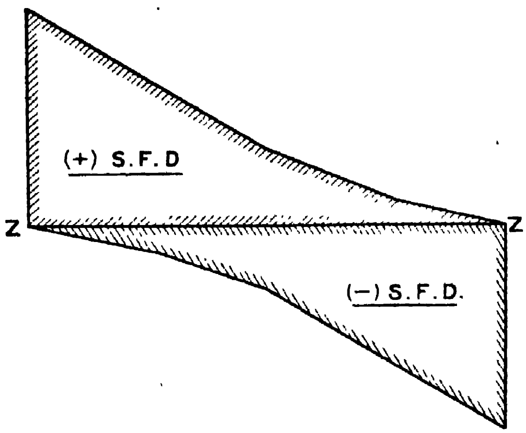


FIG. 128.

107. Reversal of Shearing Stress due to Addition of Travelling Load.—Referring to the upper part of Fig. 129, AEO and BFO are the diagrams of positive and negative shearing forces respectively due to the dead or constant load on a girder of span AB, and AHB and BKA are the diagrams of maximum positive and maximum negative shearing forces respectively due to the travelling load. The lower part of the same figure shows all the diagrams, drawn for convenience on the same side of the base AB, the lines of the negative shearing force diagrams being dotted.

An inspection of the lower part of Fig. 129 shows that with the dead load only the shearing force over the portion CO of the girder is positive, but when the travelling load is going over the girder there will be between C and O, for certain positions of the travelling load, a negative shear greater than the positive shear. Hence during a part of the time that the travelling load is moving over the girder the shear on the portion CO will change from positive to negative. Also between O and D the negative shear due to the dead load will change to a positive shear due to the travelling load. Hence the portion of the girder between C and D must be capable of resisting either

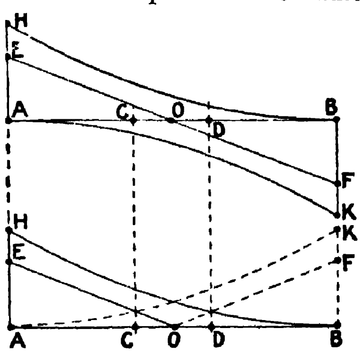


FIG. 129.

positive or negative shear. This is important in the case of open web or braced girders, and is referred to again in Art. 207, p. 233.

108. Bending and Shearing by Forces in different Planes.—Suppose a beam AB (Fig. 130) to be acted on by a force P_1 at B and a force P_2 at D, the lines of action of P_1 and P_2 being in different planes, but perpendicular to AB. Consider the bending action at a section C at a distance x_1, from P_1 and x_2 from P_2. So far as the bending action of P_2 is concerned P_2 may be replaced by a force Q acting at B in a direction parallel to P_2, the magnitude of Q being such that $Qx_1 = P_2x_2$. There are now two forces acting at B, namely, P_1 and Q, and their resultant R

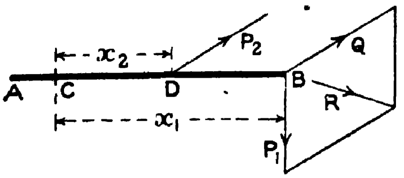


FIG. 130.

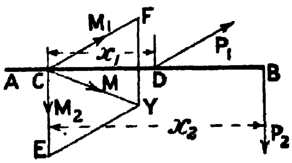


FIG. 131.

may be found by the parallelogram of forces. The resultant bending moment at C is then Rx_1, and the plane of bending is ABR.

If the plane ABP_1 be perpendicular to the plane ADP_2, then $R = \sqrt{P_1^2 + Q^2}$, and $Rx_1 = \sqrt{\{(P_1x_1)^2 + (P_2x_2)^2\}}$.

The following alternative method will in general be more convenient. Through C (Fig. 131) draw CF parallel to P_1 and equal to $M_1 = P_1x_1$. Draw CE parallel to P_2 and equal to $M_2 = P_2x_2$. Complete the parallelogram CEYF. Then $CY = M$ will be the resultant bending moment at C. Or, after drawing CF, draw FY parallel to P_2 and equal to P_2x_2, then CY, the closing side of the triangle CEY, is the resultant bending moment at C.

It is understood, of course, that the actual scale drawing of the parallelogram or triangle must be made on a plane perpendicular to the length of the beam, and not in oblique projection, as shown.

Any number of forces at right angles to the beam and in different planes may be dealt with in a similar manner. If there are more than two forces, a polygon will take the place of the parallelogram or triangle.

If any of the given forces are not perpendicular to the beam, resolve them parallel and perpendicular to the beam, and deal with the components perpendicular to the beam, as above, to find the resultant bending moment at any section.

To find the resultant shearing force at any section, consider the forces to one side of the section. Find the shearing forces at the section due to these forces separately. The resultant of these shearing forces is the resultant shearing force at the section.

Exercises VIIa.

Draw the bending moment and shearing force diagrams for the examples given. In this set of exercises the bending moments and shearing forces at a sufficient number of sections should be found by calculation.

1. Cantilever (Fig. 132). W = 2 tons. Linear scale, 1 inch to 1 foot. Force scale, 1 inch to 1 ton. Moment scale, 1 inch to 4 foot-tons.

2. Cantilever (Fig. 133). Uniform load of 2 tons per foot. Scales.—Linear, 1 inch to 1 foot. Forces, 1 inch to 4 tons. Moments, 1 inch to 8 foot-tons.

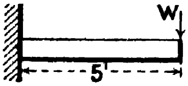


FIG. 132.

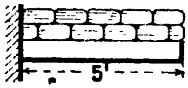


FIG. 133.

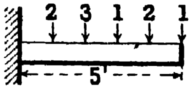


FIG. 134.

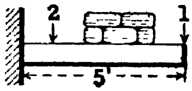


FIG. 135.

3. Cantilever (Fig. 134). Loads in tons at intervals of 1 foot. Scales.—Linear, 1 inch to 1 foot. Forces, 1 inch to 4 tons. Moments, 1 inch to 8 foot-tons.

4. Cantilever (Fig. 135). A load of 2 tons at 1 foot from the fixed end, a load of 1 ton at the free end, and a load of 4 tons uniformly distributed over the second and third feet of the length from the free end. Scales.—Linear, 1 inch to 1 foot. Forces, 1 inch to 4 tons. Moments, 1 inch to 5 foot-tons.

5. Beam resting on supports 12 feet apart. Load of 5 tons at the centre. Scales.—Linear, $\frac{1}{2}$ inch to 1 foot. Forces, 1 inch to 2 tons. Moments, 1 inch to 8 foot-tons.

6. Same as Exercise 5, except that the load is placed at 2 feet from the centre of the span.

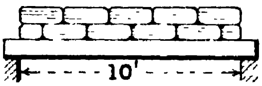


FIG. 136.

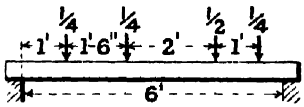


FIG. 137.

7. Beam (Fig. 136) resting on supports 10 feet apart. Uniform load of $\frac{1}{2}$ ton per foot. Scales.—Linear, $\frac{1}{2}$ inch to 1 foot. Forces, $\frac{1}{2}$ inch to 1 ton. Moments, $\frac{1}{4}$ inch to 1 foot-ton.

8. Beam resting on supports 6 feet apart. Loads in tons, at intervals, as shown in Fig. 137. Scales.—Linear, 1 inch to $1\frac{1}{2}$ feet. Forces, 2 inches to 1 ton. Moments, 2 inches to 1 foot-ton.

9. Cantilever, 10 feet long, carrying a central *downward* load of 8 tons, and an *upward* load of $2\frac{1}{2}$ tons at the free end Scales Linear, 1 inch to 2 feet. Forces, 1 inch to 4 tons. Moments, 1 inch to 8 foot-tons.

10. Beam (Fig. 138) resting on supports 12 feet apart. Load of 5000 lbs. uniformly distributed over the middle third of the span. Scales.—Linear, $\frac{1}{2}$ inch to 1 foot. Forces, 1 inch to 2000 lbs. Moments, 1 inch to 5000 ft.-lbs.

11. Same as Exercise 10, except that the load is to be moved forward until one end of it is at the centre of the span.

12. Girder, 40 feet long, resting on supports, as shown in Fig. 139. Uniform load of 1 ton per foot. $l_1 = 7$ feet 6 inches. Scales.—Linear, $\frac{1}{8}$ inch to 1 foot. Forces, 1 inch to 8 tons. Moments, 1 inch to 16 foot-tons.

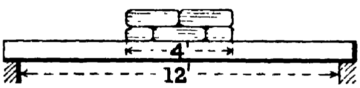


FIG. 138.

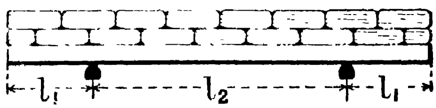


FIG. 139.

13. Same as Exercise 12, except that $l_1 = 10$ feet.

14. Same as Exercise 12, except that the supports are to be placed so as to make the maximum bending moment the least possible.

15. Same as Exercise 12, with a central load of 10 tons added.

16. Beam (Fig. 140) resting on supports 13 feet apart. The load increases at a uniform rate from nothing at one end.

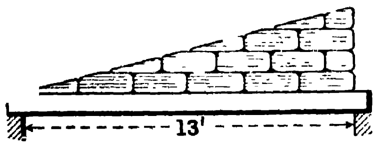


FIG. 140.

Total load, 10 tons. Scales.—Linear, $\frac{1}{2}$ inch to 1 foot. Forces, 1 inch to 3 tons. Moments, 1 inch to 10 foot tons.

17. Two cantilevers, 10 feet long, carrying a beam 20 feet long, as shown in Fig. 120, p. 91. The whole carrying a load of 40 tons uniformly distributed over the cantilevers and beam. Scales.—Linear, $\frac{1}{8}$ inch to 1 foot. Forces, 1 inch to 16 tons. Moments, 1 inch to 50 foot-tons.

18. Girder resting on supports 40 feet apart. Single travelling load of 2 tons. Find also the equivalent uniform dead load w in tons per foot. Scales.—Linear, 1 inch to 8 feet. Forces, 1 inch to 1 ton. Moments, 1 inch to 8 foot-tons.

19. Girder resting on supports 50 feet apart. A travelling uniform load of $\frac{1}{2}$ ton per foot. The length of the load being not less than 50 feet. Scales.—Linear, 1 inch to 10 feet. Forces, 1 inch to 5 tons. Moments, 1 inch to 50 foot-tons.

20. Girder resting on supports 50 feet apart. Two travelling loads of 5 tons each, and at a fixed distance of 10 feet apart. Find also the equivalent uniform dead load w in tons per foot. Scales.—Linear, 1 inch to 5 feet. Forces, 1 inch to 5 tons. Moments, 1 inch to 40 foot-tons.

21. Girder resting on supports 50 feet apart. Two travelling loads, one of 8 tons and the other of 4 tons, the fixed distance between the loads being 12 feet. Find also the equivalent uniform dead load w in tons per foot. Scales, the same as in Exercise 20.

22. Same as Exercise 21, but in addition to the travelling loads there is a uniformly distributed load of $\frac{1}{4}$ ton per foot run. Determine the portion of the girder in which the shearing force may change sign.

23. Girder resting on supports 60 feet apart. Three travelling loads W_1 10 tons, $W_2 = 15$ tons, and $W_3 = 5$ tons. W_2 is between W_1 and W_3, and is 10 feet from W_1 and 6 feet from W_3. Use the graphic tracing paper method. Find the equivalent uniform dead load w in tons per foot. Scales.—Linear, 1 inch to 10 feet. Forces, 1 inch to 10 tons. Moments, 1 inch to 100 foot-tons.

24. An axle AB rests in swivel bearings at A and B 12 feet apart. At a point 4 feet from A there is a vertical load of 600 lbs., and at a point 4 feet from B there is a horizontal load of 900 lbs. at right angles to the beam. Calculate the bending moments on the axle, in ft.-lbs., at distances of 2, 4, 6, 8, and 10 feet from A. Find also the shearing forces, in lbs., on the axle at sections 2, 6, and 10 feet from A.

109. Stresses Induced by Bending.—At (*a*) Fig. 141 is shown a portion of a straight beam before it is subjected to bending. At (*b*) is shown the same portion bent to a circular form. It is obvious that in bending this portion of beam, the plane of the paper being the plane of bending, the upper part is compressed while the lower part is stretched, and there will evidently be a surface which will separate the compressed and stretched parts; this surface is called the *neutral surface* of the beam. Let HK be the position of the neutral surface. Transverse sections AD and BC, which are perpendicular to the neutral surface, will be parallel to one another when the beam is straight, but when the beam is bent these sections will be inclined to one another and their planes will intersect at O, the axis of the cylindrical surfaces assumed by longitudinal sections of the straight beam perpendicular to the plane of bending. R, the radius of the curved neutral surface, is called the *radius of curvature* of the bent beam.

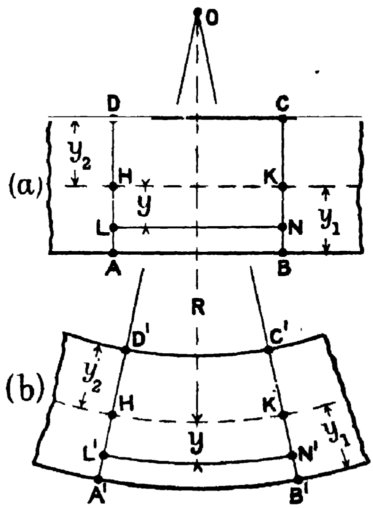


Fig. 141.

Consider an indefinitely thin layer of material LN parallel to the neutral surface, and at a distance y from it. When unstrained, LN = HK, but in the bent beam LN becomes L′N′. Now $\frac{\mathrm{L'N'}}{\mathrm{HK}} = \frac{\mathrm{R}+y}{\mathrm{R}}$, therefore $\mathrm{L'N'} = \frac{(\mathrm{R}+y)\mathrm{HK}}{\mathrm{R}}$, and the strain produced in LN is

$$\frac{\mathrm{L'N'}-\mathrm{LN}}{\mathrm{LN}} = \left\{\frac{(\mathrm{R}+y)\mathrm{HK}}{\mathrm{R}} - \mathrm{HK}\right\} \div \mathrm{HK} = \frac{y}{\mathrm{R}}.$$

But $\mathrm{E} = \frac{\text{stress}}{\text{strain}}$; hence if f is the stress produced in the layer LN in the process of bending the beam, $\frac{f}{y} = \frac{\mathrm{E}}{\mathrm{R}}$, that is to say, f is proportional to y. The distribution of the stress on a cross section will therefore evidently be as shown in Fig. 142, f_1 being the maximum tensile stress, and f_2 the maximum compressive stress.

The line in which the neutral surface cuts a transverse section of a beam is called the *neutral axis* of that section.

110. Moment of Resistance to Bending—Position of Neutral Axis.—The resultant of the external forces which produce pure bending is a couple, and this external couple is balanced by internal forces in the beam, and the resultant of these internal forces must therefore be a couple, because only a couple will balance a couple. The two forces which form the internal couple are the resultants of the tensile and compressive stresses, therefore these resultants must be equal and parallel.

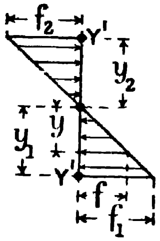


FIG. 142.

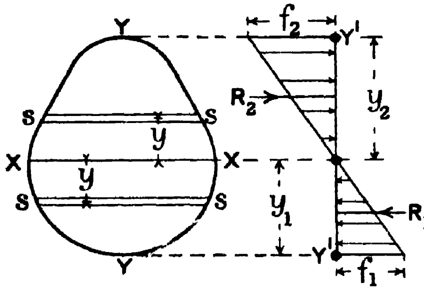


FIG. 143.

Let YY (Fig. 143) represent a face view and Y′Y′ an edge view of a transverse section of a beam, and let XX be the neutral axis. Consider an indefinitely narrow strip *ss* of the section parallel to XX and at a distance y from it. Let a denote the area of the strip *ss*. The stress f on the strip *ss* is such that $\frac{f}{y} = \frac{f_1}{y_1}$, therefore $f = \frac{f_1 y}{y_1}$. The resultant of the stress on *ss* is $fa = \frac{f_1 ay}{y_1}$. The resultant R_1 of the stress on the part of the section below XX must be $\Sigma fa = \Sigma \frac{f_1 ay}{y_1} = \frac{f_1}{y_1}\Sigma ay$. In like manner $\mathrm{R}_2 = \frac{f_2}{y_2}\Sigma ay$. Therefore if $\mathrm{R}_1 = \mathrm{R}_2$, $\frac{f_1}{y_1}\Sigma ay = \frac{f_2}{y_2}\Sigma ay$. But $\frac{f_1}{y_1} = \frac{f_2}{y_2}$, therefore Σay for the area below XX must be equal to Σay for the area above

XX. Hence *the neutral axis XX must pass through the centre of gravity of the section.*

The moment of resistance is found as follows. The moment of the resultant stress on *ss* is $\frac{f_1}{y_1}ay^2$ or $\frac{f_2}{y_2}ay^2$, and the total moment for the whole section is $\frac{f_1}{y_1}\Sigma ay^2$ or $\frac{f_2}{y_2}\Sigma ay^2$, which is equal to $\frac{f_1}{y_1}I$ or $\frac{f_2}{y_2}I$, where I is the moment of inertia of the whole section about the axis XX.

For equilibrium the bending moment must be equal to the moment of resistance, therefore $M = \frac{f_1}{y_1}I = \frac{f_2}{y_2}I$. Putting $Z_1 = \frac{I}{y_1}$ and $Z_2 = \frac{I}{y_2}$, $M = f_1Z_1 = f_2Z_2$. Z_1 and Z_2 are called the *moduli of the section.*

Since $\frac{E}{R} = \frac{f}{y} = \frac{f_1}{y_1} = \frac{f_2}{y_2}$, therefore $M = \frac{EI}{R}$.

111. Moments of Inertia, and Moduli of Various Sections.—The moments of inertia and the moduli of the more common sections required in connection with the moment of resistance to bending are tabulated on p. 106. The axis of moments is the neutral axis of the section, and passes through the centre of gravity of the section. y = distance of axis of moments from the top or bottom of the section. Where no value is given for y, it is equal to half the total depth of the section. Where the section is not symmetrical about the neutral axis, there are two values for the modulus, $Z_1 = I \div y_1$, and $Z_2 = I \div y_2$.

112. Equivalent Beam Sections.—Since the moment of resistance of an element of a beam section is equal to its area multiplied by the stress on it and by its distance from the neutral axis, and since the stress on the element is proportional to its distance from the neutral axis, it follows that if the element be moved parallel to the neutral axis into another position its moment of resistance will not be altered. Hence if a beam section be divided into indefinitely narrow strips parallel to the neutral axis, these strips may be moved parallel to the neutral axis so as to form another section, which will have the same moment of resistance as the original section. An example is shown in Fig. 144, where the original section, a hollow semicircle, shown in full lines, is converted into an equivalent solid section by collecting the area about a central axis.

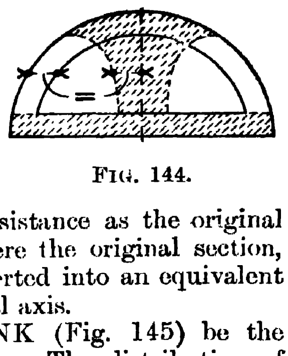
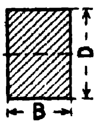


FIG. 144.

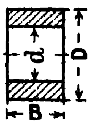
113. Section Modulus Figures.—Let MHNK (Fig. 145) be the cross section of a beam, and XX its neutral axis. The distribution of stress on the section due to the bending is shown at (*a*), f_1 being the stress at N, and f_2 the stress at M. Take an indefinitely narrow strip HK of the section parallel to XX. Draw the base line Y_1Y_1 parallel to XX and passing through N, the lowest point of the section. Draw H*h* and K*k* perpendicular to Y_1Y_1. Select a point O in XX. If the section is symmetrical about an axis perpendicular to XX, then O is preferably where this axis cuts XX. In Fig. 145, MON is an axis of symmetry perpendicular to XX. Join *h* and *k* to O by lines cutting HK at *m* and *n*.

The stress f at HK is such that f/f_1 = OL/ON. By similar triangles mn/hk = OL/ON, but hk = HK, therefore mn/HK = OL/ON = f/f_1, and

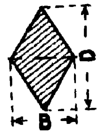
Moments of Inertia, etc., of Various Beam Sections (see Art. 111, p. 105).



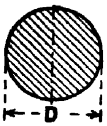
$$I = \frac{1}{12}BD^3. \qquad Z = \frac{1}{6}BD^2.$$



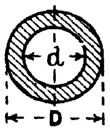
$$I = \frac{B}{12}(D^3 - d^3). \qquad Z = \frac{B(D^3 - d^3)}{6D}.$$



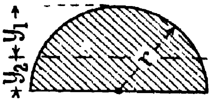
$$I = \frac{BD^3}{48}. \qquad Z = \frac{BD^2}{24}.$$



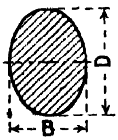
$$I = \frac{\pi}{64}D^4. \qquad Z = \frac{\pi}{32}D^3.$$



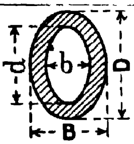
$$I = \frac{\pi}{64}(D^4 - d^4). \qquad Z = \frac{\pi}{32}\left(\frac{D^4 - d^4}{D}\right).^*$$



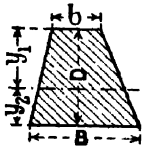
$$y_1 = 0{\cdot}5756r. \quad y_2 = 0{\cdot}4244r. \quad I = 0{\cdot}1098r^4. \quad Z_1 = 0{\cdot}1907r^3. \quad Z_2 = 0{\cdot}2586r^3.$$



$$I = \frac{\pi}{64}BD^3. \qquad Z = \frac{\pi}{32}BD^2.$$

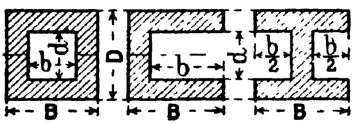


$$I = \frac{\pi}{64}(BD^3 - bd^3). \qquad Z = \frac{\pi}{32}\left(\frac{BD^3 - bd^3}{D}\right).$$

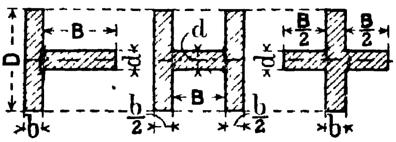


$$y_1 = \frac{D(2B + b)}{3(B + b)}. \qquad y_2 = \frac{D(B + 2b)}{3(B + b)}. \qquad I = \frac{(B^2 + 4Bb + b^2)D^3}{36(B + b)}.$$

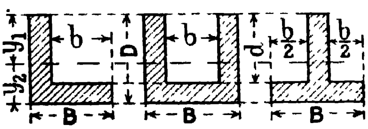
$$Z_1 = \frac{I}{y_1} = \frac{(B^2 + 4Bb + b^2)D^2}{12(2B + b)}. \qquad Z_2 = \frac{I}{y_2} = \frac{(B^2 + 4Bb + b^2)D^2}{12(B + 2b)}.$$



$$I = \frac{1}{12}(BD^3 - bd^3). \qquad Z = \frac{BD^3 - bd^3}{6D}.$$



$$I = \frac{1}{12}(bD^3 + Bd^3). \qquad Z = \frac{bD^3 + Bd^3}{6D}.$$

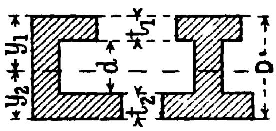


$$y_1 = \frac{BD^2 - bd^2}{2(BD - bd)}. \qquad y_2 = \frac{BD^2 - 2bdD + bd^2}{2(BD - bd)}.$$

$$I = \frac{(BD^2 - bd^2)^2 - 4BDbd(D - d)^2}{12(BD - bd)}.$$

$$Z_1 = \frac{I}{y_1} = \frac{(BD^2 - bd^2)^2 - 4BDbd(D - d)^2}{6(BD^2 - bd^2)}. \qquad Z_2 = \frac{I}{y_2} = \frac{(BD^2 - bd^2)^2 - 4BDbd(D - d)^2}{6(BD^2 - 2bdD + bd^2)}.$$

a_1 = area of top flange. a_2 = area of bottom flange. a = area of web.



$$y_1 = \frac{a_2(2D - t_2) + a_1t_1 + a(d + 2t_1)}{2(a_1 + a_2 + a)}.$$

$$y_2 = \frac{a_1(2D - t_1) + a_2t_2 + a(d + 2t_2)}{2(a_1 + a_2 + a)}.$$

$$I = \frac{a_1t_1^2 + a_2t_2^2 + ad^3}{12} + \frac{a_1a_2(D + d)^2 + a_1a(t_1 + d)^2 + a_2a(t_2 + d)^2}{4(a_1 + a_2 + a)}. \qquad Z_1 = \frac{I}{y_1}. \qquad Z_2 = \frac{I}{y_2}.$$

In actual practice it is often sufficiently accurate to take $Z_1 = a_1h$, and $Z_2 = a_2h$, where h is the total depth of the section.

* If d does not differ much from D, then $Z = \frac{\pi}{32}(D^3 - d^3)$ nearly.

$f_1 \cdot mn = f \cdot HK$. Hence the strip HK subjected to the stress f will have the same resistance as the strip mn subjected to the stress f_1.

If the above construction be repeated for a sufficient number of strips (the construction for a strip above XX is shown by dotted lines), and the points joined up, a figure shown to the right in Fig. 145, is obtained, which is called the *section modulus figure.* The section modulus figure has the property that the sum of the moments of the areas of all the strips parallel to XX about XX multiplied by f_1 will be the moment of resistance of the section. Hence if d_1 is the distance between the centres of gravity of the parts of the section modulus figure on opposite sides of XX measured perpendicular to XX, and if a_1 is the area of the figure, then $f_1 Z_1 = f_1 a_1 d_1$, and $Z_1 = a_1 d_1$.

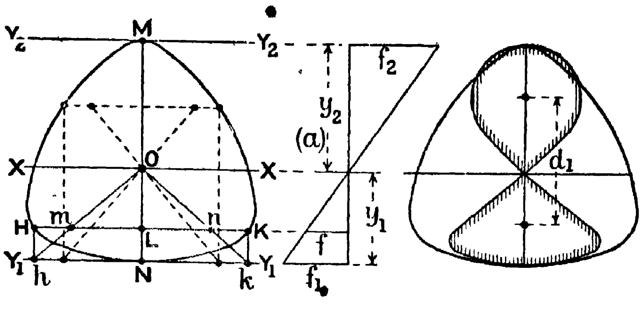


FIG. 145.

If instead of projecting the various strips on to the base line Y_1Y_1 they be projected on to Y_2Y_2, which is parallel to XX, and passes through the highest point of the section, the construction will give another section modulus figure whose area is a_2, and the distance corresponding to d_1 will be d_2, and then $Z_2 = a_2 d_2$. But since $Z_1 = I/y_1$, and $Z_2 = I/y_2$, it follows that $Z_2 = Z_1 y_1 / y_2$. Hence if Z_1 is found from a modulus figure, Z_2 can readily be deduced from it without drawing another figure.

The positions of the centres of gravity of the parts of the modulus figure on opposite sides of the neutral axis may be determined by one of the methods described in Chapter V.

A section modulus would only be determined in practice from a section modulus figure when the section was such that, not having definite, or sufficiently simple, mathematical properties, its moment of inertia could not be determined by the usual method. Even then it may be quicker to find the moment of inertia by one of the methods described in Chapter V.

114. Beams of Uniform Strength. — Considering resistance to bending, a beam is said to be of uniform strength when the maximum stress is the same at every cross section. If f is the maximum stress, Z the modulus of the section, and M the bending moment, then for every section $M = fZ$, and for uniform strength M/Z is constant.

In practice it is seldom possible to make M/Z constant for the whole length of a beam. For example, in a beam supported at the ends and loaded on the top, M vanishes at the ends where the shearing force is a maximum, and sufficient area of cross section must be provided to resist the shearing force.

115. Strength of Wheel Teeth.—A tooth of a wheel is a cantilever on which the load acts at different points of the length during the time of its contact with another tooth on another wheel. The direction of the

load on the tooth also varies during contact, and the investigation of the strength of the tooth is still further complicated by the variation in the form of the tooth, due to variations in the diameters of the pitch and rolling circles, and also to variations in the number of pairs of teeth in contact at one time.

An approximate general solution is obtained by assuming that the load, Q (in lbs.), is two-thirds of the load, P (in lbs.), at the pitch line due to the horse-power, H, transmitted when the velocity of the pitch line is V feet per minute, and that Q acts at the outer end of the tooth. $H = \frac{PV}{33000}$.

There are two cases to consider: (1) where Q is distributed over the whole width of the tooth; (2) where Q acts at one corner of the tooth. It will also be assumed that t, the thickness of the tooth at the root, is the same as at the pitch line.

Case I.—Load Q distributed over the whole width of the tooth (Fig. 146). This will obtain when the directions of the axes of the wheels are properly fixed and maintained, and the teeth are truly shaped.

The greatest bending moment is at the root, and is equal to Qh. The moment of resistance to bending is $\frac{1}{6}bt^2f$. Hence $Qh = \frac{1}{6}bt^2f$. t is generally about $0{\cdot}48\ p$, but allowing for wear t may be taken at $0{\cdot}4\ p$, where p is the pitch of the teeth. Taking $h = 0{\cdot}7p$, $b = np$, and $Q = \frac{2}{3}P$, then $P = \frac{2}{35}np^2f$. Taking f at 3500 for cast-iron, the following simple formula is obtained, $P = 200np^2$.

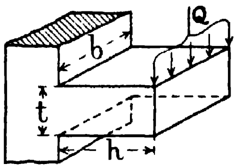


Fig. 146.

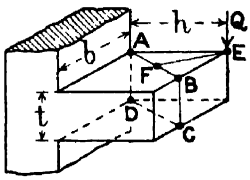


Fig. 147.

Case II.—Load Q acts at one corner of the tooth (Fig. 147). This may result from inaccurate mounting of the shafts in the first instance, or through unequal wear of the bearings, or from want of truth in the shape of the teeth.

The tooth may break at a section ABCD, which makes an angle θ with the side of the wheel. EF being perpendicular to AB, $EF = h \sin \theta$. $AB = \frac{h}{\cos \theta}$. Then $Qh \sin \theta = \frac{1}{6}\frac{h}{\cos \theta}t^2f$, therefore

$$Q = \frac{t^2f}{6 \sin \theta \cos \theta} = \frac{t^2f}{3 \sin 2\theta}.$$

This shows that Q will be least when $\sin 2\theta$ is greatest, that is, when $\theta = 45°$. Hence if the tooth breaks at an oblique section, that section will be inclined at 45° to the side of the wheel, and $Q = \frac{t^2f}{3}$. If $Q = \frac{2}{3}P$, $t = 0{\cdot}4p$, then $P = 0{\cdot}08p^2f$, and if $f = 3500$ for cast-iron, $P = 280p^2$.

If while Q acts at one corner the tooth breaks at the root, then $Q = \frac{bt^2f}{6h}$, and if the tendency to break at the root is the same as at the weakest oblique section, $\frac{bt^2f}{6h} = \frac{t^2f}{3}$ or $b = 2h$. This shows that if b is less

than $2h$, the tooth will break at the root, and then $Q=\frac{bt^2f}{6h}$, but if b is greater than $2h$, the tooth will break at an oblique section, and then $Q=\frac{t^2f}{3}$.

116. Bending beyond the Elastic Limit—Modulus of Rupture.—The expression for the moment of resistance of a beam to bending, viz. $f_1\frac{I}{y_1}$ or $f_2\frac{I}{y_2}$, was deduced on the assumptions that the stress varied uniformly from zero at the neutral axis, and that the material was not strained beyond the elastic limit. In the case of a ductile material, such as wrought-iron or mild steel, permanent set will first take place either at the top or bottom of the section, and as the strain increases the distribution of stress will change, tending to become more uniform.

The change in the distribution of the stress as the beam is strained beyond the elastic limit is shown approximately in Fig. 148. At (*a*) is shown the distribution of stress before permanent set takes place. At (*b*) the material has taken a permanent set in tension and compression, the portions AC and BD being strained beyond the elastic limit, but the portions OC and OD still obey Hooke's law. At (*c*) the whole of the material has been strained beyond the elastic limit. If the material is very plastic beyond the elastic limit the distribution of stress approximates to that shown at (*d*), the tensile and compressive stresses being both uniformly distributed. If permanent set takes place at A before it takes place at B, the distribution of stress will be as shown at (*e*), the neutral axis moving nearer to B.

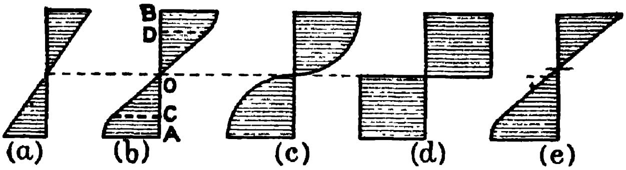


FIG. 148.

The value of f in the formula $M=fZ$, when M is the bending moment which fractures the beam, is called the *modulus of rupture* of the beam. The modulus of rupture is generally greater than the value of f determined by experiments on bars in direct tension or compression, and the difference depends on the form of the cross section of the beam, being small for a flanged section and greatest for a circular section.

117. Reinforced Concrete Beams.—Good concrete, having the composition, cement 1, sand 2, and broken stone 4, will carry safe working stresses of 60 and 500 lbs. per square inch in tension and compression respectively. Beams made of this material offer small resistance to bending on account of the low value of the allowable tensile stress.

The resistance of a concrete beam to bending may, however, be greatly increased by embedding steel bars in the concrete near the face of maximum tension. The cross section of such a beam is shown to the right in Fig. 149, the black squares representing the reinforcement.

In calculating the moment of resistance of a reinforced concrete beam it is usual to assume that the steel reinforcement carries the whole of the

tension, and that the concrete carries the compression. It is also assumed that as the beam bends the strains in the various layers parallel to the neutral surface are proportional to their distances from that surface, as in the case of homogeneous beams discussed in Art. 109, page 103.

From the latter assumption it is obvious that the neutral axis of a cross section will not pass through the centre of gravity of that section, since the moduli of elasticity of the steel and concrete are not equal.

Referring to Fig. 149, y_1 is the depth of the neutral surface below the top or compression surface of the beam, h is the depth of the axes of the steel bars from the top surface, and b is the breadth of the beam. Let a = total area of cross section of the steel bars, f_1 = maximum compressive stress in the concrete, and f_2 = stress in the steel. The depth of the section of the steel bars being small compared with the depth of the beam, it is assumed that f_2 is uniform over the section of these bars.

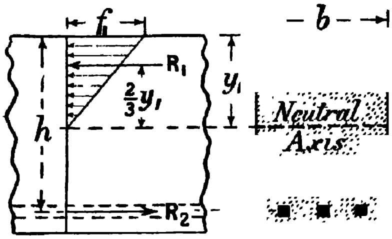


FIG. 149.

The resultant of the compressive stress in the concrete is $R_1 = \frac{1}{2} f_1 b y_1$, and this resultant acts at a distance $\frac{2}{3} y_1$ from the neutral surface.

The resultant of the tensile stress in the steel is $R_2 = a f_2$, and this resultant acts at a distance $h - y_1$ from the neutral surface.

The assumption of proportionality of strain to distance from the neutral surface already mentioned leads to the equation $\frac{f_1}{E_1 y_1} = \frac{f_2}{E_2(h - y_1)}$, where E_1 and E_2 are the Young's moduli for the concrete and steel respectively.

Since R_1 and R_2 are the forces which form the couple whose moment is the moment of resistance of the section, it follows that $R_1 = R_2$, therefore $\frac{1}{2} f_1 b y_1 = a f_2$.

Dividing the equation $\frac{f_1}{E_1 y_1} = \frac{f_2}{E_2(h - y_1)}$ by the equation $\frac{1}{2} f_1 b y_1 = a f_2$,

the result is $\frac{2}{E_1 b y_1^2} = \frac{1}{E_2 a (h - y_1)}$, which may be written

$$y_1^2 + \frac{2a}{b} \cdot \frac{E_2}{E_1} y_1 = \frac{2ah}{b} \cdot \frac{E_2}{E_1},$$

which is a quadratic equation for determining y_1.

Putting $\frac{E_2}{E_1} = n$, the solution of the equation gives

$$y_1 = \frac{\sqrt{(a^2 n^2 + 2abhn)} - an}{b}.$$

Having found y_1, the moment of resistance of the section is

$$R_1(h - y_1 + \tfrac{2}{3} y_1) = \tfrac{1}{2} f_1 b y_1 (h - \tfrac{1}{3} y_1),$$

or $$R_2(h - y_1 + \tfrac{2}{3} y_1) = a f_2 (h - \tfrac{1}{3} y_1).$$

In designing a reinforced beam, if either f_1 or f_2 is assumed, the other is found from the equation $\frac{1}{2}f_1by_1 = af_2$.

The value of the ratio $\frac{E_2}{E_1} = n$ is generally taken at 15.

Exercises VIIb.

1. Taking the moment of resistance to bending of section A (Fig. 150) as unity, find the numbers which will represent the moments of resistance of the sections B, C, and D. (In section C the metal is 1 inch thick.)

2. Denoting the moment of resistance to bending of section A, per square inch of section, by 1, determine the numbers which will express the moments of resistance of the sections B, C, and D per square inch of section.

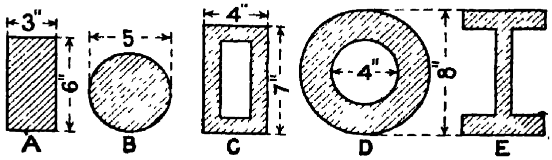


FIG. 150.

3. The section E (Fig. 150) has a total depth of 8 inches. The flanges are 5 inches wide and $1\frac{1}{4}$ inches thick. The web is 1 inch thick. What is the moment of resistance of this section to bending when the maximum stress is 5 tons per square inch? What will the answer be if the section is placed with the web horizontal?

4. A solid circular section is 6 inches diameter. A hollow circular section is 8 inches diameter outside. Find the internal diameter of the hollow section so that it shall have the same area as the solid section; then, denoting the moment of resistance of the solid section by 1 determine the number which will represent the moment of resistance of the hollow section.

5. A steel joist has a total depth of 18 inches. The flanges are 7 inches wide and 0·94 inch thick. The web is 0·55 inch thick. Determine the section modulus, Z, in inch units (*a*) by the correct formula, (*b*) by the formula $Z = ah$, where *a* is the area of one flange, and *h* is the total depth.

6. Construct, half full size, the modulus figures for the following sections. (*a*) Circle 6 inches diameter. (*b*) Hollow circle, external diameter 6 inches, internal diameter 3 inches. (*c*) Flanged section 6 inches deep, flanges $3\frac{1}{2}$ inches wide and $1\frac{1}{2}$ inches thick, web $1\frac{1}{4}$ inches thick. (*d*) Isosceles triangle, base 5 inches, height 6 inches. (*e*) Flanged section 6 inches deep, top flange $2\frac{1}{2}$ inches wide and $1\frac{1}{4}$ inches thick, bottom flange 4 inches wide and $1\frac{1}{2}$ inches thick, web $1\frac{1}{4}$ inches thick. From these figures determine in cases (*a*), (*b*), and (*c*) the values of Z, and in cases (*d*) and (*e*) the values of Z_1 and Z_2. Compare the results with those obtained by calculation from the correct formulæ.

7. AEB and CFD (Fig. 151) are semicircles whose diameters AB and CD are parallel and $2\frac{1}{4}$ inches apart. $AB = 2\frac{1}{4}$ inches, $CD = 2\frac{1}{4}$ inches. AD and BC are straight lines which are perpendicular to one another. The whole figure AEBCFD is the modulus figure of a beam section. Construct the beam section.

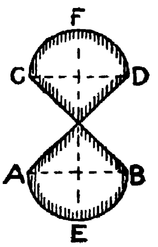


FIG. 151.

8. A cantilever 50 inches long carries a load of 4000 lbs. at its free end. The maximum stress due to bending is to be 3000 lbs. per square inch at every cross section. The cross section is a rectangle, breadth $= x$, depth $= y$. Determine the cross section, $x \times y$ at 10, 20, 30, 40, and 50 inches from the free end, and draw a plan and side elevation of the cantilever (scale, 1 inch to 1 foot) in each of the following cases:—

(*a*) $y = 6$ inches, and the lever to be symmetrical about a vertical longitudinal section.

(*b*) $x = 3$ inches, and the top surface of the lever to be horizontal.

(*c*) $y = 3x$, the top surface to be horizontal, and the lever to be symmetrical about a vertical longitudinal section.

9. Same as Exercise 8, except that the cross section is a circle of diameter *y*.

and the lever is symmetrical about vertical and horizontal longitudinal sections.

10. Same as Exercise 8, except that the load is 6000 lbs., and is uniformly distributed.

11. Same as Exercise 9, except that the load is 6000 lbs., and is uniformly distributed.

12. An overhung steel crank-pin journal has a diameter d and length l. The total load on the journal is 62,500 lbs. uniformly distributed. The pressure on the journal is to be 600 lbs. per square inch of projected area (projected area $= dl$). The maximum bending stress is to be 10,000 lbs. per square inch. Find d and l.

13. Same as Exercise 12, except that there is a hole through the pin having a diameter equal to $\frac{1}{3}d$, the axis of the hole coinciding with the axis of the pin.

14. A cast-iron flanged beam resting on supports 12 feet apart carries a central load of 12 tons, and a load of 5 tons uniformly distributed over the whole length. The total depth of the beam is 13 inches. The area of the cross section of the bottom flange is to be four times the area of that of the top flange, and the stress in the bottom flange is to be 2 tons per square inch. Find the area of the top flange and the stress in it, assuming that the modulus of the cross section is equal to the area of one flange multiplied by the total depth of the beam.

15. Professor Goodman in his "Mechanics Applied to Engineering" gives the proportions for cast-iron flanged beams shown in Fig. 152. Show that, for this form of section, neglecting the fillets between the flanges and the web, the moment of resistance to bending is $0{\cdot}077d^3f$, where d is the total depth, and f the maximum tensile stress in the larger flange.

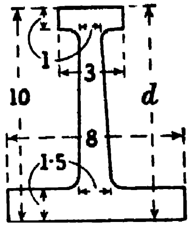


FIG. 152.

16. Determine the horse-power which may be safely transmitted by a spur-pinion 12 inches diameter, having 20 teeth, when running at 200 revolutions per minute. Breadth of teeth $2\frac{1}{2}$ times the pitch.*

17. Find the pitch and number of teeth for a spur-wheel 4 feet in diameter, which when running at 90 revolutions per minute transmits 150 horse-power. Breadth of teeth 3 times the pitch.*

18. At what speed, in revolutions per minute, must a spur-wheel 3 feet in diameter run when transmitting 80 horse-power. Number of teeth 10, breadth of teeth 7 inches.*

19. A horizontal steel shaft 5 inches in diameter projects 36 inches beyond a supporting bearing. At the free end there is a vertical load of P lbs., and at a point 9 inches from the free end there is an equal load acting in a horizontal direction at right angles to the shaft. If the maximum tensile stress in the shaft is 12,000 lbs. per square inch, find the force P.

20. A ferro-concrete beam, rectangular in section, is 12 inches wide and 24 inches deep. The reinforcement consists of four steel bars, each $\frac{3}{4}$ inch in diameter, their axes being at a depth of 22 inches below the top or compression face of the beam. Taking the modulus of elasticity of the steel as 10 times that of the concrete, find the depth of the neutral axis of the section from the top. If the beam rests on supports 18 feet apart, what load, in tons, uniformly distributed, will this beam carry when the maximum compressive stress produced in the concrete is 600 lbs. per square inch, and what will then be the tensile stress in the steel in lbs. per square inch?

21. A concrete beam of rectangular section, 11 inches wide, is to be reinforced by steel bars whose axes are to be 20 inches from the compression face of the beam. If the modulus of elasticity of the steel is 15 times that of the concrete, find the total area of the steel bars, so that when the tensile stress in the steel is 11,000 lbs. per square inch, the maximum compressive stress in the concrete is 500 lbs. per square inch? Find also the distance of the neutral surface of the beam from the compression face.

* Use the formula $P = 200np^2$, where P is the driving force at the pitch line in lbs., p the pitch of the teeth in inches, and n the breadth of the teeth divided by the pitch.

CHAPTER VIII

DEFLECTION OF BEAMS

118. Bending to Circular Arc.—It was shown in Arts. 109 and 110, pp. 103–105, that $\frac{f_1}{y_1}=\frac{E}{R}=\frac{M}{I}$ or $R=\frac{Ey_1}{f_1}=\frac{EI}{M}$. These equations show that a beam will bend to a circular form when y_1/f_1 is constant, or when I/M is constant throughout the length of the beam. For a beam of uniform cross section y_1 and I are constant, and therefore f_1 and M must also be constant for circular bending. If M is variable, then, for circular bending, I must be proportional to M.

The equations $R=\frac{Ey_1}{f_1}=\frac{EI}{M}$ may be used for non-circular bending if an indefinitely short length of the beam be considered; R will then be the radius of curvature at a point in the length, M will be the bending moment at that point, and f_1, y_1, and I will refer to the section at the same point.

119. Deflection due to Circular Bending.—Let a beam (Fig. 153) resting on supports A and B, whose distance apart is l, be bent to a circular arc ACB. The point O is the centre of curvature, D is the middle point of AB, and CD is the maximum deflection u_1. In the triangle OAD, $OA^2 = AD^2 + OD^2$. But $OA = R$, $AD = \frac{1}{2}l$, and $OD = R - u_1$. Therefore $R^2=\frac{1}{4}l^2+R^2-2u_1R+u_1^2$, that is, $2u_1R=\frac{1}{4}l^2+u_1^2$. But since u_1 is a very small quantity compared with R and l, the term u_1^2 may be neglected. Hence, $2u_1R=\frac{1}{4}l^2$, and $u_1=\frac{l^2}{8R}$. But $R=\frac{EI}{M}$, therefore $u_1=\frac{Ml^2}{8EI}$.

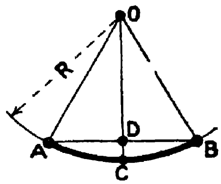


FIG. 153.

For a cantilever of length l it is easy to show that $u_1=\frac{Ml^2}{2EI}$.

120. Cantilever Loaded at Free End.—The cantilever (Fig. 154) is supposed to be of uniform cross section throughout. Consider an indefinitely small portion HK of the length at a distance x from the free end A. Let θ be the angle between the radii drawn from H and K to the centre of curvature of HK, and let R be the radius of curvature of HK. Let HC and KD be tangents to HK at H and K, meeting the vertical through A at C and D. Then CD is the amount of deflection of

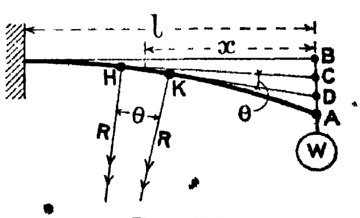


FIG. 154.

H

the cantilever due to the curvature of the part HK. Let $HK = dx$, and $CD = du$. Then, $\theta = \frac{dx}{R} = \frac{du}{x}$ therefore $du = \frac{xdx}{R} = \frac{Mxdx}{EI}$. But $M = Wx$, therefore $du = \frac{Wx^2dx}{EI}$. The total deflection

$$AB = u_1 = \int_0^l \frac{Wx^2dx}{EI} = \frac{W}{EI}\int_0^l x^2dx = \frac{Wl^3}{3EI},$$

where l is the length of the cantilever.

121. Cantilever Loaded Uniformly.—Let the load be w per unit of length. Using the same notation, and proceeding in the same way as in the preceding Article, $du = \frac{Mxdx}{EI}$, but $M = \frac{1}{2}wx^2$, therefore $du = \frac{wx^3dx}{2EI}$, and $u_1 = \frac{w}{2EI}\int_0^l x^3dx = \frac{wl^4}{8EI} = \frac{Wl^3}{8EI}$, where $W = wl =$ total load. The cantilever is supposed to be of uniform cross section.

122. Beam Supported at the Ends and Loaded at the Centre.—Referring to Fig. 155, and proceeding as in Art. 120, $du = CD = \frac{Mxdx}{EI}$, but $M = \frac{1}{2}Wx$, therefore $du = \frac{Wx^2dx}{2EI}$, $u_1 = AB = \frac{W}{2EI}\int_0^l x^2dx = \frac{Wl^3}{6EI} = \frac{WL^3}{48EI}$.

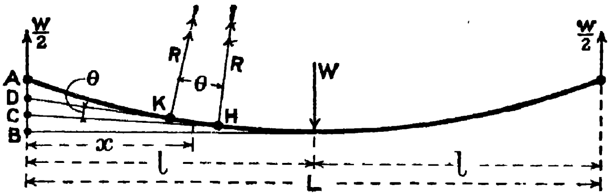


FIG. 155.

123. Beam Supported at the Ends and Loaded Uniformly.—Referring to Fig. 155, but remembering that the load W is uniformly distributed over the length, and that its intensity is w per unit of length.

$du = CD = \frac{Mxdx}{EI}$, but $M = \frac{1}{2}wLx - \frac{1}{2}wx^2$, therefore $du = \frac{wLx^2dx}{2EI} - \frac{wx^3dx}{2EI}$.

Hence $$u_1 = \frac{wL}{2EI}\int_0^l x^2dx - \frac{w}{2EI}\int_0^l x^3dx = \frac{wL}{2EI}\cdot\frac{l^3}{3} - \frac{w}{2EI}\cdot\frac{l^4}{4}$$

$$= \frac{wL^4}{48EI} - \frac{wL^4}{128EI} = \frac{wL^4}{16EI}\left(\frac{1}{3} - \frac{1}{8}\right) = \frac{5wL^4}{384EI} = \frac{5WL^3}{384EI}.$$

124. Slope of Bent Beam at any Point.—Referring to Figs. 154 and 155, θ is the change in the slope of the beam between the points

H and K, and $\theta = \frac{dx}{R} = \frac{M dx}{EI}$. Hence the change in the slope between the points $x = l$ and $x = x_1$ is equal to $\int_{x_1}^{l} \frac{M dx}{EI} = \theta_1$, and if the beam is horizontal where $x = l$, then the slope at the point where $x = x_1$ is θ_1.

125. Stiffness of a Beam.—The ratio of the maximum deflection of a beam to its span is called the *stiffness* of the beam. The stiffness may be denoted by $\frac{1}{n}$, where n varies from 1000 to 2000 for steel girders of large span, and from 500 to 700 for short spans. For timber beams, n should not be less than 360.

126. General Method of Determining Deflection from Bending Moment Diagram.—The analytical method of finding the deflection of a beam used in the preceding Articles is simple in simple cases, but in many cases in practice it becomes difficult and complicated. The method now to be discussed will be found to be comparatively simple in cases where the analytical method would be troublesome.

In what follows, the beam or cantilever is assumed to be of uniform cross section.

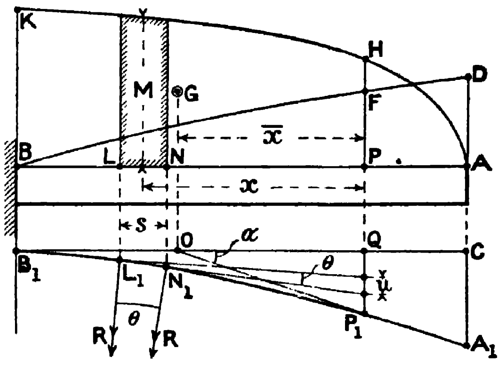


FIG. 156.

Consider first the case of a cantilever AB (Fig. 156) under any system of loads. Let AHKB be the bending moment diagram, and let A_1B_1 be the curve in which the cantilever bends. Take two points L and N on the cantilever near to one another, their distance apart being s. If the distance s be small enough, the bending moment M may be considered as uniform over the length LN. Let R be the radius of curvature of L_1N_1, then $\frac{1}{R} = \frac{M}{EI}$, and θ the angle between the radii to the centre of curvature from L_1 and N_1 will be the change in the slope of the beam in passing from L_1 to N_1. But $\theta = \frac{s}{R}$, therefore $\theta = \frac{Ms}{EI}$, which shows that the change in the slope of the cantilever in passing from L_1 to N_1 is equal to the area of the vertical strip of the bending moment diagram over LN divided by EI. Hence if the bending moment diagram over the portion BP be divided into a large number of vertical strips, it follows that the total change in the slope of the cantilever between B and P will be equal to the sum of the areas of these strips divided by EI, that is, equal to the area of the part of the bending moment diagram lying between the verticals through B and P divided by EI. Hence if a tangent P_1O be drawn to the curve $A_1P_1B_1$, at P_1 it will be inclined to the horizontal B_1C at an angle α, whose tangent or circular measure, the angle being very small, is equal to the area of the figure PHKB

divided by EI, and therefore $a = \frac{A}{EI}$, where A is the area of the figure PHKB.

In measuring the area of the bending moment diagram for the purpose of finding the change of slope, the unit of area is a rectangle, whose base is the unit of length, and whose height is the unit of bending moment.

If on AB as base a curve BFD be constructed such that the ordinate PF at any point P in AB is equal to the area of the figure PHKB divided by EI, then the ordinate of this curve at any point in AB will represent the slope of the bent cantilever at that point, and this curve may be called the *curve of slope*.

Now let the mean distance of LN from P be denoted by x, and let u be the deflection at P, due to the curvature of the part LN, then $u = x\theta = \frac{xa'}{EI}$, where a' is the area of the strip of bending moment diagram over LN. The total deflection P_1Q at P will be the sum of all such quantities as $\frac{xa'}{EI}$ between B and P, therefore $P_1Q = \frac{A\bar{x}}{EI}$, where $\bar{x}$ is the distance of G, the centre of gravity of the figure PHKB from PH. This simple rule may therefore be used for constructing the curve $A_1P_1B_1$, the work being done graphically, or in part graphically and in part by calculation.

Again, $P_1Q = OQ \tan a$, but since a is a small angle, $\tan a$ may be taken equal to a, therefore $P_1Q = OQ \cdot a$. But $a = \frac{A}{EI}$ and $P_1Q = \frac{A\bar{x}}{EI}$, hence $OQ\frac{A}{EI} = \frac{A\bar{x}}{EI}$, and therefore $OQ = \bar{x}$. This shows that the tangent to the curve $A_1P_1B_1$ at P_1 meets BC at a point vertically under G, the centre of gravity of the figure PHKB.

The deflection A_1C at the free end of the cantilever is obviously equal to $\frac{A_1\bar{x}_1}{EI}$, where A_1 is the area of the whole diagram AHKB, and $\bar{x}_1$ is the horizontal distance of the centre of gravity of the figure AHKB from A. Also, the tangent to the curve $A_1P_1B_1$ at A_1 will meet B_1C at a point vertically under the centre of gravity of the figure AHKB.

Suppose that the area of the figure PHKB, on a scale drawing, is A' square inches. Let the scale for the base PB be 1 inch to m inches, and let the scale for the ordinates or bending moments be 1 inch to n inch-pounds, then $A = A'mn$. $\bar{x}$ must be measured with the scale 1 inch to n inches. Then if E is in lbs. per square inch and I is in inch-units, the deflection $P_1Q = \frac{A\bar{x}}{EI}$ will be in inches.

If the bending moment diagram AHKB (Fig. 156) be considered as a diagram, showing the intensity of a load distributed over the cantilever, and if the cantilever be fixed at the end A instead of at the end B, as shown in Fig. 157, then at any section P of the cantilever (Fig. 157) the shearing force is proportional to the area of the figure PHKB, and therefore the ordinate PF of the curve BFD in Fig. 156,

which measures the slope of the bent cantilever at P, will in Fig. 157 represent the shearing force on the cantilever at P, that is, the curve of slope in Fig. 156 is the shearing force curve in Fig. 157.

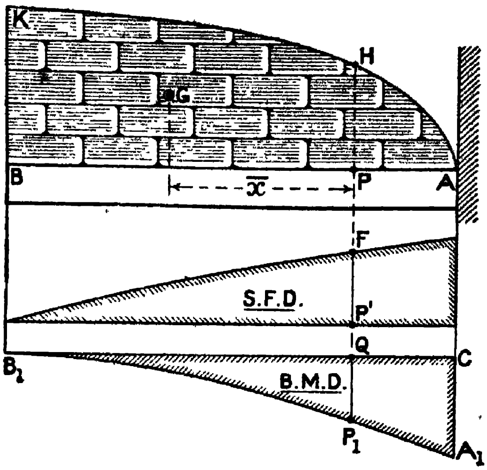


FIG. 157.

Again, at any section P of the cantilever (Fig. 157) the bending moment is proportional to the area of the figure PHKB multiplied by $\bar{x}$, the horizontal distance of G, the centre of gravity of PHKB from P, and therefore the ordinate P_1Q of the curve $A_1P_1B_1$ in Fig. 156, which measures the deflection of the bent cantilever at P, will in Fig. 157 represent the bending moment on the cantilever at P, that is, the curve of deflection in Fig. 156 is the bending moment curve in Fig. 157.

Hence having constructed the bending moment diagram for any system of loads, the curve of slope and the deflection curve may be constructed by the rules for constructing the shearing force and bending moment diagrams, the original bending moment diagram being considered as a load diagram.

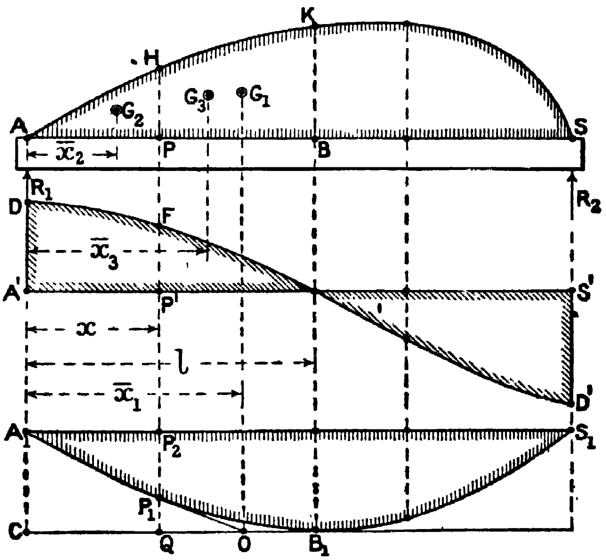


FIG. 158.

Consider next the case of a beam ABS (Fig. 158) supported at the ends, and under any given system of loads. Let AHKS be the bending moment diagram for the given system of loads, and let $A_1P_1B_1S_1$ be the curve in which the beam bends, B_1 being the lowest point in that curve.

Let a_1, a_2, and a_3 be the areas of the figures PHKB, AHP, and AHKB respectively. G_1, G_2, and G_3 are the centres of gravity of these figures respectively, and the horizontal distances of the points G_1, G_2, and G_3 from A are $\bar{x}_1$, $\bar{x}_2$, and $\bar{x}_3$ respectively. $AP = x$, and $AB = l$.

The portion AB of the beam may be looked upon as a cantilever fixed at B, like that in Fig. 156, the bending moment diagram being AHKB, and the deflection P_1Q measured from the horizontal B_1C will be equal to $ka_1(\bar{x}_1 - x)$, where k is a constant. The total deflection A_1C is equal to $ka_3\bar{x}_3$. Hence $P_1P_2 = A_1C - P_1Q = ka_3\bar{x}_3 - ka_1(\bar{x}_1 - x)$, and

$$\frac{P_1P_2}{A_1C} = \frac{a_3\bar{x}_3 - a_1(\bar{x}_1 - x)}{a_3\bar{x}_3} = \frac{a_1\bar{x}_1 + a_2\bar{x}_2 - a_1(\bar{x}_1 - x)}{a_3\bar{x}_3} = \frac{a_2\bar{x}_2 + a_1x}{a_3\bar{x}_3}.$$

Consider the figure AHKS to be a load diagram. Let A′DD′S′ be the shearing force diagram, and $A_1P_1B_1S_1$ the bending moment diagram corresponding to the load diagram AHKS. Then the point where the shearing force is zero must be in a vertical line through B_1, the lowest point in the bending moment diagram, and the reaction R_1 must equal the load represented by the area AHKB, therefore $R_1 = qa_3$, where q is a constant. Then the bending moment at P is equal to

$$R_1x - qa_2(x - \bar{x}_2) = qa_3x - qa_2(x - \bar{x}_2) = P_1P_2.$$

Also, the bending moment at B is equal to

$$R_1l - qa_3(l - \bar{x}_3) = qa_3l - qa_3(l - \bar{x}_3) = qa_3\bar{x}_3 = A_1C,$$

and $$\frac{P_1P_2}{A_1C} = \frac{a_3x - a_2(x - \bar{x}_2)}{a_3\bar{x}_3} = \frac{(a_1 + a_2)x - a_2(x - \bar{x}_2)}{a_3\bar{x}_3} = \frac{a_1x + a_2\bar{x}_2}{a_3\bar{x}_3}.$$

Hence whether the curve $A_1P_1B_1S_1$ be considered as a deflection curve or a bending moment curve, the ratio of P_1P_2 to A_1C is the same, and therefore the bending moment curve will represent the deflection at every point.

127. Beam of Uniform Section Supported at the Ends and Loaded at any Intermediate Point.—AB (Fig. 159) is a beam resting on supports whose distance apart is L. This beam carries a load W at a point C at distances a and b from A and B respectively. The bending moment at C is $\frac{Wab}{L}$, and making the ordinate CD equal to this bending moment, and joining A and B to D, the figure ADB is the bending moment diagram for the beam carrying the load W at C.

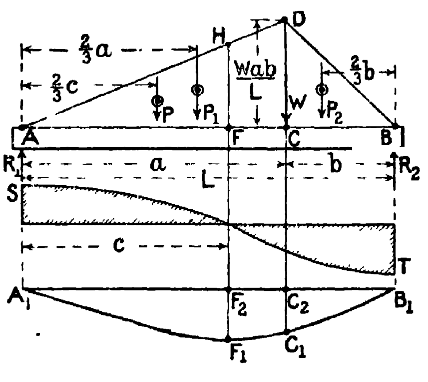


FIG. 159.

Now consider ADB to be a load diagram. The resultant P_1 of the load represented by the triangle ADC is equal to $\frac{Wa^2b}{2L}$, and P_1 acts in a vertical line through the centre of gravity of the triangle ADC. The resultant P_2 of the load represented by the triangle BDC is equal to $\frac{Wab^2}{2L}$, and P_2 acts in a vertical line through the centre of gravity of the

triangle BDC. Let R_1 and R_2 be the reactions at the supports due to the load represented by ABD, then

$$R_1 = \frac{Wab(a+2b)}{6L}, \text{ and } R_2 = \frac{Wab(2a+b)}{6L}.$$

If $A_1F_1B_1$ is the bending moment diagram corresponding to the load diagram ADB, then the ordinate of the curve $A_1F_1B_1$ at any point will represent the deflection at that point, and the maximum deflection will be at the section of the beam where the shearing force due to the load ADB is zero. The complete shearing force diagram ST for the load ADB is shown, but it is not necessary to draw this to find the deflection of the beam, but it is necessary to find the point F, where the shearing force is zero. Let $AF = c$, then the resultant P of the load represented by the triangle AHF is equal to $\frac{Wa^2b}{2L} \cdot \frac{c^2}{a^2} = \frac{Wbc^2}{2L}$, and P acts in a vertical line through the centre of gravity of the triangle AHF.

The shearing force at $F = R_1 - P$, and if this is zero $R_1 = P$, hence

$$\frac{Wbc^2}{2L} = \frac{Wab(a+2b)}{6L}, \text{ and therefore } c = \left\{\frac{a(a+2b)}{3}\right\}^{\frac{1}{2}}.$$

The bending moment at $F = R_1c - P\frac{c}{3} = \frac{2}{3}Pc = \frac{Wbc^3}{3L} = \frac{Wb}{3L}\left\{\frac{a(a+2b)}{3}\right\}^{\frac{3}{2}}$,

and the deflection at $F = \frac{Wb}{3LEI}\left\{\frac{a(a+2b)}{3}\right\}^{\frac{3}{2}}$. If $a = nL$, then $b = (1-n)L$,

and the deflection at F is given by the expression $\frac{WL^3}{3EI}(1-n)\left(\frac{2n-n^2}{3}\right)^{\frac{3}{2}}$.

The bending moment at $C = R_1a - P_1\frac{a}{3} = \frac{Wa^2b(a+2b)}{6L} - \frac{Wa^3b}{6L} = \frac{Wa^2b^2}{3L}$,

and the deflection at $C = \frac{Wa^2b^2}{3LEI} = \frac{WL^3}{3EI}n^2(1-n)^2$.

128. Beam of Uniform Section Fixed at the Ends and Loaded at the Middle.—The beam AB (Fig. 160) is held at the ends in such a way that the tangents to the bent beam at A and B are horizontal. The load W at the centre of the beam will obviously bend the middle part of the beam so that it sags, that is, it becomes concave on the top, and the tangent to the bent beam at C will be horizontal. Hence the curve $A_1C_1B_1$ into which the beam bends must have points of inflexion E_1 and F_1 between the centre of the beam and its ends, and the positions of these points have to be determined.

If the beam AB were simply supported at the ends it would be concave on its upper surface for the whole of its length, and the bending moment diagram would be the triangle acb, the altitude of which would be equal to the bending moment at the centre, namely, $\frac{1}{4}$WL. Also the bending moment would be everywhere positive. In order that the beam may be concave on its under surface at A and B there must be negative

bending moments at these points, and these are supplied by the method of fixing. In Fig. 160 the beam is shown with flanges, which are supposed to be bolted to the walls, and the forces PP shown produce the necessary negative bending moments just referred to.* Also these forces PP produce a uniform bending action over the whole of the beam. Let aa' be the bending moment at A. Draw $a'b'$ parallel to ab, cutting ac and bc at e and f respectively, then the shaded figure will be the actual bending moment diagram for the beam AB, with fixed ends and loaded at the centre, $a'b'$ being the base of the diagram. This diagram shows that the portion EF of the beam is subjected to positive bending, and that the parts AE and BF are subjected to negative bending; also, that there is no bending moment at either E or F. Hence if the beam be cut at E and F, and the parts be again connected by pin joints, the axes of the pins being perpendicular to the plane of bending, the jointed beam will behave exactly as the solid beam. Hence the original beam is equivalent to two cantilevers AE and BF loaded at E and F, and a beam EF supported at E and F, and loaded at the centre, as shown in the lower part of Fig. 160.

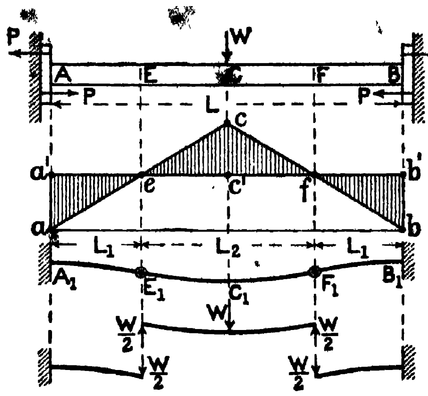


FIG. 160.

The slope of the cantilever AE at E is represented by the area of the triangle aea' (Art. 126), and the slope of the beam EF at E is represented by the area of the triangle cec'. But these two slopes must be equal, therefore the triangles aea' and cec' are equal in area, and as they are also similar, it follows that $a'e = c'e$. Therefore the points of inflexion and the middle point of the beam divide the span into four equal parts, and $L_1 = \frac{1}{4}L$, also $L_2 = \frac{1}{2}L$.

The cantilever A_1E_1 of length $= \frac{1}{4}L$ carries a load $= \frac{1}{2}W$ at E_1, hence by Art. 120 the deflection at $E_1 = \left(\frac{W}{2}\right)\left(\frac{L}{4}\right)^3 \div 3EI = \frac{WL^3}{384EI}$.

The beam E_1F_1 of length $= \frac{1}{2}L$ carries a load W at its centre C_1, hence by Art. 122 the deflection of C_1 below E_1

$$= W\left(\frac{L}{2}\right)^3 \div 48EI = \frac{WL^3}{384EI}.$$

The total deflection of the whole beam at the centre is therefore equal to $\frac{WL^3}{192EI}$.

* In order that the theory developed in this Article and the next may be strictly applicable, the method of fixing must not hinder any horizontal movement of the beams as a whole at the ends. The fixing is only supposed to keep the beam horizontal at the ends.

129. Beam of Uniform Section, Fixed at the Ends, and Loaded Uniformly.—The reasoning in this case, which is illustrated by Fig. 161, is similar to that in the preceding Article, and corresponding points in Figs. 160 and 161 have the same letters attached to them.

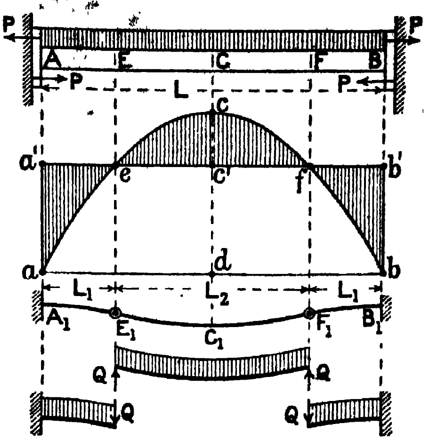


FIG. 161.

Let w denote the load on the beam per unit of length. The bending moment diagram for the whole beam considered as supported at the ends is a parabola acb, the height of the middle ordinate being equal to $\frac{1}{8}wL^2$, the bending moment at the centre.

As in the preceding Article, it may be shown that the area of the figure aea' is equal to the area of the figure cec', and therefore the area of the rectangle $adc'a'$ is equal to the area of the semi-parabola $aecd$. But by the well-known property of the parabola, area $aecd = \frac{2}{3}\overline{ad}\cdot\overline{cd}$, hence $\overline{aa'}\cdot\overline{ad} = \frac{2}{3}\overline{ad}\cdot\overline{cd}$, therefore $\overline{aa'} = \frac{2}{3}\overline{cd}$, and $\overline{cc'} = \frac{1}{3}\overline{cd}$; but $\overline{cd} = \frac{1}{8}wL^2$, therefore $\overline{cc'} = \frac{1}{24}wL^2$.

Considering now the middle portion EF as a beam supported at the ends and loaded uniformly. $\overline{cc'} = \frac{1}{24}wL^2 = \frac{1}{8}wL_2^2$, therefore $L_2^2 = \frac{1}{3}L^2$, and $L_2 = \frac{1}{3}L\sqrt{3} = 0{\cdot}577L$. Also $L_1 = \frac{1}{2}(L - L_2) = \frac{1}{6}L(3 - \sqrt{3}) = 0{\cdot}211L$.

The cantilever A_1E_1 of length $L_1 = \frac{1}{6}L(3 - \sqrt{3})$ carries a load $= \frac{1}{2}wL_2 = \frac{1}{6}wL\sqrt{3}$ at E_1, and a uniform load of w per unit of length. The deflection at E_1 due to the first load is, by Art. 120,

$$= \frac{\frac{1}{6}wL\sqrt{3}L_1^3}{3EI} = \frac{wL^4(9\sqrt{3} - 15)}{648EI}.$$

The deflection at E_1 due to the second load is, by Art. 121,

$$= \frac{wL_1^4}{8EI} = \frac{wL^4(7 - 4\sqrt{3})}{288EI}.$$

The total deflection of the cantilever A_1E_1 at E_1 is therefore

$$\frac{wL^4(9\sqrt{3} - 15)}{648EI} + \frac{wL^4(7 - 4\sqrt{3})}{288EI} = \frac{wL^4}{864EI}.$$

The beam E_1F_1 of length $L_2 = \frac{1}{3}L\sqrt{3}$ carries a uniform load of w per unit of length, hence by Art. 123 the deflection of C_1 below E_1

$$= \frac{5wL_2^4}{384EI} = \frac{5wL^4}{3456EI}.$$

The total deflection of the whole beam at the centre is therefore

$$\frac{wL^4}{864EI} + \frac{5wL^4}{3456EI} = \frac{wL^4}{384EI}.$$

130. Beam of Uniform Section, Fixed at one End, Supported at the Other, and Loaded at the Centre.– AB (Fig. 163) is a beam fixed at B, resting on a support at A, and carrying a load W at the centre C. The first step is to determine the reaction P of the support on the beam at A. Suppose the support at A removed, as shown in the upper

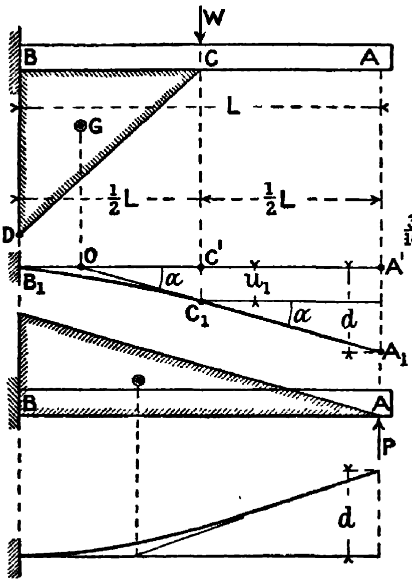


FIG. 162.

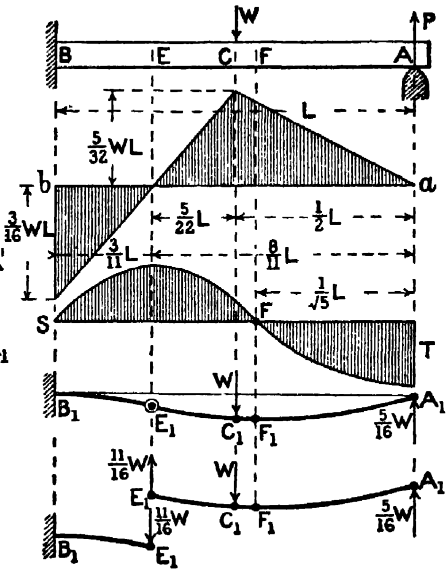


FIG. 163.

part of Fig. 162, then the load W will produce a deflection in AB at $C = u_1 = \frac{W(\frac{1}{2}L)^3}{3EI} = \frac{WL^3}{24EI}$. The bending moment diagram due to W on BC will be a triangle BCD, and the tangent to the bent cantilever B_1C_1 at C_1 will meet the horizontal through B_1 at O, which is vertically under G, the centre of gravity of the triangle BCD. Hence $OC' = \frac{2}{3} \cdot \frac{L}{2} = \frac{L}{3}$.

If α is the inclination of OC_1 to the horizontal, then $\tan \alpha = u_1 \div \frac{L}{3} = \frac{3u_1}{L}$. The portion C_1A_1 of the deflected cantilever will remain straight, but will be inclined to the horizontal at an angle α. Hence the deflection A_1A' of the cantilever at $A_1 = d = u_1 + \frac{1}{2}L \tan \alpha = \frac{5}{2}u_1 = \frac{5WL^3}{48EI}$.

Next suppose that the load W is removed, and the reaction P at the support to act as shown in the lower part of Fig. 162. An upward deflection will be produced at the free end of the cantilever $= u_2 = \frac{PL^3}{3EI}$.

Now if P and W act together, the deflection at A due to W will be neutralised by the deflection due to P. Hence $d = u_2$, that is,

$$\frac{5WL^3}{48EI} = \frac{PL^3}{3EI}, \text{ therefore } P = \tfrac{5}{16}W.$$

The bending moment diagram for the beam AB (Fig. 163) may now be constructed. At C the bending moment is $\frac{1}{2}\mathrm{PL} = \frac{5}{32}\mathrm{WL}$. At B the bending moment is $\frac{5}{16}\mathrm{WL} - \frac{1}{2}\mathrm{WL} = -\frac{3}{16}\mathrm{WL}$. Let E be the point where the bending moment is zero, and let $\mathrm{AE} = x$, then $\frac{5}{16}\mathrm{W}x - \mathrm{W}(x - \frac{1}{2}\mathrm{L}) = 0$, therefore $x = \frac{8}{11}\mathrm{L}$. The shaded figure on the base *ab* is the bending moment diagram. As there is no bending moment at E the beam may be supposed to be hinged at that point, and the whole beam is equivalent to a cantilever $\mathrm{B_1E_1}$ fixed at $\mathrm{B_1}$, and a beam $\mathrm{E_1A_1}$ supported at its ends. Since the supporting force at $\mathrm{A_1}$ is $\frac{5}{16}\mathrm{W}$, it follows that the supporting force at $\mathrm{E_1}$ is $\frac{11}{16}\mathrm{W}$, and this latter force will also be equal to the load on the cantilever $\mathrm{B_1E_1}$ at $\mathrm{E_1}$.

The deflection of the cantilever $\mathrm{B_1E_1}$ at $\mathrm{E_1} = \dfrac{\frac{11}{16}\mathrm{W}(\frac{3}{11}\mathrm{L})^3}{3\mathrm{EI}} = \dfrac{9\mathrm{WL}^3}{1936\mathrm{EI}}$.

By Art. 127 the deflection of $\mathrm{C_1}$ below $\mathrm{A_1E_1} = \dfrac{\mathrm{W}a^2b^2}{3(a+b)\mathrm{EI}}$. In this case $a = \frac{5}{22}\mathrm{L}$, and $b = \frac{1}{2}\mathrm{L}$, therefore the deflection of $\mathrm{C_1}$ below $\mathrm{A_1E_1}$ $= \dfrac{25\mathrm{WL}^3}{4224\mathrm{EI}}$. But $\mathrm{E_1}$ deflects $\dfrac{9\mathrm{WL}^3}{1936\mathrm{EI}}$, and this will lower the point $\mathrm{C_1}$ a distance equal to $\dfrac{11}{16} \times \dfrac{9\mathrm{WL}^3}{1936\mathrm{EI}} = \dfrac{9\mathrm{WL}^3}{2816\mathrm{EI}}$, the multiplier $\frac{11}{16}$ being the ratio of AC to AE.

The total deflection of $\mathrm{C_1}$ below the horizontal through $\mathrm{B_1}$ is therefore

$$= \frac{9\mathrm{WL}^3}{2816\mathrm{EI}} + \frac{25\mathrm{WL}^3}{4224\mathrm{EI}} = \frac{7\mathrm{WL}^3}{768\mathrm{EI}}.$$

If the bending moment diagram on the base *ab* be considered as a load diagram, the part below *ab* representing a load acting upwards, the reaction at the right-hand support will be found to be equal to $\frac{1}{32}\mathrm{WL}^2$, and the point F, where the shearing force is zero, is easily shown to be at a distance from the right-hand support equal to $\dfrac{1}{\sqrt{5}}\mathrm{L}$. The complete shearing force diagram ST is shown, but this need not be drawn.

Still considering the bending moment diagram on the base *ab* as a load diagram, the bending moment due to this load at a point in AC at a distance x from A is equal to $\dfrac{\mathrm{WL}^2}{32}\left(x - \dfrac{5x^3}{3\mathrm{L}^2}\right)$, and the deflection at this point is therefore equal to $\dfrac{\mathrm{WL}^2}{32\mathrm{EI}}\left(x - \dfrac{5x^3}{3\mathrm{L}^2}\right)$. Putting $x = \dfrac{1}{\sqrt{5}}\mathrm{L}$, the deflection at F, where the deflection is greatest, is equal to

$$\frac{\mathrm{WL}^3}{48\sqrt{5}\mathrm{EI}} = \frac{\mathrm{WL}^3}{107\mathrm{EI}} \text{ nearly.}$$

Putting $x = \frac{1}{2}\mathrm{L}$ in the same expression, the deflection at C is found to be $\dfrac{7\mathrm{WL}^3}{768\mathrm{EI}}$, a result which has already been found in another way.

131. Beam of Uniform Section, Fixed at one End, Supported at the Other, and Loaded Uniformly.—The treatment of this case is similar to that of the case in the preceding Article, and the steps will be here stated briefly, and the results given. The load is w per unit length. Removing the support at A (Fig. 164), the downward deflection at that point due to the uniform load will be $\frac{wL^4}{8EI}$. An upward force P at A, the uniform load being removed, will produce an upward deflection at that point equal to $\frac{PL^3}{3EI}$. Hence $\frac{PL^3}{3EI} = \frac{wL^4}{8EI}$, therefore $P = \frac{3}{8}wL$.

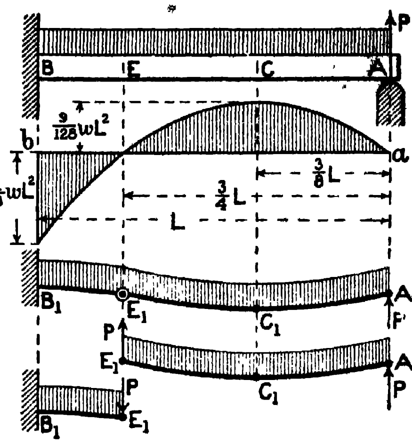


FIG. 164.

The bending moment at a distance x from A is $\frac{3}{8}wLx - \frac{1}{2}wx^2$, and when this is zero, $\frac{3}{8}wLx = \frac{1}{2}wx^2$, and $x = \frac{3}{4}L$. This gives the point of inflexion E. When $x = L$, the bending moment is

$$\frac{3}{8}wL^2 - \frac{1}{2}wL^2 = -\frac{1}{8}wL^2.$$

Between A and E the bending moment is greatest at C, where $x = \frac{3}{8}L$, and is then equal to $\frac{9}{128}wL^2$.

The beam AB may now be considered as a cantilever B_1E_1 fixed at B_1, and a beam E_1A_1 supported at the ends. The load on the cantilever B_1E_1 is $P = \frac{3}{8}wL$ at E_1, and a uniform load of w per unit length. The load on the beam E_1A_1 is a uniform load of w per unit length.

$$\text{Deflection at } E_1 = \frac{\frac{3}{8}wL(\frac{1}{4}L)^3}{3EI} + \frac{\frac{1}{4}wL(\frac{1}{4}L)^3}{8EI} = \frac{5wL^4}{2048EI}.$$

$$\text{Deflection of } C_1 \text{ below } E_1A_1 = \frac{5(\frac{3}{4}wL)(\frac{3}{4}L)^3}{384EI} = \frac{135wL^4}{32768EI}.$$

Total deflection of C_1 below A_1B_1

$$= \frac{5wL^4}{2 \times 2048EI} + \frac{135wL^4}{32768EI} = \frac{wL^4}{187EI} \text{ nearly.}$$

The greatest deflection is $\frac{wL^4}{185EI}$ at a point whose distance from A is 0·4215L.

Observe that this beam is not strengthened by fixing it at one end, the maximum bending moment being $\frac{1}{8}wL^2$, the same as when the beam is simply supported at the ends.

132. Beam of Uniform Section, Resting on Three Equidistant Supports and Uniformly Loaded.—AB (Fig. 166) is a beam resting on three supports, one at each end and the other at the centre, and carrying a uniform load of w per unit length. Let each of the equal spans be denoted by L, and let the reactions at the ends be Q, and the reaction at the centre P.

First consider the case where the supports are at the same level. Suppose the middle support to be removed, as shown in the upper half of Fig. 165. The deflection at the centre will then be $\frac{5w(2L)^4}{384EI} = \frac{5wL^4}{24EI}$. Now suppose that the load is removed and that a force P acts upwards at the centre, forces acting downwards being applied at the ends, as shown

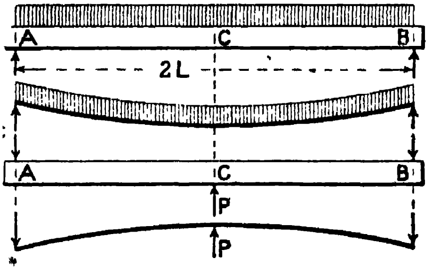


FIG. 165.

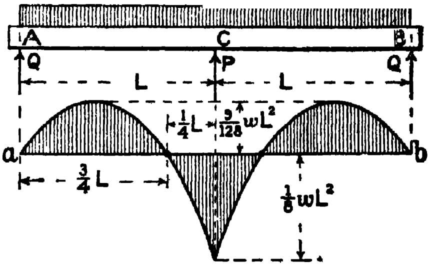


FIG. 166.

in the lower half of Fig. 165. The upward deflection at the centre will be $\frac{P(2L)^3}{48EI} = \frac{PL^3}{6EI}$. If the conditions in the upper and lower halves of Fig. 165 be applied simultaneously, and there is no resulting deflection at the centre C, then $\frac{PL^3}{6EI} = \frac{5wL^4}{24EI}$, therefore $P = \frac{5}{4}wL$, and consequently $Q = \frac{3}{8}wL$. The bending moment diagram may now be constructed, and will be as shown on the base ab in Fig. 166. The beam AB (Fig. 166) may evidently be considered as two cantilevers AC and BC fixed at C, loaded uniformly and supported at their free ends.

Next, suppose that the middle support is 1–nth of d below the level of the other supports, where $d = \frac{5wL^4}{24EI}$, the downward deflection at the centre when the middle support is removed. The upward deflection due to P is now $\frac{5wL^4}{24EI}\left(1 - \frac{1}{n}\right) = \frac{PL^3}{6EI}$, therefore $P = \frac{5}{4}wL\left(1 - \frac{1}{n}\right)$, and $Q = \frac{1}{8}wL\left(3 + \frac{5}{n}\right)$.

Lastly, suppose that the middle support is 1-nth of d above the level of the other supports. The upward deflection due to P is now $\frac{5wL^4}{24EI}\left(1 + \frac{1}{n}\right) = \frac{PL^3}{6EI}$, therefore $P = \frac{5}{4}wL\left(1 + \frac{1}{n}\right)$, and $Q = \frac{1}{8}wL\left(3 - \frac{5}{n}\right)$.

The case discussed in this Article is the simplest case of a *continuous beam*; the general case, where there are any number of supports and any combination of loads, is considered in the next Article.

133. Continuous Beams and Theorem of Three Moments.—A beam which rests on more than two supports is called a *continuous beam*. Let BCD (Fig. 167) be a portion of a continuous beam, BC $= L_2$ and CD $= L_3$ being two consecutive spans. Let $b'ec'$ and $c'fd'$ be the bending moment diagrams on the base $b'c'd'$ for BC and CD considered as separate beams supported at their ends. The separate beams BC and CD would have no bending moments at their supports, but the continuous beam BCD will have bending moments M_B, M_C, and M_D at the supports B, C, and D

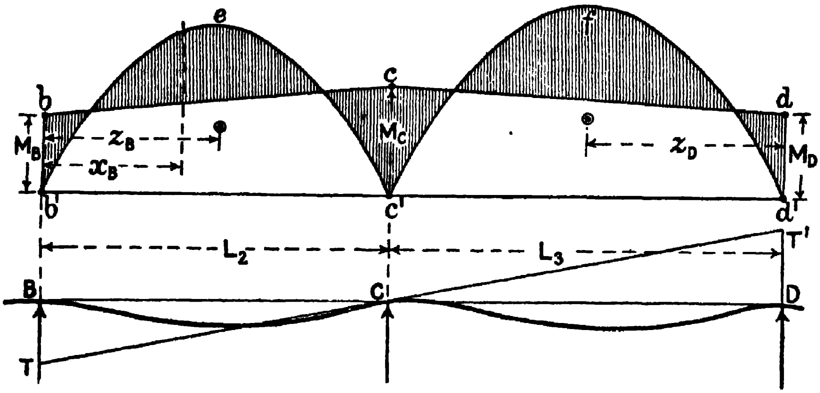


FIG. 167.

respectively, but at present these bending moments are unknown. Suppose, however, that M_B, M_C, and M_D are known. Make $b'b = M_B$, $c'c = M_C$, and $d'd = M_D$. Considering the portion BC, the bending moments M_B and M_C may be considered as arising from the loading to the left of B and to the right of C, and these bending moments will affect the whole of BC, as shown by the diagram $b'bcc'$, where bc is a straight line. The resulting bending moment diagram for BC as a part of the continuous beam will be the shaded diagram $b'ec'cb$, the base of which is bc.

Let a_2 = area of bending moment diagram $b'ec'$, and a_3 = area of bending moment diagram $c'fd'$. Let z_B = horizontal distance of the centre of gravity of the diagram $b'ec'$ from B, and z_D = horizontal distance of centre of gravity of the diagram $c'fd'$ from D. Also let x_B = horizontal distance of the centre of gravity of the shaded diagram $b'ec'cb$ from B, parts below bc being reckoned as negative, and parts above bc as positive. Lastly, let S_2 denote the effective area of $b'ec'cb$, that is, the algebraical sum of the positive and negative parts.

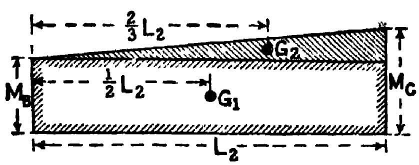


FIG. 168.

Now consider the shaded bending moment diagram to be a load diagram. Let TCT′ be the tangent to the bent continuous beam at C, and let it meet the verticals through B and D at T and T′ respectively. By Art. 126, BT $= S_2 x_B \div$ EI. Dividing the figure $b'bcc'$ into a rectangle and a triangle, as shown in Fig. 168, where G_1 is the centre of gravity of

the rectangle and G_2 is the centre of gravity of the triangle, it follows that

$$S_2x_B = a_2z_B + L_2M_B \cdot \tfrac{1}{2}L_2 + \tfrac{1}{2}L_2(M_C - M_B) \cdot \tfrac{2}{3}L_2$$
$$= a_2z_B + \tfrac{1}{6}L_2^2M_B + \tfrac{1}{3}L_2^2M_C.$$

[Note that in the foregoing general expression, as applied to Fig. 167, if a_2 is positive, M_B and M_C would be negative.]

Hence $$BT = \frac{1}{EI}(a_2z_B + \tfrac{1}{6}L_2^2M_B + \tfrac{1}{3}L_2^2M_C).$$

In like manner $$DT' = -\frac{1}{EI}(a_3z_D + \tfrac{1}{6}L_3^2M_D + \tfrac{1}{3}L_3^2M_C).$$

But if the three supports are at the same level as in Fig. 167,

$$\frac{BT}{L_2} = -\frac{DT'}{L_3}.$$

Therefore $$\frac{a_2z_B}{L_2} + \frac{a_3z_D}{L_3} + \tfrac{1}{6}L_2M_B + \tfrac{1}{3}(L_2 + L_3)M_C + \tfrac{1}{6}L_3M_D = 0,$$

which is the form of the *theorem of three moments* for the case where the supports are at the same level.

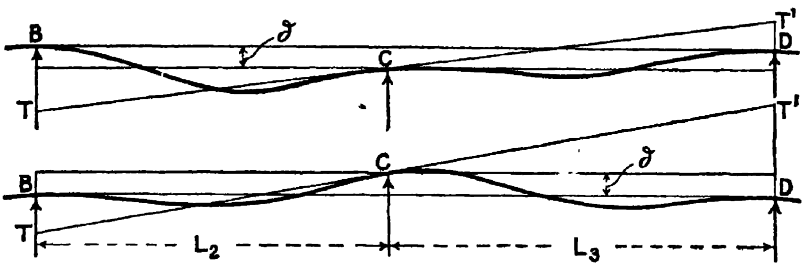


FIG. 169.

If the intermediate support at C is at a distance δ below or above the supports at B and D, as shown in Fig. 169, then

$$\frac{BT \mp \delta}{L_2} = -\frac{DT' \pm \delta}{L_3}, \text{ and } \frac{BT}{L_2} - \frac{DT'}{L_3} = \pm\delta\left(\frac{1}{L_2} + \frac{1}{L_3}\right),$$

where the plus sign applies to the case where C is below BD, and the minus sign applies to the case where C is above BD. It then follows that $\frac{a_2z_B}{L_2} + \frac{a_3z_D}{L_3} + \tfrac{1}{6}L_2M_B + \tfrac{1}{3}(L_2 + L_3)M_C + \tfrac{1}{6}L_3M_D = \pm\delta\left(\frac{1}{L_2} + \frac{1}{L_3}\right)EI$, which is the most general form of the theorem of three moments.

Consider the common and simple case in which the three supports are at the same level, and the load over the span BC is uniformly distributed and equal to w_2 per unit of length, and the load over the span CD is also uniformly distributed and equal to w_3 per unit of length. Here

$$a_2 = \tfrac{2}{3} \cdot \tfrac{1}{8}w_2L_2^2L_2 = \tfrac{1}{12}w_2L_2^3, \quad z_B = \tfrac{1}{2}L_2.$$

$$a_3 = \tfrac{2}{3}\cdot\tfrac{1}{8}w_3L_3^2L_3 = \tfrac{1}{12}w_3L_3^3,\quad \text{and } z_D = \tfrac{1}{2}L_3.$$

Hence $\frac{1}{24}w_2L_2^3 + \frac{1}{24}w_3L_3^3 + \frac{1}{6}L_2M_B + \frac{1}{3}(L_2+L_3)M_C + \frac{1}{6}L_3M_D = 0,$

or, multiplying both sides by 6,

$$\tfrac{1}{4}w_2L_2^3 + \tfrac{1}{4}w_3L_3^3 + L_2M_B + 2(L_2+L_3)M_C + L_3M_D = 0.$$

For a continuous beam on n supports the theorem of three moments furnishes $n-2$ equations, and the conditions of support at the two ends furnish another two equations. These n equations are sufficient for determining the bending moments over the n supports. Most commonly the beam is free over the end supports, and the bending moments there are then zero.

134. Reactions at the Supports of a Continuous Beam.—Consider the reaction R_C (Fig. 170) at the intermediate support of two consecutive spans $BC = L_2$ and $CD = L_3$. Let M_B, M_C, and M_D be the bending moments over the supports B, C, and D respectively. Let F_C and F'_C be the shearing forces on the beam immediately to the left and right respectively of the support at C. Then $R_C = F_C + F'_C$.

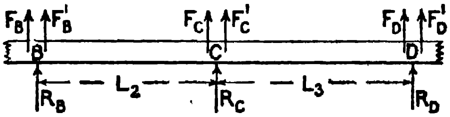


FIG. 170.

Consider the span BC, and take moments about B.

$M_B = F_CL_2 + M_C - W_2Z_B$. Therefore $F_C = \dfrac{1}{L_2}(M_B - M_C + W_2Z_B)$, where W_2 is the sum of the loads on the span BC, and Z_B is the horizontal distance of their centre of gravity from B.

Consider the span CD, and take moments about D.

$M_D = F'_CL_3 + M_C - W_3Z_D$. Therefore $F'_C = \dfrac{1}{L_3}(M_D - M_C + W_3Z_D)$, where W_3 is the sum of the loads on the span CD, and Z_D is the horizontal distance of their centre of gravity from D.

$$\text{Hence}\qquad R_C = F_C + F'_C = \frac{M_B - M_C}{L_2} + \frac{M_D - M_C}{L_3} + \frac{W_2Z_B}{L_2} + \frac{W_3Z_D}{L_3}.$$

For uniform loading of w_2 per unit run on BC and w_3 per unit run on CD.

$$R_C = \frac{M_B - M_C}{L_2} + \frac{M_D - M_C}{L_3} + \frac{w_2L_2}{2} + \frac{w_3L_3}{2}.$$

If the beam is free over the end supports, then the reaction at either end is equal to the shearing force at that end.

135. Example of Continuous Beam.—A bridge ABCD (Fig. 171) consists of two continuous girders having a central span BC of 200 feet, and two side spans AB and CD each of 160 feet. There is a uniform dead load of $\frac{1}{2}$ ton per foot run on the whole of each girder, and on each girder of the span AB there is an additional load equivalent to $\frac{3}{4}$ ton per foot run. The four piers are at the same level, and the ends of the girders

are free. It is required to construct the bending moment and shearing force diagrams for one girder.

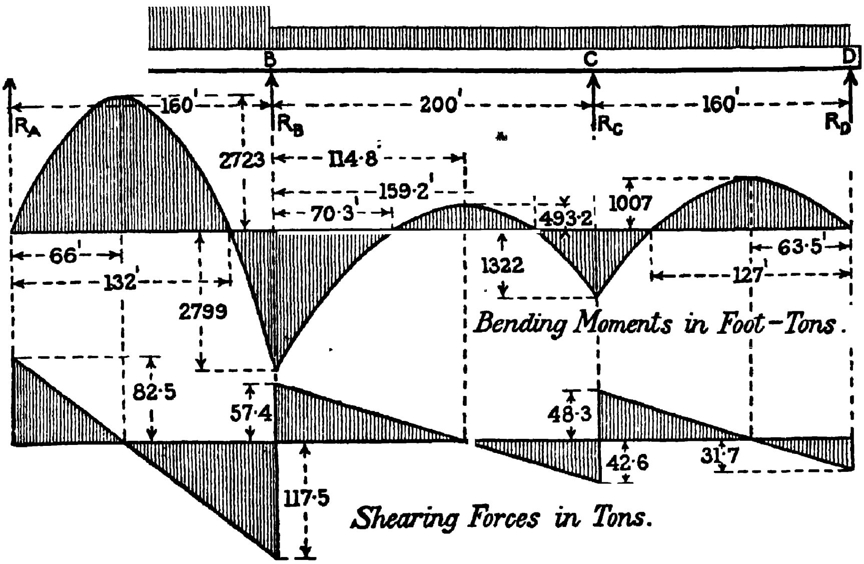


FIG. 171.

Bending Moments at Supports.—Using the notation of the preceding Articles, the theorem of three moments gives the equations :—

$\frac{1}{4}w_1L_1^3 + \frac{1}{4}w_2L_2^3 + L_1M_A + 2(L_1 + L_2)M_B + L_2M_C = 0$ for the spans AB and BC, and $\frac{1}{4}w_2L_2^3 + \frac{1}{4}w_3L_3^3 + L_2M_B + 2(L_2 + L_3)M_C + L_3M_D = 0$ for the spans BC and CD, where $w_1 = 1\frac{1}{4}$, $w_2 = \frac{1}{2}$, $w_3 = \frac{1}{2}$, $L_1 = 160$, $L_2 = 200$, $L_3 = 160$, $M_A = 0$, and $M_D = 0$. Loads being in tons, lengths in feet, and bending moments in foot-tons.

Solving the above equations, $M_B = -2799$, and $M_C = -1322$.

Shearing Forces at Supports.—

$$F'_A = \frac{M_B - M_A}{L_1} + \frac{w_1L_1}{2} = -\frac{2799}{160} + \frac{5 \times 160}{8} = 82{\cdot}5 \text{ tons.}$$

$$F_B = \frac{M_A - M_B}{L_1} + \frac{w_1L_1}{2} = \frac{2799}{160} + \frac{5 \times 160}{8} = 117{\cdot}5 \text{ tons.}$$

$$F'_B = \frac{M_C - M_B}{L_2} + \frac{w_2L_2}{2} = -\frac{1322 - 2799}{200} + \frac{200}{4} = 57{\cdot}38 \text{ tons.}$$

$$F_C = \frac{M_B - M_C}{L_2} + \frac{w_2L_2}{2} = -\frac{2799 - 1322}{200} + \frac{200}{4} = 42{\cdot}62 \text{ tons.}$$

$$F'_C = \frac{M_D - M_C}{L_3} + \frac{w_3L_3}{2} = \frac{1322}{160} + \frac{160}{4} = 48{\cdot}26 \text{ tons.}$$

$$F_D = \frac{M_C - M_D}{L_3} + \frac{w_3L_3}{2} = -\frac{1322}{160} + \frac{160}{4} = 31{\cdot}74 \text{ tons.}$$

Reactions at Supports.—

$R_A = F'_A = 82{\cdot}5$ tons. $R_B = F_B + F'_B = 117{\cdot}5 + 57{\cdot}38 = 174{\cdot}88$ tons.
$R_C = F_C + F'_C = 42{\cdot}62 + 48{\cdot}26 = 90{\cdot}88$ tons. $R_D = F_D = 31{\cdot}74$ tons.

Bending Moments.—

For span AB. $M = 82{\cdot}5x - \frac{5}{8}x^2$, where x is the distance of the section from A. M is zero when $x = 0$, or 132. Maximum positive value of $M = 2723$ when $x = 66$.

For span BC. $M = -2799 + 57{\cdot}38x - \frac{1}{4}x^2$, where x is the distance of the section from B. M is zero when $x = 70{\cdot}3$, or $159{\cdot}2$. Maximum positive value of $M = 493{\cdot}2$ when $x = 114{\cdot}8$.

For span CD. $M = 31{\cdot}74x - \frac{1}{4}x^2$, where x is the distance of the section from D. M is zero when $x = 0$, or 127. Maximum positive value of $M = 1007$ when $x = 63{\cdot}5$.

All the bending moment curves are parabolas whose axes are vertical and pass through the points of zero shear and maximum positive bending moment, as shown in Fig. 171.

136. Advantages and Disadvantages of Continuous Girders.—The chief advantage which a continuous girder has over separate girders for each span will be clearly seen by reference to Fig. 172, in which the bending moment and shearing force diagrams for the example of a continuous girder, discussed in the preceding Article, are reproduced with the addition of the bending moment and shearing force diagrams for the three spans when covered with separate girders. The diagrams for the continuous girder are shaded, while the boundaries of the diagrams for the separate girders are shown dotted where they do not coincide with the boundaries of the others.

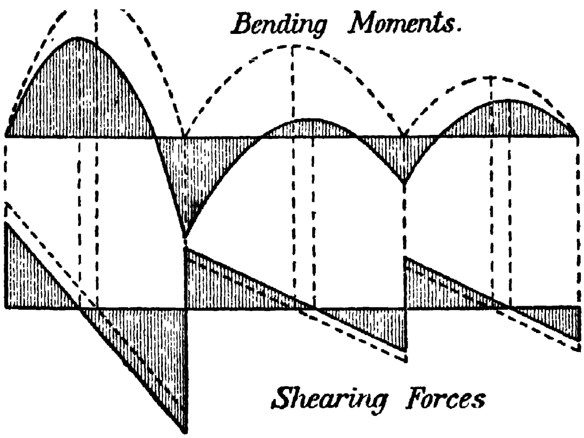


Fig. 172.

It will be seen that one effect of converting the separate girders into one continuous girder is to considerably reduce the bending moments in the neighbourhood of the middle of each span, and to produce bending moments at the supports, and also to increase the bending moments in the neighbourhoods of the supports. If the bending moments at the supports of the continuous girder are greater than at the other points, which is generally the case, the girders may be strengthened in the neighbourhoods of the supports, and the increase in weight will not, to any considerable extent, affect the bending moment diagram, the additional weight coming over or near the piers. A continuous girder will therefore

be lighter than a series of separate girders covering the same spans. It will be noticed that the maximum shear is still at the supports when the separate girders are converted into a continuous girder, and that the change in the values of the maximum shearing forces in the various spans is not very great.

The disadvantages of continuous girders are, however, serious. In the first place, the level of the piers is liable to changes due to unequal settlement or variations of temperature, and comparatively small inequalities of level may cause considerable changes in the bending moment diagram. In the second place, travelling loads will cause the points of inflexion to change, and there will be portions of the girder in the vicinities of those points on which the bending moments will be alternately positive and negative. This disadvantage will obviously be greater the greater the moving loads are compared with the permanent load due to the weight of the structure. The advantage of continuity is therefore greater in long spans, where the permanent load is the most important one.

A continuous girder requires more care in construction than separate girders, because any want of straightness in the unloaded girder will upset the results of the designer's calculations.

The shearing force diagrams in Fig. 172 show that for a uniformly distributed load the web of a continuous girder must be slightly heavier than the webs of a series of separate girders covering the same spans.

137. Cantilever Bridges.—Let the shaded diagram in Fig. 173 be the bending moment diagram for a continuous girder covering three spans. If the continuous girder be cut at the points of inflexion E and F, *or* at the points of inflexion G and H, and joints be made at these points which are capable of resisting shear but not bending, the resulting girders form what is called a *cantilever bridge*, and they will have all the advantages of the original continuous girder, but the bending moment diagram will not now be affected by any settlement of the piers. Moreover, the bending moment and shearing force diagrams for a cantilever bridge may be constructed by applying the simple principles of statics without any reference to the elasticity and deflection of the structure. The cantilever bridge has therefore the advantage of being simple to design and, what is most important, there is the further advantage that there need be no doubt about the results of the designer's calculations.

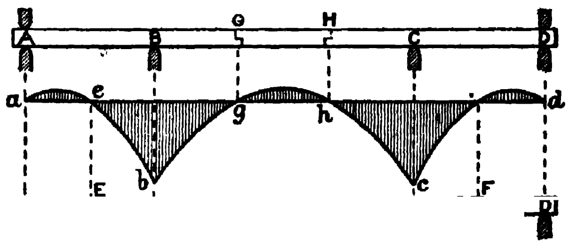


FIG. 173.

It should be noticed that when the flexible joints are made at G and H, in the centre span, the cantilevers AG and DH may need to be *anchored down* but not *fixed* at A and D, as the reactions at these points may become negative.

Cantilever bridges have been used with great success for very large

spans. The Forth Bridge, one of the greatest achievements of the engineer, is a cantilever bridge. This bridge, which crosses the Firth of Forth, consists of three great double cantilevers, one at each end and one in the centre, the centre cantilever being connected to the others by independent girders. Fig. 174 is a skeleton diagram of one of the end canti-

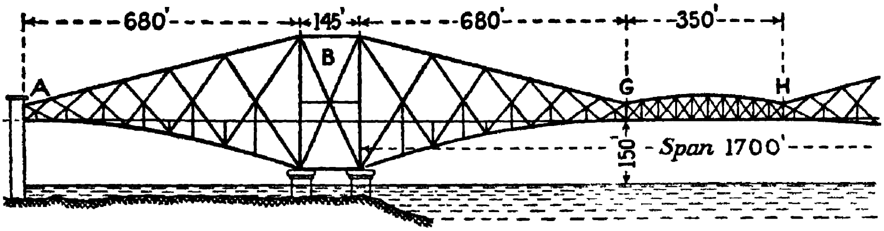


FIG. 174.

levers ABG, and the bridging girders GH in the centre of one of the large spans. As the arm BG has to carry half the weight of the central girders GH and of the train loads which may be passing over them the arm AB is made heavier than the arm BG, and at the extremity A there is an additional weight sufficient to counterpoise with an excess of 200 tons half the weight of GH when carrying a full train load.

138. Resilience of a Beam.—Consider a very short portion LN of length s of a beam, and let M be the mean bending moment over LN, also let θ be the change of slope of the beam in passing from L to N. The work done in bending LN is equal to $\frac{1}{2}M\theta$. But by Art. 126, $\theta = \frac{Ms}{EI}$, therefore work done in bending LN $= \frac{M^2 s}{2EI}$.

Referring to Fig. 175, let ACB be the bending moment diagram for a beam. Construct another curve AC′B on the same base AB, the ordinates of AC′B being equal to the squares of the corresponding ordinates of ACB. The work done in bending the portion of the beam lying between A and B will evidently be equal to the area of the figure AC′B divided by 2EI. In measuring the area of the figure AC′B the unit of area is a rectangle, whose base is the unit of length, and whose height is the unit of bending moment.

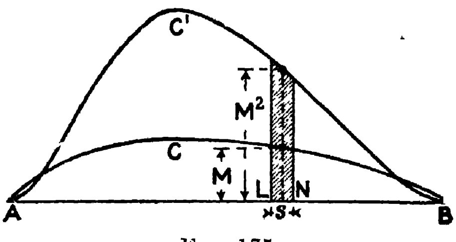


FIG. 175.

In the simple case where a beam of length L is subjected to a uniform bending moment M, the work done in bending it is obviously $\frac{M^2 L}{2EI}$. If f_1 is the greatest stress at the elastic limit, and y_1 the distance at which it acts from the neutral axis, then $M = \frac{f_1 I}{y_1}$, and the resilience of the beam is $\frac{f_1^2 LI}{2Ey_1^2}$.

For a beam of length L supported at its ends and loaded with a weight W at an intermediate point dividing L into two parts a and b, the bending moment diagram (Fig. 176) is a triangle ACB and the curves AC′ and BC′, whose ordinates are the squares of the ordinates of the bending moment diagram, are semi-parabolas whose axes are vertical and whose vertices are at A and B respectively. Hence the area of

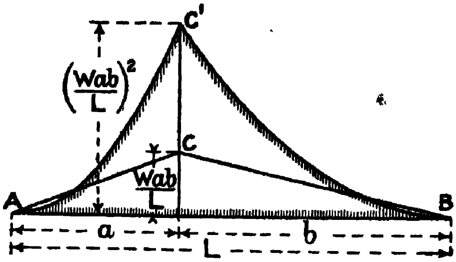


Fig. 176.

$AC'B = \frac{1}{3} \cdot \frac{W^2a^2b^2}{L^2}(a+b) = \frac{W^2a^2b^2}{3L}$, and the work done in deflecting the beam is $\frac{W^2a^2b^2}{6LEI}$. But the work done is equal to $\frac{1}{2}W\delta$, where δ is the deflection at the load. Hence $\delta = \frac{Wa^2b^2}{3LEI}$, a result which was proved in another way in Art. 127.

The work done in deflecting a beam of length L may be found analytically as follows. Let x = mean distance of LN (Fig. 175) from A, and let $dx = s$, then work done in deflecting beam $= \int_0^L \frac{M^2dx}{2EI} = \frac{1}{2EI}\int_0^L M^2dx$. Applying this to the case of a beam supported at the ends and carrying a uniformly distributed load of w per unit of length, $M = \frac{w}{2}(Lx - x^2)$, and work done

$$= \frac{1}{2EI}\int_0^L \frac{w^2}{4}(L^2x^2 - 2Lx^3 + x^4)dx = \frac{w^2}{8EI}\int_0^L \left(L^2x^2dx - 2Lx^3dx + x^4dx\right)$$

$$= \frac{w^2}{8EI}\left(\frac{L^5}{3} - \frac{2L^5}{4} + \frac{L^5}{5}\right) = \frac{w^2L^5}{240EI} = \frac{W^2L^3}{240EI},$$

where $W = wL$ = total load.

Exercises VIII.

1. A pitch pine beam rests on supports 15 feet apart, and carries a uniformly distributed load of 2 tons per foot run. The cross section is a rectangle 15 inches deep, and the maximum stress is 3000 lbs. per square inch at every cross section. The breadth b of the section is to vary so that the beam will bend to a circular arc. Find b at the centre of the span, and at 2 feet and 4 feet from the centre. Find also the deflection at the centre, and the radius of curvature of the neutral surface. E = 1,900,000 lbs. per square inch.

2. A cast-iron cantilever, 54 inches long, carries a load of 3000 lbs. at its outer end. The cross section is a rectangle 2 inches broad. At the fixed end the depth is 8 inches. The depth at other points is to be such that the lever will bend to a circular arc, and the lever is to be symmetrical about the neutral surface. Find the depth at 9, 27, and 45 inches from the fixed end. Find also the deflection at the free end, and the radius of curvature of the neutral surface when the lever is bent. What are the maximum stresses at the fixed end and at the other sections mentioned? E = 17,000,000 lbs. per square inch.

3. A beam, instead of being straight when free from bending moment, is curved

to a radius r. On applying a bending moment the radius of curvature is altered from r to R. Show that $\frac{E}{R}=\frac{E}{r}\pm\frac{f}{y}\left(\frac{r+y}{r}\right)$, and therefore if r is large compared with y, $\frac{f}{y}=\pm E\left(\frac{1}{R}-\frac{1}{r}\right)$ nearly.

4. The cross section of a cantilever is a circle of diameter d. Length of lever, 4 feet. Load at free end, 5000 lbs. Maximum stress, 9000 lbs. per square inch. E = 29,000,000 lbs. per square inch. Find d and the deflection at the free end.

5. A cylindrical cantilever is 30 inches long, and 5 inches in diameter. There is a load at the free end which causes a maximum stress of 2500 lbs. per square inch. Taking E at 1,800,000 lbs. per square inch, what is the deflection at the free end?

6. A vertical mild steel tube of 6 inches external diameter, and ½ inch thick, is securely bedded in the ground. Its height above ground is 10 feet, and it is subjected at the upper end to a horizontal pull of 1500 lbs. Calculate the maximum stress at the ground section and the deflection at the top. (Take E as 30,000,000 lbs. per square inch.) [Inst.C.E.]

7. A beam 12 feet long, 1 foot deep, and 5 inches wide rests on supports at its ends, and carries a load of W lbs. at its centre. The maximum stress being 2000 lbs. per square inch, find W and the deflection at the centre. E = 1,800,000 lbs. per square inch.

8. Referring to the beam of the preceding exercise, what load, in pounds, distributed uniformly over the length, will cause a deflection at the centre equal to 1-500th of the span?

9. A steel joist, 10 inches deep and 10 feet long, is supported at the ends. The joist has equal flanges 5 inches wide and 0·54 inch thick, and a web 0·35 inch thick. The weight of the joist is 29 lbs. per foot. What central load, in addition to its own weight, will this joist carry when its deflection at the centre is 1-1000th of the span, and what will then be the maximum stress? E = 30,000,000 lbs. per square inch.

10. A wooden plank, 12 inches wide and 3 inches deep in section, rests freely on two supports, in the same horizontal level, which are 20 feet apart. A man weighing 12 stone stands in the middle of this plank carrying on his shoulder a hod of bricks which weighs 84 lbs. Find:—(a) The maximum stress at the central section due to this load, and the weight of the plank. (1 cubic foot of wood weighs 46 lbs.) (b) The deflection in the centre, if Young's modulus of elasticity, is 1,600,000 lbs. per square inch. [B.E.]

11. In connection with a contract for the supply of cast-iron pipes, certain bending tests were specified on bars (cast at the same time) 40 inches long, 2 inches deep, and 1 inch thick. The following results were obtained when one of these bars was tested on edge on a 36-inch span:—

Load at centre of beam, pounds	100	400	800	1200	1600	2000	2400
Deflection at centre of beam, inches	0·012	0·048	0·098	0·150	0·204	0·256	0·314

(a) Plot on squared paper a curve to show the relation between the load at the centre of the beam and the deflection at the centre of the beam. (b) From your curve determine the load which will be required at the centre of the beam in order to give a deflection of one-eighth of an inch. (c) Calculate in lbs. per square inch Young's modulus of elasticity for this cast-iron. (d) Calculate in inch-pounds the total work done in bending this beam up to a load of 2400 lbs. in the centre of the span. (e) The beam eventually broke with a load of 3200 lbs. in the centre. Assuming that the ordinary beam formula holds up to the breaking point in cast-iron beams, what was the maximum intensity of tensile stress in the metal at the instant of rupture? [B.E.]

12. A steel girder, having a uniform depth of 13 feet, rests on piers which are 150 feet apart, and carries a uniformly distributed load. Find the deflection at the centre in inches; (a) when the area of the flanges is proportioned so that there is a uniform flange stress of 6½ tons per square inch; (b) when the girder

is of uniform cross section throughout and the maximum flange stress is $6\frac{1}{2}$ tons per square inch. $E = 13{,}400$ tons per square inch.

13. Show that if a cantilever of length L carries a load W at a point $\frac{L}{n}$ from the fixed end, the deflection at the free end is $\frac{(3n-1)WL^3}{6n^3EI}$.

14. A steel shaft AB, 3 inches in diameter, rests on supports at C and D, and is loaded at the ends, as shown in Fig 177. If the maximum stress due to bending is 10,000 lbs. per square inch, what is the total deflection u_1 at the centre? $E = 30{,}000{,}000$ lbs. per square inch.

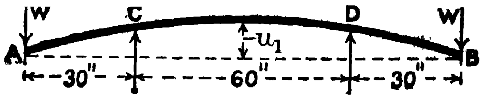


FIG. 177.

15. A continuous girder, built for crossing two equal spans, has a uniform section whose moment of inertia is I, while its uniform weight per foot lineal is w. The girder is launched across the spans from one end, and, when its centre comes nearly over the central pier, the leading end will droop downwards under its unsupported weight. Write the expression for the extreme deflection of the leading end, the length of each span being denoted by L. [Inst.C.E.]

16. A cast-iron pipe, internal diameter 18 inches, and thickness 1 inch, rests on supports 40 feet apart. Find the maximum bending stress and the deflection at the centre when the pipe is full of water. Take weight of cast-iron = 450 lbs. per cubic foot, weight of water = 62·3 lbs. per cubic foot, and $E = 6000$ tons per square inch.

17. If any beam of uniform section deflects 1 inch in a span of 100 inches under a central load, what will be the slope of the beam at each end? [Inst.C.E.]

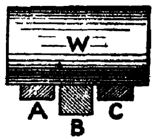


FIG. 178.

18. Suppose that three beams or planks, A, B, and C (Fig. 178), of the same material, are laid side by side across a span $L = 100$ inches, and a load $W = 600$ lbs. is laid across them at the centre of the span so that they must all bend together. The beams are all 6 inches wide, but while A and C have a depth of 3 inches, the depth of the middle beam B is twice as great. How much of the weight W will be carried by each of the three beams, and what will be the extreme fibre stress in each? [Inst.C.E.]

19. A flitch beam is made up of two timbers, each 6 inches wide and 14 inches deep, and a steel plate $\frac{5}{8}$ inch thick and 12 inches deep, as shown in Fig. 179. Taking the modulus of elasticity of the steel as 21 times that of the timber, find the maximum tensile stress in the steel when the maximum tensile stress in the timber is 1000 lbs. per square inch. Find also the percentage increase in the strength of the timber beam to resist bending due to the addition of the steel plate, allowing the same stresses.

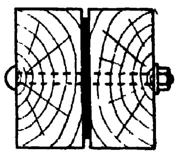


FIG. 179.

20. A beam of uniform section rests on supports whose distance apart is L, and carries two loads each equal to W, one at a distance a from one support, and the other at a distance a from the other support. Show that the deflection under each load is equal to $\frac{Wa^2}{6EI}(3L - 4a)$, and that the maximum deflection (at the centre) is equal to $\frac{Wa}{24EI}(3L^2 - 4a^2)$.

21. A single line railway bridge is carried by two main girders, each of 40 feet span. The total weight of a locomotive standing on the bridge in the position shown in Fig. 180 is 68 tons, distributed upon 4 axles, the leading axle carrying 8 tons, and each of the others 20 tons. Find the maximum deflection of the girders, and where it occurs. [U.L.]

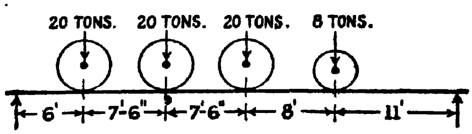


FIG. 180.

22. A beam of uniform section is built into a wall at one end, and rests on a support at a distance of 20 feet from the wall. A load of 26 tons rests on the beam at a point 12 feet from the wall. Taking $E=13{,}000$ tons per square inch, and $I=1000$ in inch units, determine the reaction of the support on the beam and the deflection of the beam at the point where the load is applied. Draw the bending moment and shearing force diagrams for this beam.

23. A beam of uniform section is built into a wall at one end and supported on a column, as shown in Fig. 181. The beam carries a load of 54 tons uniformly distributed. Find the vertical thrust on the column, and draw the bending moment and shearing force diagrams. At what points is the bending moment zero?

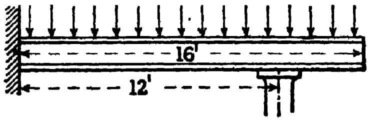


FIG. 181.

24. A beam of uniform section is rigidly fixed at its ends to two walls, which are 24 feet apart. Two loads, each of 10 tons, are applied to this beam at points 6 feet from the walls. Determine the bending moments at the ends and at the centre, and find the positions of the points of zero bending moment. Draw the bending moment and shearing force diagrams.

25. A continuous beam of uniform section covers two spans, each equal to L, and carries a uniform load of w per unit of length. Show that the middle support must be below the level of the outer supports by an amount $\frac{7wL^4}{72EI}$ in order that the pressures on the three supports may be equal.

26. A continuous girder of uniform section consists of two spans, each of 50 feet, and carries over both spans a uniformly distributed load of 1 ton per foot run. Both ends of the girder are free. Calculate the bending moment over the middle support, and the maximum positive bending moment between the centre and one end. Find also the reactions at the supports.

27. A continuous girder of uniform section and of two equal spans carries a uniformly distributed load of w tons per foot run. Find the bending moment over the central pier when the height of the three piers is the same, and also when the central pier, owing to temperature effects, is raised or lowered by an amount equal to x inches. [U.L.]

28. A rolled steel joist 40 feet in length, of **I** section, 10 inches deep, and 5 inches wide, has a thickness which is equivalent to ½ inch in both flanges and web. It is continuous over three supports, forming two spans of 20 feet each. What uniformly distributed load would produce a maximum stress of 5½ tons per square inch? Sketch the diagrams of bending moments and shearing forces. [Inst.C.E.]

29. Apply the theorem of three moments to find the reactions when there are three level piers supporting a continuous girder carrying a uniformly distributed load of 2 tons per foot run, the two spans being 200 feet and 150 feet respectively. [U.L.]

30. A continuous girder consists of two spans. One span of 100 feet is loaded with 1¾ tons per foot run, the second span of 80 feet is loaded with 2½ tons per foot run. Find the values of the supporting forces, and the maximum bending moment for the whole girder. Both ends of the girder are free. [Inst.C.E.]

31. Work out the example of Article 135, pp. 128–130, assuming that, owing to settlement of the pier, the support at B is ¼ inch below the level of the other supports. Take $E=13{,}000$ tons per square inch, and $I=432{,}000$ in inch units.

32. A cantilever bridge ABCD has supports at A, B, C, and D. AB = CD = 100 feet. BC = 300 feet. There are hinge joints at E and F in the centre span. BE = CF = 100 feet. Assuming that there is a permanent dead load of 2 tons per foot run, and a live load of 1 ton per foot run, construct the bending moment and shearing force diagrams for this bridge when the live load covers (*a*) the span AB only, (*b*) the cantilever AE only, (*c*) the girder EF only.

33. A cantilever bridge has three spans, each of 200 feet. There are hinge joints in the side spans at points 120 feet from the shore ends. Assuming a dead load of 2 tons per foot run, and a live load of 1 ton per foot run, construct the bending moment and shearing force diagrams when the live load covers (*a*) one side span only, (*b*) the centre span only.

34. Show that the resilience of a bar of uniform circular section when subjected to a uniform bending moment is one-quarter of its resilience when subjected to simple tension or simple crushing.

35. A steel bar of uniform rectangular section, 3 inches deep, 2 inches wide, and 30 inches long, is supported at the ends and loaded at the centre with a weight W. Taking the maximum stress at 20,000 lbs. per square inch, and E = 30,000,000 lbs. per square inch, find the number of ft.-lbs. of elastic energy stored up in this loaded bar. What would the answer be if this bar were placed in simple tension under the same stress?

CHAPTER IX

COMPOUND STRAINS AND STRESSES

139. Directions of Stresses Generally Parallel to one Plane.—If a portion of a strained body be taken which is a right prism it will be found that in the majority of practical cases this prism may be selected so that on its parallel ends there is no stress whatever, and if this is so it is easily shown that the directions of the stresses on the remaining faces must be parallel to the planes of the ends of the prism. In the web of a girder, for instance, there is usually no stress on the sides or on plane sections parallel to the sides, and the tensile, compressive, and shearing stresses are all in directions parallel to the sides of the web.

In considering the equilibrium of a right prismatic element, on the ends of which there is no stress, it is most convenient to represent this element with its ends parallel to the plane of the paper upon which it is projected; the directions of the stresses considered are then all parallel to that plane.

In the articles and exercises of this chapter it will be assumed, unless otherwise stated, that the directions of the stresses considered are parallel to the plane of the paper, and that the plane sections upon which the stresses act are perpendicular to that plane.

In proving the propositions connected with stresses in a strained body it is convenient to consider an element of it which is a right prism, selected as described above, and in many cases it is necessary to assume that the element is *indefinitely small* to allow of the stresses being of varying intensities, because if the stress on a surface is not of uniform intensity, the stress on an indefinitely small area of that surface may be considered as of uniform intensity.

140. Stresses on an Oblique Section of a Bar subjected to Direct Tension or Compression.—Let AB (Fig. 182) be a bar subjected to a direct pull or push by a load P which is uniformly distributed over its ends. If a is the area of the cross section of the bar, and p the intensity of the stress on it, then $P = pa$. Consider an oblique section CD inclined at an angle θ to the cross section. The area of this oblique section is $a/\cos\theta$. Considering the equilibrium of the part ACD, the force P is balanced by a force N perpendicular to CD, and a force Q in the plane of CD. The force N is the resultant

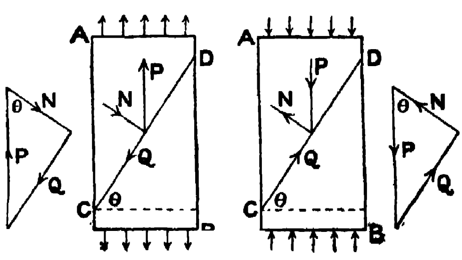


FIG. 182.

of a normal stress on CD, and the force Q is the resultant of a tangential or shear stress on CD. By the triangle of forces $N = P \cos\theta$, and $Q = P \sin\theta$.

If n is the intensity of the normal stress, and q the intensity of the tangential stress on CD, then $N = \frac{na}{\cos\theta} = P\cos\theta = pa\cos\theta$, therefore $n = p\cos^2\theta$. When $\theta = 0$, n is a maximum, and is then equal to p. Also $Q = \frac{qa}{\cos\theta} = P\sin\theta = pa\sin\theta$, therefore $q = p\sin\theta\cos\theta$, but $\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$, therefore $q = \frac{1}{2}p\sin 2\theta$. When $\sin 2\theta = 1$, *i.e.* when $\theta = 45°$, q is a maximum and is then equal to $\frac{1}{2}p$.

If a section be taken perpendicular to CD its inclination to the cross section will be $90° - \theta$, and the shear stress on this section will be $\frac{1}{2}p\sin 2(90 - \theta) = \frac{1}{2}p\sin(180 - 2\theta) = \frac{1}{2}p\sin 2\theta$, which is the same as the shear stress on CD. It will be shown in the next Article that in all cases where there is a shear stress on one section there is always an equal shear stress on a section perpendicular to it.

The fact that there is a shear stress q on the section CD having a maximum value equal to $\frac{1}{2}p$ when θ is 45°, suggests that if the resistance of a material to rupture by shearing be less than half its resistance to rupture by direct tension or compression, it will give way by shearing when subjected to tension or compression. This is what really happens with several materials, and examples will be found in Art. 166, p. 175.

141. Equality of Shear Stresses on Planes at Right Angles.—Consider an indefinitely small rectangular portion ABCD (Fig. 183) of a strained body, and let h be the height, b the breadth, and t the thickness of this portion. Assume that there is no stress on the face ABCD or on any interface parallel to it. The portion of material ABCD being at rest, the stresses on the faces which are perpendicular to the face ABCD must balance one another. The stresses on the faces AD and BC may be resolved into normal stresses p, and shear stresses q. In Fig. 183 the arrows representing the normal stresses are omitted. The normal stresses on AD and BC must evidently balance one another.

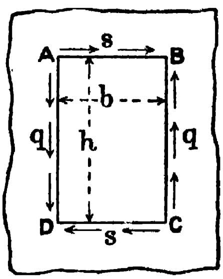


FIG. 183.

The resultant of the shear stress on AD equals qht, and this will also be the magnitude of the resultant shear stress on BC. These two resultants will form a couple whose moment is $qhtb$. Now no system of forces but a couple will balance a couple, therefore the stresses on AB and CD must have components which are shear stresses s on these faces. The normal components of the stresses on AB and CD must balance one another. The resultants of the shear stresses on AB and CD will form a couple whose moment is $sbth$, and if this couple is to balance the other couple, then $sbth = qhtb$, therefore $s = q$. Hence if at any point of a strained body there is a shear stress in one plane there must be a shear stress of equal intensity in another plane at right angles to the first, but these two planes must be perpendicular to a plane which is parallel to the directions of the stresses.

142. Pure Shear Stress Equivalent to two Normal Stresses.—Consider an indefinitely small cube ABCD (Fig. 184) of a strained body. Let b be the length of the edges of this cube. Assume that there is no stress on the face ABCD or on any interface parallel to it. Suppose that the faces AD and BC are subjected to pure shear stress of intensity q, the direction of which is parallel to the face ABCD, then by the preceding

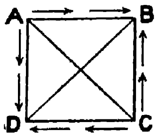


FIG. 184.

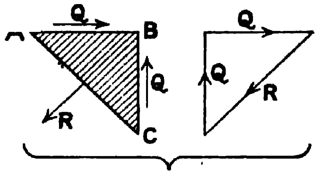


FIG. 185.

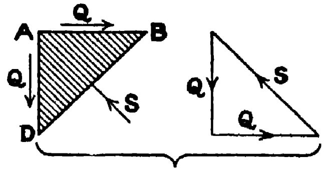


FIG. 186.

Article there must be shear stresses of intensity q on the faces AB and CD, as shown. Imagine the cube divided into two equal parts by a plane AC perpendicular to the face ABCD. Consider (Fig. 185) the equilibrium of ABC, one of these parts. The resultants QQ of the stresses on AB and BC must balance the resultant R of the stress on AC, and by the triangle of forces it is seen that R is perpendicular to AC and equal to Q $\sqrt{2}$. The stress on AC is evidently a tensile stress. Let r be the intensity of the stress on AC, then

$$R = rb^2\sqrt{2} = Q\sqrt{2} = qb^2\sqrt{2}, \text{ therefore } r = q.$$

In like manner, by dividing the cube into two equal parts by a plane BD perpendicular to the face ABCD, and considering (Fig. 186) the equilibrium of the part ABD, it can be shown that there is a compressive stress of intensity q on the face BD.

Hence a pure shear stress is equivalent to two normal stresses at 45° to the shear stress, and each equal in intensity to the shear stress, but one is a tensile and the other a compressive stress.

It is evident that all sections of the cube parallel to the plane AC will be subjected to tensile stress, of intensity q, and all sections parallel to BD will be subjected to compressive stress of intensity q. Hence if a part of the interior of the cube be mapped out so as to form a rectangular solid EF having faces parallel to AC and BD, as shown in Fig. 187, this solid will be subjected to tensile and compressive stresses of intensity equal to that of the shear stresses on the faces of the cube ABCD.

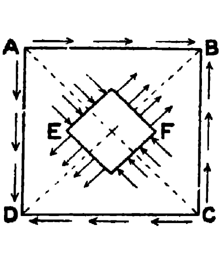


FIG. 187.

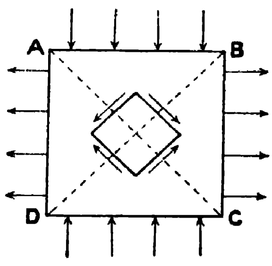


FIG. 188.

Conversely, it is easy to show that if a cube ABCD (Fig. 188) have its faces AD and BC subjected to tensile stress, and also have its faces AB and CD subjected to an equal compressive stress, sections parallel to AC and BD will be subjected to shear stress of the same intensity.

The theorem which has just been proved has an important application in the case of a shaft subjected to twisting. Let ABCD (Fig. 189) be a square traced on the surface of a shaft, the sides AD and BC being perpendicular to the axis of the shaft, and suppose that this square represents an indefinitely thin prism of the material. The faces AD and BC are subjected to pure shear stress, and by Art. 141 the faces AB and CD must be subjected to an equal shear stress. Hence by the theorem of the present Article the diagonal face AC is subjected to pure tension, and the diagonal face BD is subjected to pure compression, also the intensities of the tensile and compressive stresses will be the same as that of the shear stress. Now, if the resistance of the material to tension be less than its resistance to shearing the shaft will give way along AC, which is part of a helix whose inclination to the axis of the shaft is 45°. This is what actually occurs when a cast-iron shaft is broken in torsion, except that the inclination of the helix is not exactly 45°. Further reference to this matter will be found in Art. 167, p. 176.

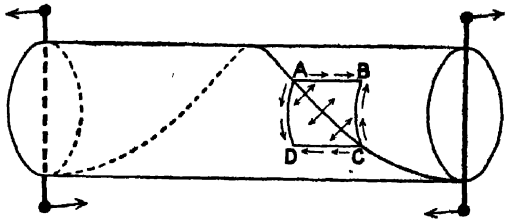


FIG. 189.

As an illustration of the presence of tensile and compressive stresses whose directions are inclined at 45° to the directions of the shear stresses in a shaft under torsion, it will be found that a spiral spring whose coils are close together and inclined at 45° to the axis will be as stiff and strong when twisted in one direction as a tube of the same material having the same outside and inside diameters, but under torsion in the opposite direction the spring will be very weak. When twisted in the first direction the surfaces of the coils in contact are subjected to pure compression, and therefore the fact that the material is divided at the surfaces in contact will not affect the power of the coils to resist compression. When twisted in the opposite direction the coils will separate.

143. Principal Stresses—Principal Axes of Stress.—Let ABCD (Fig. 190) be an indefinitely small cube in a strained body, the face ABCD and all interfaces parallel to it being free from stress. The stresses on the faces AB, BC, CD, and DA may be resolved into normal and shear stresses. The normal stress on AD must balance the normal stress on BC; let the intensity of these stresses be denoted by p. The normal stress on AB must balance the normal stress on CD; let the intensity of these stresses be denoted by q.

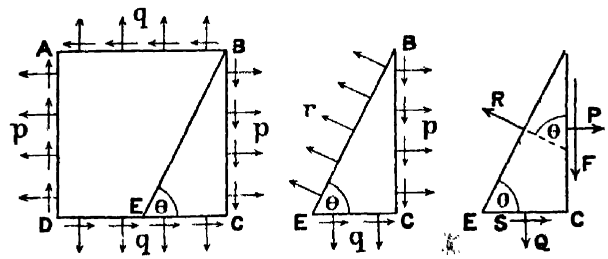


FIG. 190.

By Art. 141 the intensities of the shear stresses on AB, BC, CD, and DA must be equal; let these be denoted by f.

Let BE be a plane section, upon which there is pure normal stress of intensity r. It is required to find r, and θ, the inclination of BE to CD, in terms of p, q, and f.

Let the edge of the cube be denoted by x. Consider the equilibrium of the prism BCE.

P the resultant of the stress p on BC $= px^2$.
Q " " " q " CE $= qx^2 \cot \theta$.
R " " " r " BE $= rx^2/\sin \theta$.
F " " " f " BC $= fx^2$.
S " " " f " CE $= fx^2 \cot \theta$.

Resolving vertically.

$$R \cos \theta = F + Q.$$
$$rx^2 \cot \theta = fx^2 + qx^2 \cot \theta.$$
$$r \cot \theta = f + q \cot \theta \quad . \quad . \quad (1)$$

Resolving horizontally.

$$R \sin \theta = P + S.$$
$$rx^2 = px^2 + fx^2 \cot \theta.$$
$$r = p + f \cot \theta \quad . \quad . \quad . \quad (2)$$

Solving equations (1) and (2), $r = \frac{1}{2}\{p + q \pm \sqrt{(p-q)^2 + 4f^2}\}$

$$\tan 2\theta = \frac{2f}{q-p} \text{ or } \tan \theta = \frac{f}{r-p} = \frac{r-q}{f} = \frac{p-q \pm \sqrt{(p-q)^2 + 4f^2}}{2f}.$$

There are two values of θ which satisfy the above equations, and these values differ by 90°. Corresponding to the two values of θ there are two values of r.

The two values of r acting in directions at right angles to one another are called *principal stresses*, and two lines parallel to the directions of the principal stresses are called *principal axes of stress*.

As a numerical example, let $p = 6$, $q = 3$, and $f = 2$, all in tons per square inch. Then $r = \frac{6 + 3 \pm \sqrt{(6-3)^2 + 4 \times 2^2}}{2} = 7$, or 2 tons per square inch, $\tan \theta = \frac{r-q}{f} = \frac{7-3}{2} = 2$, or $\tan \theta = \frac{2-3}{2}$

To show the positions of the principal axes of stress, draw HK (Fig. 191) parallel to the direction of p, and make it equal to unity on any convenient scale. Draw KL at right angles to KH, and make it $= 2$. Join HL, then the angle LHK is one value of θ. Produce KH to M, making HM $= 2$. Draw MN at right angles to MH, and make it $= 1$. Join NH, then the angle NHK is the other value of θ, and it is easily seen that the angle NHL is a right angle. Through any point O in the original cube ABCD draw OX and OY perpendicular to HL and HN respectively. OX and OY are principal axes of stress, and if a rectangular element be taken at O, with its faces parallel to OX and OY, the stresses on the faces of this element will be entirely normal, the stress in the direction OX being a tension of 7 tons per square inch, and the stress in the direction OY a tension of 2 tons per square inch.

In Fig. 190, p and q are shown as tensile stresses. If p or q, or both,

be altered to compressive stresses, this will be equivalent to making p or q, or both, negative instead of positive. It would be well for the student to demonstrate the proposition, first, taking q as compressive stress and p as tensile stress, and then again, taking both p and q as compressive stresses.

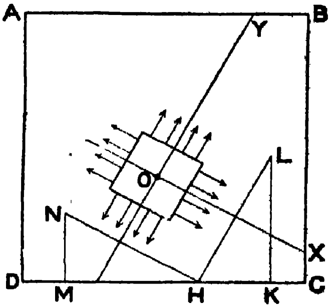


FIG. 191.

The case where p or q is equal to zero is one of great practical importance, and will be considered in the next Article.

144. Principal Stresses due to Combined Bending and Twisting.—When a shaft of diameter d is subjected to a bending moment B, the maximum tensile and compressive stresses produced are given by the equation $p=\frac{32\text{B}}{\pi d^3}$, and these stresses are in direction parallel to the axis of the shaft. If the same shaft is subjected to a twisting moment T, the maximum shear stress produced is given by the equation $f=\frac{16\text{T}}{\pi d^3}$. Let ABCD (Fig. 192) be an indefinitely small square prism of the material of the shaft in the neighbourhood where the tensile or compressive stresses are a maximum, the face ABCD being on the surface of the shaft, and AB parallel to its axis. Then by the preceding Article, putting $q=0$, there will be pure normal stresses on planes at right angles to one another, the intensities of these normal stresses being given by the equation $r=\frac{1}{2}\{p\pm\sqrt{p^2+4f^2}\}$. Inserting the values of p and f given above, $r=\frac{1}{2}\left\{\frac{32\text{B}}{\pi d^3}\pm\sqrt{\left(\frac{32\text{B}}{\pi d^3}\right)^2+4\left(\frac{16\text{T}}{\pi d^3}\right)^2}\right\}=\frac{16}{\pi d^3}\{\text{B}\pm\sqrt{\text{B}^2+\text{T}^2}\}$. The greater of these two values of r is $\frac{16}{\pi d^3}\{\text{B}+\sqrt{\text{B}^2+\text{T}^2}\}$, and when p is a tensile stress the greater value of r is a tensile stress, but when p is a compressive stress the greater value of r is a compressive stress. Now, since the resistance of the material of shafts to compression is greater than the resistance to tension, the maximum value of r should be considered as a tensile stress.

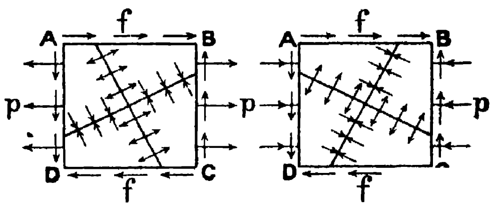


FIG. 192.

The equation $r=\frac{16}{\pi d^3}\{\text{B}+\sqrt{\text{B}^2+\text{T}^2}\}$ may be put in the form

$\frac{\pi}{16}d^3r=\text{B}+\sqrt{\text{B}^2+\text{T}^2}$. Now, a simple twisting moment $\text{T}_e=\frac{\pi}{16}d^3r$ will produce a pure shear stress and also a pure tensile stress (Art. 142) of intensity r, therefore a twisting moment $\text{T}_e=\text{B}+\sqrt{\text{B}^2+\text{T}^2}$ will produce the same maximum normal stress as is produced by the combined action

of the bending moment B and twisting moment T. T_e is called the *equivalent twisting moment.*

In using the formula $T_e = B + \sqrt{B^2 + T^2} = \frac{\pi}{16}d^3r$, it must be remembered that r is not a shear stress, but a tensile stress. This is a point which is frequently misunderstood, and it may be well to restate the case as follows:—When a shaft of diameter d is subjected to a pure twisting moment T, then $T = \frac{\pi}{16}d^3f$, where f is the shear stress on the faces AB, BC, CD, and DA (Fig. 193) of a square element of the skin of the shaft, AB being parallel to the axis, *but f is also the tensile stress on AC.* If a bending moment B is added, this shifts the plane of tensile stress to EF (Fig. 194), and increases its value from f to r, the value of r being $\frac{16}{\pi d^3}\{B + \sqrt{B^2 + T^2}\}$. Hence it may be said that in the formulæ $T = \frac{\pi}{16}d^3f$ and $T_e = \frac{\pi}{16}d^3r$, f and r are both tensile stresses.

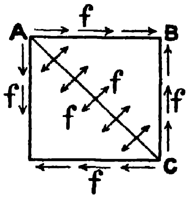


FIG. 193.

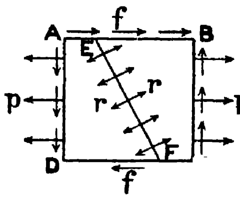


FIG. 194.

It is easy to show that if B_e is a bending moment which will produce the same maximum normal stress as a bending moment B and a twisting moment T acting together, then $B_e = \frac{1}{2}B + \frac{1}{2}\sqrt{B^2 + T^2}$. In applying this to a shaft, B_e must be equated to $\frac{\pi}{32}d^3f$, where f is the maximum normal stress.

145. Maximum Shear Stress due to Combined Twisting and Bending. — Let ABCD (Fig. 195) be an indefinitely small square prism of the material of a shaft of diameter d which is subjected to a bending moment B and a twisting moment T, the face ABCD being on the surface of the shaft in the neighbourhood of the greatest bending stress. AB is parallel to the axis of the shaft. The faces AD, CB, AB, and CD are subjected to shear stress $f = \frac{16T}{\pi d^3}$. The faces AD and CB are also subjected to bending stress. In Fig. 195 the bending stress is a tensile stress $p = \frac{32B}{\pi d^3}$. These stresses produce a pure normal stress r_1 on planes parallel to CJ, and a pure normal stress r_2 on planes parallel to DK, which is perpendicular to CJ. By Art. 143, putting $q = 0$,

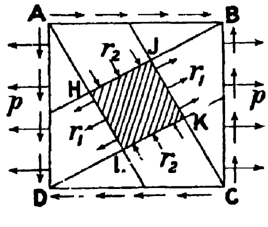


FIG. 195.

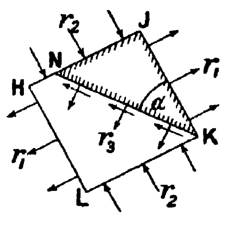


FIG. 196.

$$r_1 = \tfrac{1}{2}p + \tfrac{1}{2}\sqrt{p^2 + 4f^2} \text{ and } r_2 = \tfrac{1}{2}p - \tfrac{1}{2}\sqrt{p^2 + 4f^2}.$$

Consider the indefinitely small square prism HJKL, shown enlarged in Fig. 196. Let $JK = x$, and let the depth of the prism at right angles to the plane of the paper also be x. Cut off from the prism HJKL a wedge KNJ, the angle JKN being α. Consider the equilibrium of the wedge KNJ. On the face JK there is a normal stress r_1, the resultant of which is r_1x^2. On the face JN there is a normal stress r_2, the resultant of which is $r_2x^2 \tan \alpha$. On the face KN there is a stress equivalent to a normal stress r_3 and a shear stress f_3. The resultant of the normal stress on the face KN is $r_3 \dfrac{x^2}{\cos \alpha}$, and the resultant of the shear stress on this face is $f_3 \dfrac{x^2}{\cos \alpha}$. Resolving these resultant forces parallel to KN, $f_3 \dfrac{x^2}{\cos \alpha} = r_1x^2 \sin \alpha + r_2x^2 \tan \alpha \cos \alpha$, or

$$f_3 = r_1 \sin \alpha \cos \alpha + r_2 \sin \alpha \cos \alpha = \tfrac{1}{2}(r_1 + r_2) \sin 2\alpha.$$

Hence f_3 is a maximum when $\alpha = 45°$, then $f_3 = \frac{1}{2}(r_1 + r_2)$. If tensile stress is positive and compressive stress is negative, and if r_1 and r_2 carry their proper signs with them, then $f_3 = \frac{1}{2}(r_1 - r_2)$. Inserting the values of r_1 and r_2 in terms of p and f, then the maximum value of f_3 is $\frac{1}{2}\sqrt{p^2 + 4f^2}$. But $p = \dfrac{32B}{\pi d^3}$, and $f = \dfrac{16T}{\pi d^3}$, therefore $f_3 = \dfrac{16}{\pi d^3}\sqrt{B^2 + T^2}$ and $\dfrac{\pi}{16}d^3 f_3 = \sqrt{B^2 + T^2}$. But a simple twisting moment $T_e = \dfrac{\pi}{16}d^3 f_3$ would produce the same shear stress f_3. Hence a simple twisting moment $T_e = \sqrt{B^2 + T^2}$ will produce the same maximum shear stress as the bending moment B and twisting moment T acting together.

A bending moment $B_e = T_e = \sqrt{B^2 + T^2}$ would produce a maximum normal stress equal to $2f_3$, and therefore (Art. 140, p. 138) a maximum shear stress f_3 at 45° to the direction of the normal stress.

There is little doubt that in the case of ductile materials, such as mild steel, it is the resistance to shear which determines the strength (see Art. 166, p. 175). Hence in designing shafts made of ductile material, and which are subjected to bending and twisting, the formula $T_e = \sqrt{B^2 + T^2}$ should be used in preference to the one $T_e = B + \sqrt{B^2 + T^2}$. But, for mild steel shafts, in equating $B + \sqrt{B^2 + T^2}$ to $\dfrac{\pi}{16}d^3 f_1$, it must be remembered that f_1 is a *tensile* stress, while in equating $\sqrt{B^2 + T^2}$ to $\dfrac{\pi}{16}d^3 f_3$ the stress f_3 is a *shear* stress, and $f_3 = \frac{1}{2}f_1$.

Guest was the first to demonstrate that mild steel shafts subjected to bending and twisting gave way by shear,* and his theory and the results of his experiments have been confirmed by Hancock, Scoble C. A. Smith, and others.

$T_e = \sqrt{B^2 + T^2}$ is generally called the "Guest" formula, and

$T_e = B + \sqrt{B^2 + T^2}$ is generally called the "Rankine" formula.

Shafts designed by the Rankine formula are weaker than those designed by the Guest formula.

* "Strength of Ductile Materials under Combined Stress," *Phil. Mag.*, July 1900.

$= \sqrt{B^2+T^2}$ gives maximum shear stress in the m[illegible]
[illegible] $T_e = B + \sqrt{B^2+T^2}$ gives the maximum
[illegible] stress in the material.

146. Stresses in a Cranked Shaft.—A cranked shaft is a good example of a structure subjected to both bending and twisting, and particular attention should be directed to the fact that the crank pins are generally subjected to twisting as well as the shaft itself. The forces acting on a cranked shaft usually vary in magnitude and direction as the shaft revolves, and each part of the shaft must be designed to withstand the greatest straining action which may come upon it.

A simple example will serve to indicate how the stresses in a cranked shaft are determined. Fig. 197 shows a cranked shaft turning in bear-

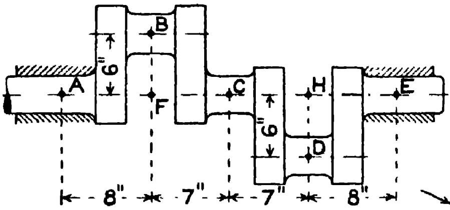


FIG. 197.

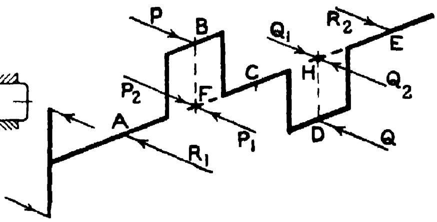


FIG. 198.

ings at A and E. The shaft has two crank pins B and D, the axes of which are in the same plane as AE, the axis of the shaft. The parts A, B, C, D, and E are each $3\frac{1}{2}$ inches in diameter. Fig. 198 shows how the shaft is loaded when the cranks are in a vertical position. There is a pure torque on the left-hand end of the shaft, a force P of 4800 lbs. on the crank pin B, and a force Q of 6000 lbs. on the crank pin D; the lines of action of P and Q are perpendicular to the plane containing the axes of the crank pins. It is required to find the maximum stresses in the pins B and D, and in the shaft at C.

Imagine the shaft produced to the points F and H directly opposite to the centres of the crank pins B and D respectively. The equilibrium of the shaft will not be affected by applying at F forces P_1 and P_2 acting in opposite directions and each parallel and equal to P. Nor will the equilibrium be disturbed by applying at H forces Q_1 and Q_2 each equal and parallel to Q, as shown.

P and P_1 being equal and parallel forces acting in opposite directions form a couple, and since a couple can only have a turning effect, there can be no pressure on the bearings due to these forces. The forces Q and Q_1 also form a couple. The reactions on the shaft at the bearings at A and E must therefore be due to the forces P_2 and Q_2. Taking moments about A, R_2 is found to be 3120 lbs., and taking moments about E, R_1 is found to be 1920 lbs.

Consider the straining actions on the crank pin D. The only force to the right of D is R_2, and this produces a bending moment $= 3120 \times 8 = 24{,}960$ inch lbs., and a twisting moment $= 3120 \times 6 = 18{,}720$ inch-lbs. Using the Rankine formula, the equivalent twisting moment at D due to these is

$$24960 + \sqrt{24960^2 + 18720^2} = 56{,}160 \text{ inch-lbs.}$$

If f is the maximum stress in the pin D, then $\frac{\pi}{16}(3\frac{1}{2})^3 f = 56{,}160$, from which $f = 6671$ lbs. per square inch.

Consider next the straining actions on the shaft at C. Taking the

forces to the right of C, the forces R_2 and Q_2 produce a bending moment $=3120 \times 15 - 6000 \times 7 = 4800$ inch-lbs., and the forces Q and Q_1 produce a twisting moment $= 6000 \times 6 = 36{,}000$ inch-lbs. The equivalent twisting moment at C due to these is

$$4800 + \sqrt{4800^2 + 36000^2} = 41{,}119 \text{ inch-lbs.}$$

If f is the maximum stress in the shaft at C, then $\frac{\pi}{16}(3\frac{1}{2})^3 f = 41{,}119$, from which $f = 4884$ lbs. per square inch.

Consider lastly the straining actions on the crank pin B. Taking the forces to the right of B, the forces Q_2 and R_2 produce a bending moment $= 6000 \times 14 - 3120 \times 22 = 15{,}360$ inch-lbs., and the forces Q, Q_1, Q_2, and R_2 produce a twisting moment $= 6000 \times 12 - 3120 \times 6 = 53{,}280$ inch-lbs. The equivalent twisting moment at B due to these is

$$15360 + \sqrt{15360^2 + 53280^2} = 70{,}810 \text{ inch-lbs.}$$

If f is the maximum stress in the pin B, then $\frac{\pi}{16}(3\frac{1}{2})^3 f = 70{,}810$, from which $f = 8411$ lbs. per square inch.

147. Ellipse of Stress.—ABCD (Fig. 199) is an indefinitely small cube, edges of length l. On the faces AD and BC there is pure normal stress of intensity p, and on the faces AB and CD there is pure normal stress of intensity q. It is required to find the direction and intensity of the stress on any interface LN inclined at an angle θ to AB. Draw MN parallel to AB. Consider the equilibrium of the element LMN. The resultant of the stress on LM $= pl^2 \tan\theta = $ P. The resultant of the stress on MN $= ql^2 = $ Q. Let r denote the intensity of the stress on LN, R the resultant of this stress, and ϕ its inclination to LM. $R = rl^2/\cos\theta$. Then, since R must balance P and Q,

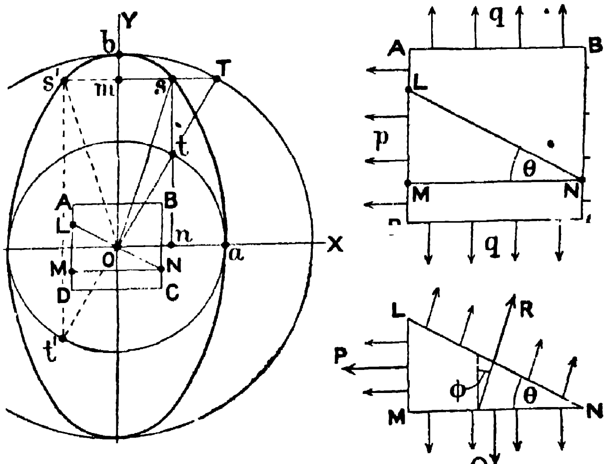


FIG. 199.

$$R \sin\phi = P, \text{ or } \frac{rl^2}{\cos\theta}\sin\phi = pl^2\tan\theta, \text{ therefore } \sin\theta = \frac{r}{p}\sin\phi.$$

$$R\cos\phi = Q, \text{ or } \frac{rl^2}{\cos\theta}\cos\phi = ql^2, \text{ therefore } \cos\theta = \frac{r}{q}\cos\phi.$$

Hence $$\frac{r^2\sin^2\phi}{p^2} + \frac{r^2\cos^2\phi}{q^2} = \sin^2\theta + \cos^2\theta = 1.$$

Also, $\sin^2\phi = \frac{p^2}{r^2}\sin^2\theta$, and $\cos^2\phi = \frac{q^2}{r^2}\cos^2\theta$,

therefore, $r^2 = p^2\sin^2\theta + q^2\cos^2\theta$, and $\tan\phi = \frac{p}{q}\tan\theta$.

Draw OX parallel to P, OY parallel to Q, and Os parallel to R Make O$s = r$. Draw sm and sn parallel to OX and OY respectively. Let O$n = x = r\sin\phi$, and O$m = y = r\cos\phi$. Then, substituting x for $r\sin\phi$, and y for $r\cos\phi$ in the equation at the foot of p. 147, $\frac{x^2}{p^2} + \frac{y^2}{q^2} = 1$, which is the equation to an ellipse whose semi-axes Oa and Ob are equal to p and q respectively.

The point s may be found graphically as follows. Draw OtT perpendicular to LN, to meet a circle with centre O and radius Oa at t, and a circle with centre O and radius Ob at T. Through T and t draw parallels to OX and OY respectively to meet at s.

OX and OY are principal axes of stress, and the ellipse, whose semi-axes are Oa and Ob, is called the *ellipse of stress*.

If the stresses p and q have opposite signs, that is, if one is tensile and the other compressive, then O$t' = p$ must be measured in the opposite direction from O. The construction being completed as before, Os' will be the direction and intensity of the resultant stress on the interface LN.

148. Shear Stresses in Beams.—The existence of a transverse shear stress in beams has been discussed in Art. 99, p. 87, and in Art. 141, p. 139, it has been shown that a shear stress in one plane is always accompanied by a shear stress of equal intensity in planes at right angles to

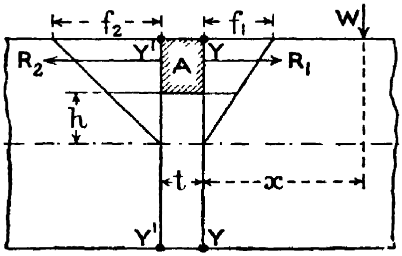


FIG. 200.

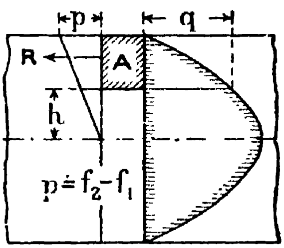


FIG. 201.

that plane; hence there is shear stress in horizontal longitudinal sections of a horizontal beam. The object of this Article is to determine the intensity of the longitudinal shear stress at any point in a beam, and also to show how the intensity of the transverse shear stress varies at different points in the depth of the beam.

Before discussing the general case of a beam of any section, it will be advantageous to first consider the simple case of a beam of rectangular section. Fig. 200 shows a portion of a rectangular beam of depth d and breadth b. YY and Y$'$Y$'$ are two transverse sections very near to one another, and W, at a distance x from YY, is the resultant of all the external forces acting on the beam to the right of YY or Y$'$Y$'$. The bending moment at YY is Wx, and if f_1 is the maximum stress at this section due to the bending moment Wx, then W$x = \frac{1}{6}bd^2f_1$. The bending moment at Y$'$Y$'$

is $W(x+t)$, and if f_2 is the maximum stress, then $Wx+Wt=\frac{1}{6}bd^2f_2$. Hence $Wt=\frac{1}{6}bd^2(f_2-f_1)$.

The distribution of the stresses due to bending on the upper half of the sections YY and Y′Y′ is shown in Fig. 200. A portion A of the beam bounded by the sections YY, Y′Y′, the top surface of the beam, and a horizontal section at a distance h above the neutral surface, is pulled to the right by a force $R_1=bf_1\left(\frac{d}{4}-\frac{h^2}{d}\right)$, and it is also pulled to the left by a force $R_2=bf_2\left(\frac{d}{4}-\frac{h^2}{d}\right)$. The resultant of these two pulls is a pull to the left by a force $R=R_2-R_1=b(f_2-f_1)\left(\frac{d}{4}-\frac{h^2}{d}\right)$, and this is balanced by the horizontal shear on the under surface of A. Let q equal the intensity of the shear stress on the under surface of A, then $btq=b(f_2-f_1)\left(\frac{d}{4}-\frac{h^2}{d}\right)$. But $f_2-f_1=\frac{6Wt}{bd^2}$, therefore $q=\frac{6W}{bd^2}\left(\frac{d}{4}-\frac{h^2}{d}\right)$, and this by Art. 141 must be the intensity of the transverse shear stress on the section YY at a distance h from the neutral axis.

The equation $q=\frac{6W}{bd^2}\left(\frac{d}{4}-\frac{h^2}{d}\right)$ connecting q and h is the equation to a parabola which, when drawn as shown in Fig. 201, represents the distribution of the shear stress on a transverse section of the beam.

The maximum shear stress in the case just considered evidently occurs at the neutral surface of the beam or neutral axis of the transverse section where its intensity is $\frac{3W}{2bd}$, but since the total transverse shear is W, the average transverse shear stress is $\frac{W}{bd}$, hence the maximum transverse shear stress is $1\frac{1}{2}$ times the mean.

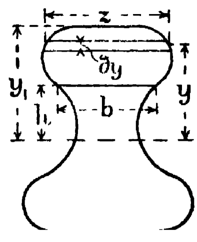


FIG. 202.

Proceeding now to the general case of a beam of any form of cross section, and referring to Figs. 200 and 201, and also to Fig. 202, which represents the section of the beam, the stress due to bending at a distance y from the neutral axis $f=\frac{f_1y}{y_1}$ at the section YY, and $f=\frac{f_2y}{y_1}$ at the section Y′Y′.

$R_1=\Sigma fz\delta y=\Sigma f_1\frac{y}{y_1}z\delta y=\frac{f_1}{y_1}\Sigma yz\delta y$ between the limits $y=h$ and $y=y_1$. But $\Sigma yz\delta y=ay_0$, where a is the area of the section beyond the line at a distance h from the neutral axis, and y_0 is the distance of the centre of gravity of that area from the neutral axis. Therefore $R_1=\frac{f_1}{y_1}ay_0$. In like manner $R_2=\frac{f_2}{y_1}ay_0$. Hence $R=R_2-R_1=(f_2-f_1)\frac{ay_0}{y_1}$. Again, $Wx=\frac{f_1I}{y_1}$ and $W(x+t)=\frac{f_2I}{y_1}$, therefore $Wt=\frac{I}{y_1}(f_2-f_1)$. Also, $R=qbt$. Hence $q=\frac{Way_0}{bI}$.

An interesting case of great practical importance is that of a flanged beam, which will now be considered, and for the sake of simplicity a numerical example will be taken. The I section shown in Fig. 203 has a total depth of 12 inches, the flanges are 8 inches wide, and the web and flanges are all 2 inches thick. The formula which gives the shear stress at a distance h from the neutral axis has just been proved to be $q=\frac{Way_0}{bI}$.

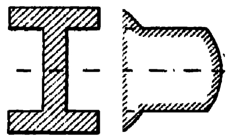


FIG. 203.

Consider the value of q at the neutral axis, where the shear stress is greatest. Here $a=24$ square inches, $y_0=4$ inches, and $b=2$ inches, therefore $q=\frac{48W}{I}$.

At the junctions of the web and flanges, $a=16$ square inches, $y_0=5$ inches, and $b=2$ inches, therefore the value of q at these places is $\frac{40W}{I}$.

In the flanges at places indefinitely near to the junctions with the web, $a=16$ square inches, $y_0=5$ inches, and $b=8$ inches, therefore q in the flanges at these places is $\frac{10W}{I}$.

For the dimensions given $I=896$ in inch units, and if $W=14$ tons, the three values of q considered above are 1680, 1400, and 350 lbs. per square inch. The diagram to the right in Fig. 203 shows the distribution of the shear stress. It will be seen that not only does the web take a large proportion of the whole of the shear stress, but that the shear stress is nearly uniform over the section of the web. It may be noted that the curves in Fig. 203 are parabolas.

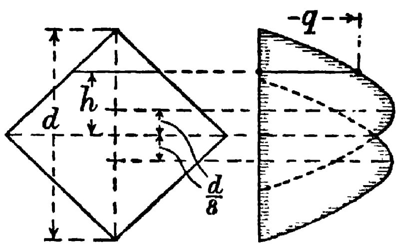


FIG. 204.

In most practical cases the maximum shear stress on a section of a beam is at its neutral axis, but this is not always the case. For example, consider a section which is a square (Fig. 204) with one diagonal vertical (in the plane of bending). Let d equal the length of a diagonal of the square. Using the same notation as before, it is easy to show that $q=\frac{2W}{d^4}(2dh-8h^2+d^2)$. Differentiating, $\frac{dq}{dh}=\frac{2W}{d^4}(2d-16h)$. Hence q is a maximum when $2d-16h=0$, or $h=d/8$.

Putting $h=d/8$, the maximum value of q is $\frac{9W}{4d^2}$.

Putting $h=0$, the value of q at the neutral axis is $\frac{2W}{d^2}$.

The mean value of q is $W \div \frac{d}{\sqrt{2}} \cdot \frac{2W}{d^2}$, the same as the value at the neutral axis.

The variation of the stress is shown plotted to the right in Fig. 204. The curved lines are portions of parabolas whose axes are horizontal, and at distances $d/8$ from the neutral axis of the section.

The distribution of a shear stress over a section of a beam may be found by making use of the section modulus figure which was described in Art. 113, p. 105. A little consideration will show that the force R (Fig. 201) is equal to $a_1(f_2 - f_1)$, where a_1 is the area of that part of the section modulus figure which lies beyond the line at a distance h from the neutral axis. Hence it follows that $q = \dfrac{Wa_1y_1}{bI}$.

Exercises IXa.

1. A cube ABCD (Fig. 205) is subjected to compressive stress of 10 tons per square inch on the faces AB and CD. Taking D as the origin, draw a polar curve showing the intensities of the shear stresses on inclined sections through D. Scale, 1 inch to 1 ton per square inch.

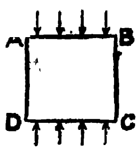


FIG. 205.

2. A cube ABCD (Fig. 206) is subjected to a compressive stress of 3 tons per square inch on the faces AD and BC, and also to a compressive stress of 5 tons per square inch on the faces AB and CD. Determine the shear stresses, in tons per square inch, on the interfaces BD and BE.

3. On A and B, the opposite faces of a cube, there is no stress, but on the remaining four faces there are normal stresses of the same kind, and of intensity p. Show that there is no shear stress on any interfaces which are perpendicular to the faces A and B, also that the intensity of the normal stresses on these interfaces is equal to p.

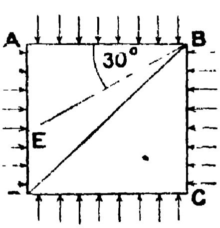


FIG. 206.

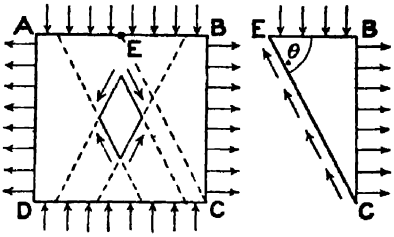


FIG. 207.

4. ABCD (Fig. 207) is a cube. On the faces AD and BC there is tensile stress of intensity p, and on the faces AB and CD there is compressive stress of intensity q. Show that there is pure shear stress of intensity f on all interfaces inclined at an angle θ to AB, and find f and θ in terms of p and q. (*Hint.*—Consider the equilibrium of the element BCE.)

5. The rhombus ABCD (Fig. 208) is one end of a right prism. There is pure shear stress of intensity f on the faces AB, BC, CD, and DA, as shown. Prove that the interfaces AC and BD are subjected to pure normal stresses of intensities p and q respectively, and that interfaces, such as BE, which are perpendicular to BC, are subjected to shear stress of intensity f, and a normal stress of intensity s. Express p, q, and s in terms of f, and θ the angle ABD.

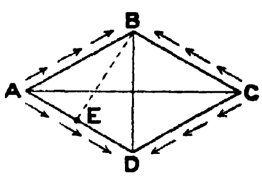


FIG. 208.

6. The maximum tensile stress on a shaft due to the bending moment is half the maximum shear stress due to the twisting moment. The maximum tensile stress due to the above two stresses combined is 12,000 lbs. per square inch. If the diameter of the shaft is 3 inches, find the twisting and bending moments in inch-lbs.

7. The maximum stress on a shaft 3 inches in diameter is 9000 lbs. per square inch, and the shaft is subjected to equal bending and twisting moments. Find the twisting moment in inch-lbs.

8. A shaft transmits 50 horse-power at 135 revolutions per minute. There is a bending moment on the shaft equal to three-fourths of the twisting moment.

Taking the maximum stress at 10,000 lbs. per square inch, find the diameter of the shaft.

9. The external diameter of a hollow shaft is 15 inches, and the internal diameter is 10 inches. The shaft is subjected to a bending moment equal to the twisting moment, and the maximum stress is 10,500 lbs. per square inch. Find the horse-power transmitted at 85 revolutions per minute.

10. A shaft 5 inches diameter is subjected to a thrust of 15 tons along its axis. There is a bending moment on the shaft equal to half the twisting moment. The maximum compressive stress is 13,000 lbs. per square inch. Find the horse-power transmitted at 120 revolutions per minute.

11. Referring to Figs. 197 and 198, page 146. If the forces P and Q are each 9000 lbs., and the angles PBF and QDH are each 60°, what must be the diameters of the crank pin B and the shaft at C if the maximum stress is 9000 lbs. per square inch?

12. The centre lines of the crank shaft of a single-cylinder engine are shown in Fig. 209. When the crank and connecting-rod are at right angles, the effective force on the rod is 30 tons. The work is entirely taken off at the right-hand end, and the bearings may be assumed to exercise no restraint on the shaft. Calculate the bending moment and twisting moment on the crank pin, and find the maximum direct stress induced, the pin being 12 inches internal and 21 inches external diameter. [U.L.]

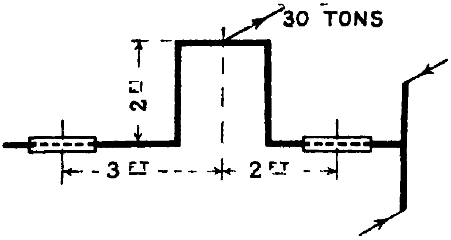


FIG. 209.

13. The cranked shaft (Fig. 210) turns in bearings at A and B. The cranks C and D are in the same plane, and the forces P and Q act at right angles to that plane. P = 2000 lbs., and Q = 2400 lbs. Find the equivalent twisting moments in inch-lbs. on the shaft at A and on the crank pin at C.

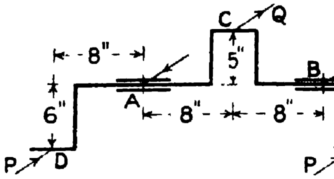


FIG. 210.

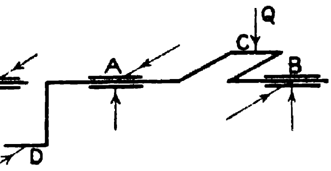


FIG. 211.

14. Same as the preceding exercise, except that the crank C and the force Q are turned through a right angle, as shown in Fig. 211.

15. A three-throw cranked shaft used for working a set of three deep well pumps is shown in Fig. 212. The rods are so long that the forces on the crank pins may be assumed as acting vertically. The shaft turns in bearings at A and E, and it is driven by pure twisting at the end A. The cranks make angles of 120° with one another. Allowing for bending and twisting, determine the diameters (in inches) of the three crank pins for the given loads when the shaft is in the given position (*a*) by the Rankine formula, and (*b*) by the Guest formula. Maximum tensile stress, 9000 lbs. per square inch.

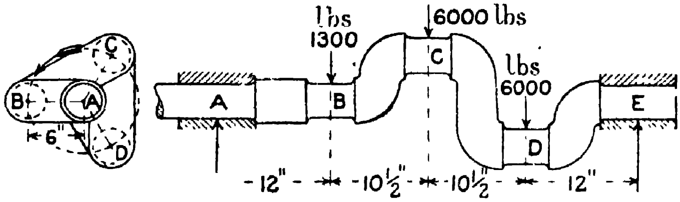


FIG. 212.

16. A wooden beam, 3 inches deep and 2 inches wide, when tested, gave way by shearing along the neutral surface when the load at the centre of the span reached 2760 lbs. What was the intensity of the longitudinal shear stress in the plane of fracture, assuming that the distribution of stress at fracture is the same as within the elastic limit?

17. Two cross sections 1 inch apart of a rectangular beam 3 inches broad by 8 inches deep are subjected to bending moments of 20 and 30 inch-tons respectively. Determine the maximum shear stress on the sections, and draw diagrams of shear stress and direct stress. [U.L.]

18. A cantilever of mild steel 10 inches long, 2 inches deep, and $1\frac{1}{2}$ inches wide, carries a load of 1000 lbs. at its free end. Construct the curve which shows the intensity of the transverse shear stress at any point of the depth. Scales.—Linear, full size. Stress, 1 inch to 500 lbs. per square inch.

19. A rolled steel joist has a total depth of 15 inches; the flanges are 6 inches wide and 0·85 inch thick, and the web is 0·54 inch thick. The total transverse shear at a certain cross section is 30 tons. What is the maximum intensity of the shear stress in the cross section, and what multiple is it of the mean shear stress, assuming that the web takes the whole of the shear. Draw the diagram which shows the actual distribution of the shear stress in the cross section.

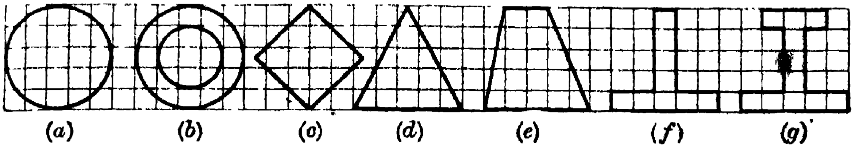


FIG. 213.

20. Cross sections of various beams are shown in Fig. 213. If the shear stress at the neutral axis in each of these sections is 6000 lbs. per square inch, calculate for each the mean shear stress in lbs. per square inch.

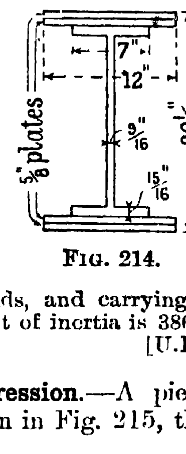


FIG. 214.

21. Referring to Fig. 203, page 150, and the dimensions in the text, and remembering that the diagram in Fig. 203 shows the variation of the *intensity* of the shear stress, determine the fraction which the total shear on the section of the web is of the total shear on the whole section.

22. Deduce an expression for the intensity of shearing stress at any place in a cross section of a beam. Draw to scale a diagram showing the intensity of shear stress throughout a cross section (Fig. 214), 2 feet from one support, of a girder of 30 feet span, supported at the ends, and carrying a uniformly distributed load of 60 tons. The greatest moment of inertia is 3860, calculated from dimensions in inches. [U.L.]

149. Bending Combined with Tension or Compression.—A piece AB has an arm CD upon which a force P acts as shown in Fig. 215, the

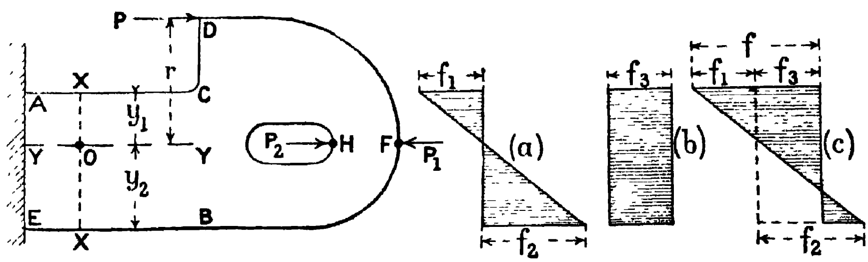


FIG. 215.

line of action of P being parallel to the line YY, which is the edge view of the neutral surface of AB when considered as a beam subjected to bending, the plane of bending being the plane of the paper.

Let XX be any cross section of AB. If forces P_1 and P_2 each equal

and parallel to P be applied as shown at points F and H in YY produced, these added forces will not affect in any way the stresses at XX, because P_1 and P_2 balance one another. P and P_1 being equal and parallel forces acting in opposite directions form a couple which produces a pure turning or bending action at XX. The bending moment is Pr, and the distribution of stress at XX due to Pr is shown at (*a*), where f_1 is the tensile stress along AC, and f_2 is the compressive stress along BE. $Pr = f_1 Z_1 = f_1 \frac{I}{y_1}$, also $Pr = f_2 Z_2 = f_2 \frac{I}{y_2}$, where I is the moment of inertia of the section XX about its neutral axis O. From these equations $f_1 = \frac{Pry_1}{I}$, and $f_2 = \frac{Pry_2}{I}$.

The force P_2 will produce a direct tension, and the tensile stress at XX due to P_2 is $\frac{P_2}{a} = \frac{P}{a} = f_3$, where a is the area of the cross section XX. The distribution of this stress is shown at (*b*).

Combining the stresses due to the pure bending and the direct tension as shown at (*c*), it is seen that there is a tensile stress f along AC equal to $f_1 + f_3$, and a compressive stress along BE equal to $f_2 - f_3$. If f_2 is less than f_3, then $f_2 - f_3$ is a tensile stress.

EXAMPLE.—Referring to Fig. 215, the section at XX is a rectangle, depth XX = 6 inches, breadth = 3 inches, $r = 5$ inches. Total tensile stress along AC = 5 tons per square inch. It is required to find P and the stress along BE.

$Pr = \frac{1}{6} bd^2 f_1$, that is, $5P = \frac{1}{6} \times 3 \times 6^2 f_1$, hence $f_1 = \frac{5P}{18}$.

$f_3 = \frac{P}{a} = \frac{P}{3 \times 6}$. $f = 5 = f_1 + f_3 = \frac{5P}{18} + \frac{P}{18} = \frac{P}{3}$, therefore P = 15 tons,

$f_2 = f_1 = \frac{5 \times 15}{18} = 4\frac{1}{6}$. $f_3 = \frac{15}{18} = \frac{5}{6}$. Hence $f_2 - f_3 = 3\frac{1}{3}$ tons per square inch, and is a compressive stress.

150. Strength of a Ring.—Fig. 216 shows a ring of uniform cross

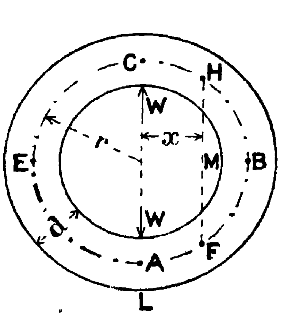


FIG. 216.

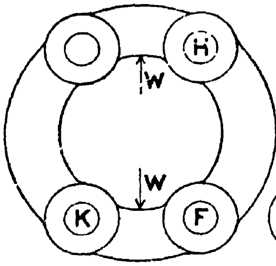


FIG. 217.

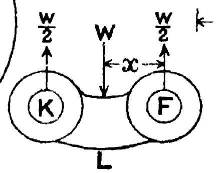


FIG. 218.

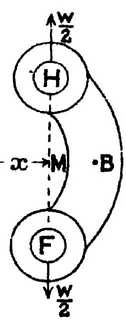


FIG. 219.

section carrying a load W. The mean radius of the ring is r. There is at the horizontal section at B a bending action tending to *diminish* the curvature of the ring at B, and at the vertical section at A there is a

bending action tending to *increase* the curvature of the ring at A. There must therefore be some intermediate point F in AFB at which there is no tendency to alter the curvature of the ring, and therefore at F there is no bending action; consequently at F the ring may be jointed without altering the stresses in the other parts. There are obviously four points at which joints may be introduced, these being symmetrically situated with reference to the vertical and horizontal diameters AC and BE, as shown in Fig. 217. The ring now consists of four links connected by pin joints.

The lower link FK, shown detached in Fig. 218, is in the condition of a beam supported at the ends and loaded in the middle. The maximum tensile stress f due to the bending moment will be at L, and will be equal to $\frac{16Wx}{\pi d^3}$, assuming the cross section of the ring to be a circle of diameter d.

The right-hand link FH, shown detached in Fig. 219, is subjected to a maximum bending moment at the horizontal section at B amounting to $\frac{W}{2}(r-x)$, causing a maximum tensile stress f_1 at M equal to $\frac{16W(r-x)}{\pi d^3}$. In addition there is a uniform tensile stress f_2 on the horizontal section at B equal to $\frac{2W}{\pi d^2}$, due to the load $\frac{W}{2}$. The total stress at M is therefore $f_1 + f_2 = \frac{16W(r-x)}{\pi d^3} + \frac{2W}{\pi d^2}$.

Now it seems reasonable to suppose that the points F and H will be so situated that the total stress at M will equal the stress at L, because when the first permanent set takes place, say at M, if there is not a simultaneous permanent set at L, the line FH would shift towards M, causing the bending moment, and therefore the stress, at L to increase. The line FH will therefore adjust itself so that permanent set takes place simultaneously at M and L, and therefore the stress at M must be the same as that at L. Making use of this,

$$f = f_1 + f_2, \text{ and } \frac{16Wx}{\pi d^3} = \frac{16W(r-x)}{\pi d^3} + \frac{2W}{\pi d^2}, \text{ from which, } x = \frac{r}{2} + \frac{d}{16}.$$

$$\text{Hence,} \qquad f = \frac{16Wx}{\pi d^3} = \frac{16W}{\pi d^3}\left(\frac{r}{2} + \frac{d}{16}\right) = \frac{W}{\pi}\left(\frac{8r}{d^3} + \frac{1}{d^2}\right).$$

$$\text{If} \qquad r = nd, \quad \text{then } d = \sqrt{\frac{W}{\pi f}(8n+1)}.$$

The foregoing results are only roughly approximate, because of the assumption that the moment of resistance to bending of the curved piece FBH is the same as for a straight piece. In the case of a curved bar subjected to bending the neutral axis of a cross section does not pass through its centre of gravity, and the stress does not vary uniformly from the neutral axis, as in the case of a straight bar. The errors in the formulæ deduced above are on the wrong side for safety.

For a full discussion of the theory of bending of curved bars the student is referred to *Morley's Strength of Materials.*

151. Poisson's Ratio.—When a bar is subjected to direct stress, either tensile or compressive, there is not only a longitudinal strain, but also

a transverse strain. If a bar of length l, breadth b, and thickness t be loaded, say in tension, in the direction of its length, the length will increase by an amount x, the breadth will decrease by an amount y, and the thickness will decrease by an amount z. The longitudinal strain is x/l, and the transverse strain is y/b or z/t.

It is found that, within the elastic limit, the ratio of the transverse strain to the longitudinal strain is constant for any given material, and this constant ratio is generally called *Poisson's ratio*. In this work Poisson's ratio is denoted by σ; thus $\sigma = \frac{\text{transverse strain}}{\text{longitudinal strain}}$. For metals σ is generally between $\frac{1}{4}$ and $\frac{1}{3}$, but values as low as 0·22 and as high as 0·45 are given by different authorities. For india-rubber, σ is about $\frac{1}{2}$.

152. Relations between σ, E, C, and K.—Let a cube ABCD (Fig. 220), whose edges are of length l, be subjected to tensile stress of intensity f on the faces AD and BC, and all interfaces parallel to them. Also let the faces AB and DC, and all interfaces parallel to them, be subjected to a crushing stress of intensity f. The cube will assume the rectangular shape A′B′C′D′.

The tension will cause a strain in the direction EG amounting to f/E, and the compression will increase this strain by the amount $\sigma f/E$. Therefore the total strain $\frac{x}{l} = \frac{f}{E}(1 + \sigma)$.

According to Art. 142, equal tensile and compressive stresses acting at right angles to one another, as in this case, are equivalent to shear stresses of the same intensity on planes inclined at 45° to the directions of the tensile and compressive stresses. If, therefore, a rectangular solid

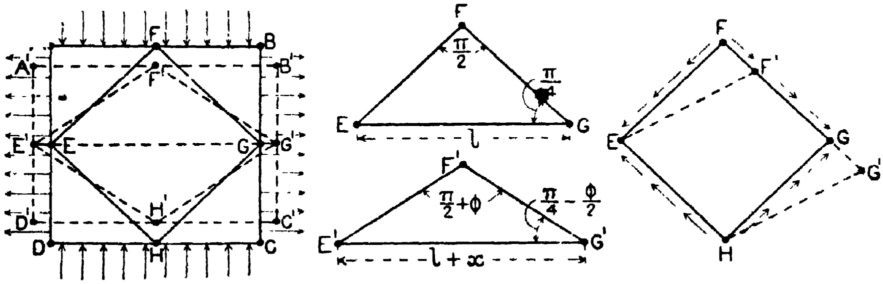


FIG. 220.

be formed whose front face is the square EFGH, having its angular points at the middle points of the sides of the square ABCD, the faces of this solid, which are perpendicular to EFGH, will be subjected to pure shear, and the intensity of the shear stress will be f. If ϕ is the angle of distortion, then $\phi = f/C$. When the block EFGH is strained it assumes the form E′F′G′H′, and the angle E′F′G′ $= \frac{\pi}{2} + \phi$. Let E′G′ $= l + x$, and let EF be denoted by a, then

$$\frac{l+x}{a} = \frac{\sin\left(\frac{\pi}{2} + \phi\right)}{\sin\left(\frac{\pi}{4} - \frac{\phi}{2}\right)} = \frac{\cos \phi}{\sin\frac{\pi}{4}\cos\frac{\phi}{2} - \cos\frac{\pi}{4}\sin\frac{\phi}{2}}$$

$$=\frac{\cos^2\frac{\phi}{2}-\sin^2\frac{\phi}{2}}{\frac{1}{\sqrt{2}}\left(\cos\frac{\phi}{2}-\sin\frac{\phi}{2}\right)}=\sqrt{2}\left(\cos\frac{\phi}{2}+\sin\frac{\phi}{2}\right).$$

But since $\frac{\phi}{2}$ is a very small angle, $\cos\frac{\phi}{2}$ may be taken $=1$, and $\sin\frac{\phi}{2}=\frac{\phi}{2}$. Therefore $\frac{l+x}{a}=\sqrt{2}\left(1+\frac{\phi}{2}\right)$. Again $\frac{l}{a}=\frac{1}{\sin\frac{\pi}{4}}=\sqrt{2}$.

Hence $\frac{x}{a}=\sqrt{2}\left(1+\frac{\phi}{2}\right)-\sqrt{2}=\frac{\phi}{2}\sqrt{2}$, and $\frac{x}{l}=\frac{\phi}{2}=\frac{f}{2C}$.

But it has already been shown that $\frac{x}{l}=\frac{f}{E}(1+\sigma)$.

Therefore $\frac{f}{2C}=\frac{f}{E}(1+\sigma)$ or $E=2C(1+\sigma)$.

If the faces AG and DH of a cube AH (Fig. 221) be subjected to a normal pressure of intensity p, the edge AD will be shortened, and the strain produced will be p/E. If, in addition, the faces AE and BH be subjected to a uniform normal pressure of intensity p, the edge AD will be lengthened, and the strain produced in AD by the pressures on AE and BH will be $\sigma p/E$.

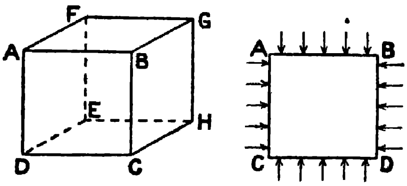


Fig. 221.

In like manner, normal pressures of intensity p on the faces AC and FH will lengthen the edge AD, and the strain in the direction AD, due to the pressures on AC and FH, will be $\sigma p/E$.

When all the faces are subjected to normal pressure of intensity p, it follows that the strain produced in the direction AD will be

$$\frac{p}{E}-\frac{2\sigma p}{E}=\frac{p}{E}\left(1-2\sigma\right).$$

But the strain in the direction of each edge will be the same, and the volume strain will be three times the above linear strain (Art. 81), therefore volume strain $=\frac{3p}{E}(1-2\sigma)$. But volume strain $=\frac{p}{K}$. Therefore $\frac{3p}{E}(1-2\sigma)=\frac{p}{K}$ or $E=3K(1-2\sigma)$.

From the equations $E=2C(1+\sigma)$ and $E=3K(1-2\sigma)$, the following relations are easily obtained:—

$$E=\frac{9CK}{C+3K}. \qquad C=\frac{3EK}{9K-E}. \qquad K=\frac{CE}{9C-3E}.$$

$$\sigma=\frac{E-2C}{2C}=\frac{3K-E}{6K}=\frac{3K-2C}{2C+6K}.$$

It is evident that if any two of the four quantities, E, C, K, and σ, be found by experiment, the other two can be calculated.

153. Thick Hollow Cylinders.—In Art. 90, p. 76, it was shown that if a thin cylindrical shell of diameter d and thickness t be subjected to an internal pressure of intensity p, the material of the shell is subjected to a tensile stress of intensity f, given by the equation $pd=2tf$. Inserting twice the radius r instead of the diameter d, then $pr=tf$. It is evident that so long as the thin shell remains circular the relation $pr=tf$ will hold if the pressure p be transferred to the outside of the shell,

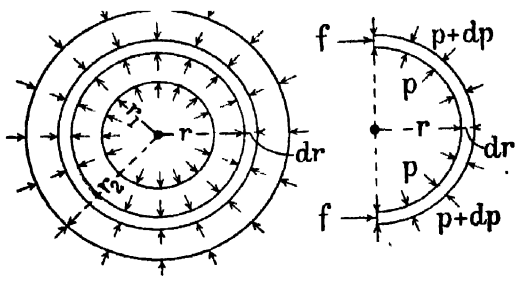


FIG. 222.

but the stress f will become compressive instead of tensile. The assumption made in proving this relation was, that the stress is uniformly distributed over the longitudinal section of the shell. This assumption is justified for a thin shell, but it cannot be used in the case of a thick hollow cylinder.

Let a thick hollow cylinder (Fig. 222) have an internal radius r_1 and an external radius r_2, and let there be an internal pressure of intensity p_1 and an external pressure of intensity p_2.

In what follows a thrust or compression will be considered as positive, and a pull or tension as negative. It will be convenient to suppose that the cylinder is in compression, the external pressure being greater than the internal pressure, but the results obtained will be of general application, the formulæ making the stress negative when the internal pressure is greater than the external pressure.

Consider a portion of the cylinder of unit length, and take an intermediate indefinitely thin ring of it of internal radius r and thickness dr. Let the internal radial pressure on this ring be p, and the external pressure $p+dp$.

Considering the equilibrium of this ring, if the external pressure $p+dp$ acted alone, the stress f produced would be given by the equation $(p+dp)(r+dr)=fdr$, and if the internal pressure acted alone, then $pr=fdr$. Hence, when both pressures act at the same time $(p+dp)(r+dr)-pr=fdr$, which reduces to $pdr+rdp=fdr$, which is one relation between p, f, and r.

Another relation involving p and f is found from a consideration of the strains produced by p and f in the direction of the axis of the cylinder. The pressure p will produce a strain in the direction of the thickness of the ring equal to p/E, and a strain in the direction of its axis equal to $\sigma p/\mathrm{E}$. The stress f will produce a strain in the material of the ring, in the direction in which it acts, equal to f/E, and a strain in the direction of the axis equal to $\sigma f/\mathrm{E}$. Hence the total strain in the direction of the axis due to p and f is $\sigma p/\mathrm{E}+\sigma f/\mathrm{E}$, and this strain will be uniform throughout, because it is reasonable to suppose that plane sections perpendicular to the axis will remain plane. If, therefore, $\sigma p/\mathrm{E}+\sigma f/\mathrm{E}$ is constant, $p+f$ is constant. Let $p+f=2a$.

From the equation $p+f=2a$, $f=2a-p$; substituting this value of f in the equation $pdr+rdp=fdr$, it follows that $2pdr+rdp=2adr$. Multi-

plying both sides by r, $2prdr + r^2dp = d(pr^2) = 2ardr$. Integrating $pr^2 = 2a\int rdr = 2a\frac{r^2}{2} + c = ar^2 + c$, where c is a constant of integration.

When $r = r_1$, $p = p_1$, therefore $p_1r_1^2 = ar_1^2 + c$.

" $r = r_2$, $p = p_2$, " $p_2r_2^2 = ar_2^2 + c$.

From these equations $a = \frac{p_2r_2^2 - p_1r_1^2}{r_2^2 - r_1^2}$, and $c = \frac{(p_1 - p_2)r_1^2r_2^2}{r_2^2 - r_1^2}$.

From the equations $pr^2 = ar^2 + c$, and $f = 2a - p$, and the values of a and c just determined,

$$f = \frac{p_2r_2^2 - p_1r_1^2}{r_2^2 - r_1^2} - \frac{(p_1 - p_2)r_1^2r_2^2}{(r_2^2 - r_1^2)r^2}.$$

Putting $r = r_1$, f becomes f_1, the stress at the inside surface,

$$f_1 = \frac{2p_2r_2^2 - p_1(r_1^2 + r_2^2)}{r_2^2 - r_1^2}.$$

Putting $r = r_2$, f becomes f_2, the stress at the outside surface,

$$f_2 = \frac{p_2(r_1^2 + r_2^2) - 2p_1r_1^2}{r_2^2 - r_1^2}.$$

The most common cases in practice are those in which p_2 is so small compared with p_1 that the former may be neglected, or $p_2 = 0$.

$$\text{If } p_2 = 0 \text{ then } f_1 = \frac{-p_1(r_1^2 + r_2^2)}{r_2^2 - r_1^2}, \quad f_2 = \frac{-2p_1r_1^2}{r_2^2 - r_1^2},$$

and

$$f = \frac{-p_1r_1^2}{r_2^2 - r_1^2}\left(\frac{r^2 + r_2^2}{r^2}\right),$$

the minus sign in front in each case showing that the stress is tensile.

The *tensile* stress f_1 has the value $\frac{p_1(r_1^2 + r_2^2)}{r_2^2 - r_1^2}$, hence it is easy to deduce

$$\frac{r_2}{r_1} = \sqrt{\left(\frac{f_1 + p_1}{f_1 - p_1}\right)}, \text{ and } \frac{t}{r_1} = \sqrt{\left(\frac{f_1 + p_1}{f_1 - p_1}\right)} - 1,$$

where t is equal to $r_2 - r_1$, the thickness of the cylinder. It is evident that for all possible values of t, f_1 must be greater than p_1.

The foregoing formulæ are known as Lamé's formulæ.

Exercises IXb.

NOTE.—*The answers given at the end of the book to Exercises 5, 6, and 8 on crane hooks were obtained on the assumption that the moment of resistance of a curved bar to bending is the same as that for a straight bar. The answers are therefore only approximate. See the latter part of Art.* 150, *p.* 155.

1. The crank shaft shown in Fig. 223 is subjected to a thrust P along its axis. Determine the magnitude of P, in tons, when the maximum tensile stress produced by it in the crank pin is 4 tons per square inch. What will then be the maximum compressive stress in the pin?

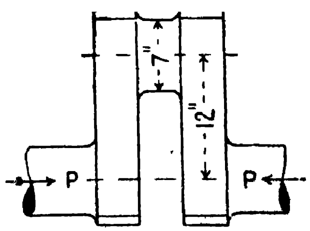


FIG. 223.

2. A link (Fig. 224) of rectangular section, 3 inches deep and 2 inches wide, is subjected to tension by a load P, the line of action of which is parallel to the axis of the central part of the link, and at a distance x above it. When $x = \frac{1}{4}$ inch, find P, if the maximum tensile stress in the link is 5 tons per square inch. Find also

the maximum value of x so that there shall be no crushing stress in the link.

3. A short cylindrical block of diameter d is subjected to crushing by a load whose line of action is parallel to the axis of the block, and at a distance x from it. Find the maximum value of x, in terms of d, so that there shall be no tensile stress in the block.

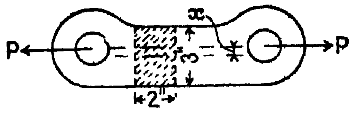


FIG. 224.

4. Show that there will be no tensile stress at the joints of a masonry arch if the resultant thrust between two adjacent blocks falls within the middle third of the joint.

5. The crane hook shown in Fig. 225 has at the level AB a circular cross section $2\frac{1}{2}$ inches in diameter, and k the distance of B from the line of action of the load W is 2 inches. Find W, in tons, when the tensile stress at B is 4 tons per square inch. Find also the nature and intensity of the stress at A.

6. Referring to the crane hook shown in Fig. 225, if d is the diameter of the section at AB, and $k=\frac{3}{4}d$, find d in terms of W, where W is in tons and the tensile stress at B is 5 tons per square inch. What is the nature and intensity of the stress at A?

7. A beam 3 inches deep and 2 inches wide projects from a wall and carries a load P, which acts as shown in Fig. 226. The maximum tensile stress in the

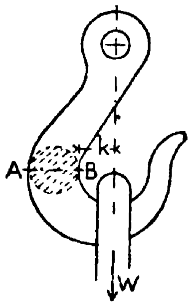


FIG. 225.

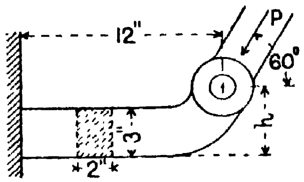


FIG. 226.

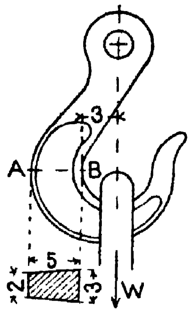


FIG. 227.

beam is to be 10,000 lbs. per square inch. Find the magnitude of P, in lbs., (*a*) when $h=1\frac{1}{2}$ inches; (*b*) when $h=4\frac{1}{2}$ inches.

8. In the crane hook shown in Fig. 227 the dimensions are given in terms of an unknown unit, and the section at AB is to be assumed as having the form shown in the lower part of the figure. Determine the unit for the dimensions in terms of W the load in tons, so that the total tensile stress at B shall be 5 tons per square inch.

9. A built-up crane jib is in the form of a curved girder, and a horizontal section near the base is a hollow rectangle. The outside dimensions of the rectangle are 54 and 36 inches, and the longer and shorter sides are 1 inch and 2 inches thick respectively. Find the maximum tensile and compressive stresses induced in the material when a load of 25 tons is suspended from the end of the crane, the horizontal distance of the load from the centre of the section being 50 feet. Show by a sketch how the intensity of stress varies across the section. [U.L.]

10. A cylindrical masonry column of diameter d feet is subjected to a horizontal force due to the wind pressure. Prove that no tensile stress will be produced in the basal cross section of this column, if the resultant of the wind pressure and the weight of the column fall inside a circle concentric to the circular cross section of the column, and of diameter $=\frac{1}{4}$ of the diameter of the column. Hence find the greatest height to which a column of granite 6 feet in diameter can be safely built, if the maximum intensity of the horizontal wind pressure is 40 lbs. per square foot, the weight of the granite being 160 lbs. per cubic foot.

Note.—You may assume that the wind pressure on the cylindrical surface is equivalent to a pressure of half the given intensity on an area equal to the diameter of the column multiplied by its height. [U.L.]

11. Using the method of Art. 150, p. 154, show that if the cross section of the ring is a square whose sides are equal to s, then $s = \frac{1}{2}\sqrt{\frac{W}{f}(6n+1)}$.

12. A steel bar 20 feet long and 2 inches square is elongated by a load of 20,000 lbs. Find by how much the volume of the bar is increased. Take $\sigma = 0{\cdot}27$ and $E = 30{,}000{,}000$ lbs. per square inch.

13. A piece of cast iron 5 inches long and 1 inch square is subjected to compression in the direction of its length by a load of 10 tons. What must be the intensity, in tons per square inch, of the lateral external pressure which must be applied to the piece to prevent lateral strain, and what will then be the alteration in length? Take $\sigma = 0{\cdot}25$ and $E = 7000$ tons per square inch.

14. The shell of a cylindrical steam boiler is 8 feet 4 inches in diameter and 1 inch thick, and the effective steam pressure is 200 lbs. per square inch. Find the circumferential strain produced, (*a*) neglecting the longitudinal strain, (*b*) taking the longitudinal strain into account. Take $\sigma = 0{\cdot}26$ and $E = 30{,}000{,}000$ lbs. per square inch.

15. A bar of mild steel 1 inch in diameter twists through an angle of 2·2 degrees in a 20-inch length when subjected to a twisting moment of 2200 inch-lbs. An exactly similar bar of the same material deflects 0·03 inch when supported horizontally at two points 20 inches apart and loaded at the centre of the span with a load of 264 lbs. Calculate the Modulus of Elasticity (E), Modulus of Transverse Elasticity (C), Modulus of Elasticity of Bulk (K), and Poisson's ratio for the material. [U.L.]

16. The cylinder of an hydraulic press has an internal diameter of 10 inches, and the water pressure is 1500 lbs. per square inch. Find the thickness of the metal of the cylinder so that the maximum stress may be 2500 lbs. per square inch. What will be the stress in the metal at the outside of the cylinder?

17. Find the safe internal pressure, in lbs. per square inch, for the cylinder of an hydraulic press which has an internal diameter of 5 inches and an external diameter of 7 inches, when the maximum safe stress is 3000 lbs. per square inch.

18. The internal diameter of a thick hollow cylinder is $5\frac{1}{2}$ inches, the internal pressure is 1120 lbs. per square inch, and the maximum stress produced is 3000 lbs. per square inch. Find the thickness of the metal.

19. The internal and external radii of a thick hollow cylinder are 5 inches and 9 inches respectively. If the tensile stress in the metal at the inner surface of the cylinder is 3000 lbs. per square inch, what is the internal pressure? Calculate the values of the tensile stress in the metal, in lbs. per square inch, at radii 6, 7, 8, and 9 inches, and plot the stresses on the thickness of the cylinder as a base. Linear scale, full size. Stress scale, 1 inch to 1000 lbs. per square inch.

20. In a thick hollow cylinder, show that if the stress f at radius r is inversely proportional to r, or that $fr^2 = $ constant, then $p_1 r_2^2 = f_1 t$ (*Barlow's formula*), where p_1 is the internal pressure, f_1 the stress at the inner surface, r_2 the external radius, and t the thickness of the metal. The external pressure is assumed to be zero.

21. The internal pressure in a thick hollow cylinder is 3 tons per square inch. The internal and external radii of the cylinder are 5 inches and 10 inches respectively. Calculate the values of the stress at radii 5, 6, 7, 8, 9, and 10 inches, (*a*) by Lamé's formula; (*b*) by Barlow's formula (see preceding exercise). Plot the results. Linear scale, full size. Stress scale, 1 inch to 2 tons per square inch.

22. Same as preceding exercise, except that there is an initial stress in the material which varies uniformly from 2 tons per square inch compression at the inner surface to 2 tons per square inch tension at the outer surface.

[redacted]

CHAPTER X

COLUMNS AND STRUTS

154. Columns and Struts.—A *Column* or *Pillar* is always vertical, and generally it is fixed rigidly at its ends. A *Strut* may be vertical or inclined, and one or both ends may be fixed rigidly, or one or both ends may be connected to the surrounding structure by flexible joints. The theory of struts will therefore evidently apply to columns. In most cases the only important load on a column or strut is one acting at its ends in the line of its axis and tending to shorten it. In some cases, however, there is a lateral load in addition.

Comparing a strut with a tie (Figs. 228 and 229), it is evident that if the strut and tie be bent by lateral forces or if they be originally bent, the load P on the strut tends to bend it still further, while the load P on

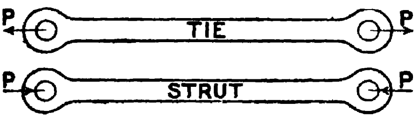


FIG. 228.

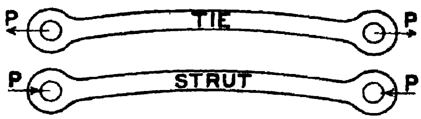


FIG. 229.

the tie tends to straighten it. The theory of a tie is obviously very simple, being expressed by the equation $P = Af$, where A is the area of the cross section and f the stress. The theory of the strut would be the same as that of the tie if the strut did not bend. Neglecting for the present the case of the laterally loaded strut, the load at its ends may bend the strut, because (1) the strut may not be perfectly straight when unloaded, (2) the load may not be applied exactly in the line of the axis of the strut, and (3) through a want of uniformity in the material one part may yield more than the other parts in compression.

The tendency of the strut to bend will evidently be greater the greater the ratio r of its length to its least transverse dimension, and the manner in which it gives way under the load will depend largely on the value of r. When r is small there is no bending, and the strut gives way by crushing (or oblique shearing, as described in Art. 166, p. 175). When r is very large the strut gives way by bending, and for moderate values of r the strut may give way by crushing and bending.

155. Critical Load for Long Column.—A simple and instructive experiment on the behaviour of a long column will here be described. A long slender lath of wood or a straight strip of steel is placed in a vertical position and loaded, the load being guided vertically, as shown in Fig. 230. The ends of this experimental strut are rounded and fit into shallow grooves in the end connections, as shown on an enlarged scale

at (*a*). A load insufficient to bend the strut is applied, and the strut is then slightly bent by pressing it sideways at the middle of its length. When this side pressure is removed the strut straightens itself. A small addition is made to the load and the side pressure is again applied and removed, the strut bends and straightens again. The experiment is continued in this way, gradually increasing the load until it is found that when the side pressure is applied and then removed the strut remains bent. When this point is reached it will be found that, whatever amount of deflection be given to the strut by the applied lateral force, the strut will retain that amount of deflection when the lateral force is removed, provided that the elastic limit is not exceeded. But if the load on the strut be further increased and the strut be slightly bent as before, the load will increase the amount of bending until the strut takes a permanent set or collapses. This load, which will keep the strut bent but will not bend it further, may be called the *critical load* for the strut.

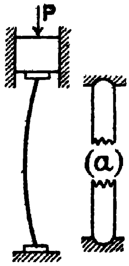


FIG. 230.

That there is a critical load for a long slender column may be demonstrated as follows. Let the strut shown at (*a*), Fig. 231, be in equilibrium under the load P_1 and lateral force Q_1, the deflection being u_1. Let Z be the modulus of the cross section, and let f_1 be the maximum stress due to bending.

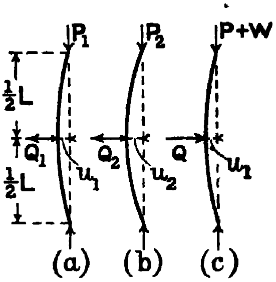


FIG. 231.

The total bending moment is $P_1u_1 + Q_1\frac{L}{4}$,

and the moment of resistance to bending is f_1Z, therefore $P_1u_1 + Q_1\frac{L}{4} = f_1Z$.

If Q_1 be now diminished to zero and P_1 be increased to P, the deflection u_1 remaining the same, it follows that $Pu_1 = f_1Z$ or $P = \frac{f_1}{u_1}Z$.

If the same strut be in equilibrium under the load P_2 and lateral force Q_2, as shown at (*b*), Fig. 231, the deflection being u_2, then decreasing Q_2 to zero, and increasing P_2 to P', the deflection u_2 remaining the same, it follows that $P' = \frac{f_2}{u_2}Z$, where f_2 is the maximum stress due to bending.

But $\frac{f_1}{u_1} = \frac{f_2}{u_2}$, therefore $P' = P$.

From the foregoing it follows that the strut will be in equilibrium under the load P for *any* deflection within the elastic limit.

For a load greater than P the force Q, acting as shown at (*c*), Fig. 231, would be necessary to prevent further deflection of the strut, because the bending moment due to P + W is $Pu_1 + Wu_1$, and the moment of resistance of the strut to bending is f_1Z, but $Pu_1 = f_1Z$, therefore the bending moment Wu_1 must be balanced by the moment of Q, hence $Q\frac{L}{4} = Wu_1$. Within the elastic limit, fZ, the moment of resistance of the

strut to bending will be proportional to f, and therefore proportional to the deflection. But when the elastic limit is passed, the moment of resistance will increase more rapidly (see Art. 116, p. 109); and with the additional load W there may be another position of equilibrium for the strut, but the strut will then have taken a permanent set.

156. Approximate Theory of Long Columns.—ACB (Fig. 232) represents a long slender column which, when unloaded, is perfectly straight, of uniform cross section, and uniform elasticity. The ends are supposed to be rounded so that the column is free to bend throughout its whole length. The critical load P is supposed to act in a fixed line which coincides with the axis of the column when the latter is unloaded.

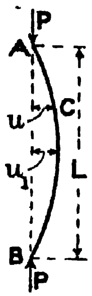


FIG. 232.

Let the loaded column be slightly bent by the application for an instant of a lateral force. At any point C there is a bending moment equal to Pu, where u is the deflection at C, and it follows that the figure ACBA is the bending moment diagram for the whole column. It is evident that the curve ACB cannot be an arc of a circle, because that would necessitate the bending moment being uniform throughout the whole length of the column. If the curve ACB be assumed to be a parabola, then the deflection of the column is the same as it would be if the column became a beam, supported at its ends, with a transverse load uniformly distributed over its length. In Art. 123, p. 114, it was shown that for a beam of length L and uniform section, supported at its ends and loaded uniformly with a total load W, the deflection u_1 at the centre is $\frac{5\text{WL}^3}{384\text{EI}}$.

If M is the bending moment at the centre of the beam, then $\text{M} = \frac{\text{WL}}{8}$ hence $u_1 = \frac{5\text{ML}^2}{48\text{EI}}$, but for the column $\text{M} = \text{P}u_1$, therefore

$$u_1 = \frac{5\text{P}u_1\text{L}^2}{48\text{EI}}, \text{ and } \text{P} = \frac{48\text{EI}}{5\text{L}^2} = \frac{9{\cdot}6\text{EI}}{\text{L}^2}.$$

By the more exact theory of Euler, discussed in the next Article, $\text{P} = \frac{9{\cdot}87\text{EI}}{\text{L}^2}$.

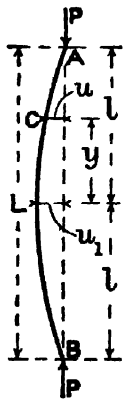


FIG. 233.

157. Euler's Theory of Long Columns.—The column ACB (Fig. 233) is supposed to be under exactly the same conditions as the column considered in the preceding Article. At any point C in the column, at a distance y from the middle point of AB, the bending moment M is equal to Pu. If R is the radius of curvature of the column at C, then by Arts. 109 and 110, pp. 103–105, $\frac{1}{\text{R}} = \frac{\text{M}}{\text{EI}}$. But $\frac{1}{\text{R}} = -\frac{d^2u}{dy^2}$ (see Art. 9, p. 9), the minus sign being used to make R positive, because as u increases $\frac{du}{dy}$ decreases. Hence $\frac{\text{P}u}{\text{EI}} = -\frac{d^2u}{dy^2}$. The general solution of this differ-

ential equation gives $u = A \sin ny + B \cos ny$, where $n^2 = \frac{P}{EI}$, and A and B are constants.

Differentiating, $\frac{du}{dy} = An \cos ny - Bn \sin ny$.

When $y = 0$, $\frac{du}{dy} = 0$, also $\sin ny = 0$; therefore $A = 0$, and $u = B \cos ny$.

But when $y = 0$, $u = u_1$, and $\cos ny = 1$, therefore $u_1 = B$.

When $y = l$, $u = 0$, therefore $B \cos nl = 0$, hence either $B = 0$ or $\cos nl = 0$. But $B = u_1$, therefore $\cos nl = 0$, hence $nl = \frac{\pi}{2}$ and $l\sqrt{\frac{P}{EI}} = \frac{\pi}{2}$.

Therefore $P = \frac{\pi^2 EI}{4l^2} = \frac{\pi^2 EI}{L^2}$.

158. Influence of End Connections on Strength of Ideal Columns. —If the ends of the ideal column ACB are fixed as shown at (*a*), Fig. 234, then when the column deflects the directions of the tangents to the curve ACB at A, C, and B must be vertical, and the line of action of the resultant load on the column will no longer pass through the centres of its ends, but must lie between the point C and the straight line AB, and will cut the curve ACB at points H and K. At the points H and K there is no bending moment, and these must therefore be points of contrary flexure.

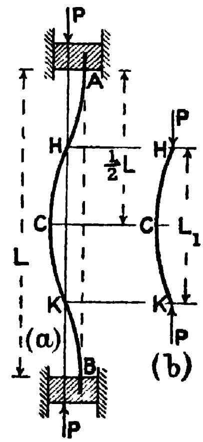


FIG. 234.

Consider the parts HA and HC of the bent column. At points in HA and HC where the deflections, measured from the vertical line through H, are equal, the bending moments are equal, and therefore the radii of curvature at these points must be equal, the column being of uniform cross-section. Also the curves have the same slope at H, and also the same slope at A and C. Hence it is evident that the curves HA and HC are similar and equal, and that the points H, C, and K divide the column into four equal parts. Hence the part HCK has a length equal to half the length of the whole column. Now the part HCK is in the condition of a column with rounded or hinged ends carrying the load P, as shown at (*b*), Fig. 234. Therefore, from the preceding Article, $P = \frac{\pi^2 EI}{L_1^2} = \frac{4\pi^2 EI}{L^2}$. Hence a column fixed at the ends is four times as strong as the same column with hinged or rounded ends.

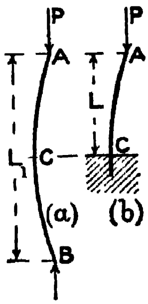


FIG. 235.

The formula for the strength of an ideal column fixed at one end and loaded at the other is easily deduced from that for the column with rounded ends. A column ACB with rounded ends is shown at (*a*), Fig. 235. At C, the middle point of its length, the tangent to the curve is vertical, and if the column be held in a clamp at C, so as to preserve the direction of the tangent to the curve at that point, the lower part of the column might be removed without affecting the upper part. The upper part will then

become a column fixed at one end and loaded at the other, as shown at (*b*), Fig. 235, where $L = \frac{1}{2}L_1$. Hence, $P = \frac{\pi^2 EI}{L_1^2} = \frac{\pi^2 EI}{4L^2}$, which shows that a column fixed at one end and loaded at the other has only one-fourth the strength of the same column with hinged or rounded ends.

159. Empirical Formulæ for Struts.—The Euler formulæ for struts is rational, but it is only applicable to struts which are very long compared with their least transverse dimension, and when applied to struts which are common in practice they give values which are too high for their strength. Numerous empirical formulæ have been devised by different authorities to give the strength of ordinary struts, the constants or coefficients in these formulæ being derived from the results of experiments on struts. The empirical formula which has been most used is that known as the *Rankine* or *Rankine-Gordon* formula.

160. Rankine-Gordon Formula for Struts.—The Rankine-Gordon formula is $p = \frac{P}{A} = \frac{f}{1 + a\left(\frac{L}{k}\right)^2}$, where

P = crushing or crippling load on strut in tons.
p = crushing or crippling load on strut in tons per square inch of cross section.
f = direct crushing strength of the material of the strut in tons per square inch.
A = area of cross section of strut in square inches.
L = length of strut in inches.
k = least radius of gyration of section of strut in inches.
a = constant.

Values of f and a commonly taken for different materials in different cases are given in the following table:—

Material.	f	Values of a. Case I.	Case II.	Case III.
Cast-iron . . .	36	$\frac{1}{6,400}$	$\frac{4}{6,400} = \frac{1}{1,600}$	$\frac{16}{9 \times 6,400} = \frac{1}{3,600}$
Wrought-iron . .	16	$\frac{1}{36,000}$	$\frac{4}{36,000} = \frac{1}{9,000}$	$\frac{16}{9 \times 36,000} = \frac{1}{20,250}$
Mild steel . . .	21	$\frac{1}{30,000}$	$\frac{4}{30,000} = \frac{1}{7,500}$	$\frac{16}{9 \times 30,000} = \frac{1}{16,875}$
Dry timber (strong kinds)	3·2	$\frac{1}{3,000}$	$\frac{4}{3,000} = \frac{1}{750}$	$\frac{16}{9 \times 3,000} = \frac{1}{1,687}$

Case I. Fixed ends. Case II. Hinged ends. Case III. One end fixed and the other hinged.

It will be observed that the values of a for Cases II. and III. are obtained by multiplying the values for Case I. by 4 and $\frac{16}{9}$ respectively.

Most text-books give a so-called proof of the Rankine-Gordon formula, but the assumptions made are not warranted in the case of actual struts. It is best to consider the formula as an empirical one, but it may be pointed out that when it is applied to very short struts it reduces to $P = Af$, which is correct, and when applied to very long struts it reduces to $P = \frac{Afk^2}{aL^2} = \frac{fI}{aL^2}$, which is the form of the expression given by Euler's theory of long columns.

Fig. 236 shows graphically the difference between the Rankine-Gordon and the Euler formulæ applied to mild steel columns with hinged ends.

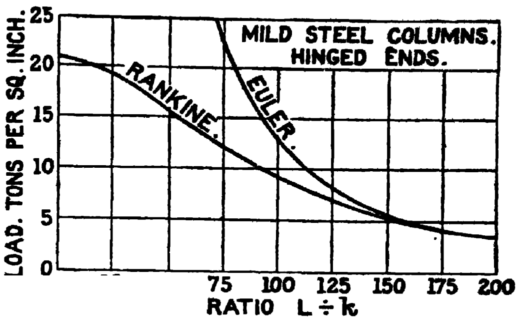


FIG. 236.

If the results of the Rankine-Gordon formula be plotted for columns of different materials, instructive curves are obtained. Fig. 237 shows the curves for columns of cast-iron, wrought-iron, and mild steel with fixed ends. It will be seen that for very short columns cast-iron is much stronger than either wrought-iron or mild steel, and that for ratios of L to k less than 85 the cast-iron columns are the stronger, but for ratios of L to k greater than 85 the mild steel columns are the stronger. Also for ratios of L to k less than 115 the cast-iron columns are stronger than the wrought-iron ones, but when the ratio of L to k is greater than 115 the wrought-iron columns are stronger than the cast-iron ones. For all values of the ratio of L to k the mild steel columns are stronger than those made of wrought-iron.

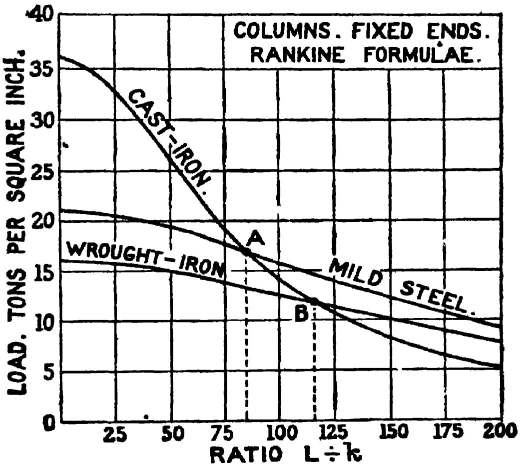


FIG. 237.

NOTE.—A defect in the theory of Arts. 155–158 is that the direct crushing stress is neglected; but in very long columns the direct stress is small compared with the bending stress, while in short columns the direct stress is large compared with the bending stress.

Exercises X.

1. A straight bar of steel 40 inches in length, 1 inch broad, and $\frac{1}{16}$ inch in thickness, is bent into the form of a bow, having an elastic deflection of 2 inches in the middle, and the ends are united by the bow string. Taking the modulus E=29,000,000 lbs. per square inch, what will be the tension on the string? [Inst.C.E.]

2. Apply the Rankine-Gordon formula to find the buckling load for a cast-iron column 8 inches external diameter, 6 inches internal diameter, and 20 feet long. The column has fixed ends.

3. A strut in a framed structure is formed of a steel pipe 6 inches external diameter, and ½ inch thick; it is 10 feet long, and has pin connections at each end. With a factor of safety of 5, to what load may it be submitted? [Inst.C.E.]

4. Determine the value of the ratio L/k for which columns of cast-iron and mild steel of the same cross section, with hinged ends, have the same strength by the Rankine formula.

5. Plot crippling load, in tons per square inch, and ratio L/k, for cast-iron columns with hinged ends, (*a*) by the Rankine-Gordon formula, (*b*) by Euler's formula, up to $L/k=200$. Take E=6000 tons per square inch.

6. Find the proper diameter for a solid mild steel strut with pin ends, if its length is 120 inches, and if it has to carry a total load of 20 tons with a factor of safety of 7. Take the values of f and a (for flat ends) in the formula as 30 tons per square inch, and $\frac{1}{30000}$ respectively. [U.L.]

7. Three solid cast-iron columns, each 3 inches in diameter, and 10 feet long, have their ends fixed, and each carries one-third of a load W. Find by the Rankine-Gordon formula the diameter of a single solid cast-iron column with fixed ends to replace these three columns, and find the percentage saving in weight of cast-iron.

8. A hollow cast-iron column has to be designed to support a total load of 130 tons; the column is to be 15 feet in length, and the internal diameter is to be four-fifths of the external diameter, and it is desired to have a factor of safety 10. Find the external and internal diameters of the column if the crushing load in tons per square inch for such a column is given by the formula

$$\text{Crushing load in tons per square inch} = \frac{36}{1+\frac{1}{800}\left(\frac{L}{D}\right)^2},$$

where L is the length of the column in inches, and D is the external diameter in inches. [B.E.]

9. A hollow cast-iron column 20 feet long, with ends rigidly fixed, has to carry safely a load of 50 tons. If the factor of safety is 6, and the external diameter of the column is 8 inches, find the internal diameter.

10. A steel strut is built up of two **T** section bars (Fig. 238) riveted back to back; the **T**'s are of the following section: 6 inches × 4⅜ inches × ⅝ inch. The ends of the strut are rigidly secured, and its over-all length is 18 feet 6 inches. What gross load can this strut carry if it is to have a factor of safety 5? By the Rankine formula the buckling load in lbs. per square inch of such a strut $=\dfrac{48{,}000}{1+\frac{1}{30000}\left(\frac{L}{k}\right)^2}$, where L is the length of the strut in inches, and k the least radius of gyration of the section. [U.L.]

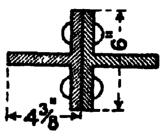


FIG. 238.

11. A column is built up of an **I** rolled joist 20 inches deep; flanges 7½ inches wide, and 1 inch thick; web ⅝-inch thick; with two ⅝-inch plates 12 inches wide riveted to each flange (Fig. 239). Find the least radius of gyration of the section. Taking a factor of safety of 6, find the working load if the column is 10 feet high with fixed ends. Use the formula $P=\dfrac{25A}{1+\frac{\lambda^2}{30000}}$, where P is the buckling load in tons, A the cross sectional area in square inches, and λ is the ratio of the length to the least radius of gyration. [Inst.C.E.]

FIG. 239.

12. A vertical mild steel post in a bridge consists of a central plate 18 inches × $\frac{3}{8}$ inch, a pair of angles 4 inches × 4 inches × $\frac{1}{2}$ inch at each end of the plate, and a plate 12 inches wide by $\frac{1}{2}$ inch thick connected to each pair of angles so as to form an I section (Fig. 240). Its length is 20 feet, and each end is firmly riveted to the flanges of the main girder. Find the safe load by Rankine's formula. [Inst.C.E.]

FIG. 240.

13. A cast-iron column 10 feet long is fixed at the ends and has an I section 6 inches deep, with flanges 5 inches wide and web 1 inch thick. Find the thickness of the flanges so that with a factor of safety of 6 this column will carry safely a load of 35 tons. Use the Rankine-Gordon formula.

14. If P denotes the buckling load of a column by Euler's formula, and W denotes the buckling load of the same column by the Rankine-Gordon formula, also if F denotes the crushing load of a very short column of the same section and same material, show that $W = \frac{PF}{P+F}$ if the constant a in the Rankine-Gordon formula is equal to $\frac{f}{4\pi^2 E}$ for a column with fixed ends.

15. A hollow cylindrical steel strut has to be designed for the following conditions. Length 6 feet, axial load 12 tons, ratio of internal to external diameter 0·8, factor of safety 10. Determine the necessary external diameter of the strut and the thickness of the metal, if the ends of the strut are firmly built in. Use the Rankine-Gordon formula, taking $f = 21$ tons per square inch, and a for rounded ends $= 1/7500$. [U.L.]

CHAPTER XI

BEHAVIOUR OF MATERIALS IN THE TESTING MACHINE

161. Nominal and Actual Stresses.—When a specimen of cross sectional area a is placed in simple tension or simple compression by a load P, then f, the direct stress produced, is given by the equation $f = \mathrm{P}/a$. The effect of the load is not only to alter the length of the specimen, but also its cross sectional area, and if in the above equation a is the *original area* of the cross section, then f is the *nominal stress* produced by the load P. If however a is the *actual area* of the cross section when the load P is on the specimen, then f is the *actual stress* produced.

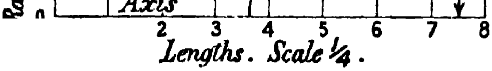


FIG. 241.

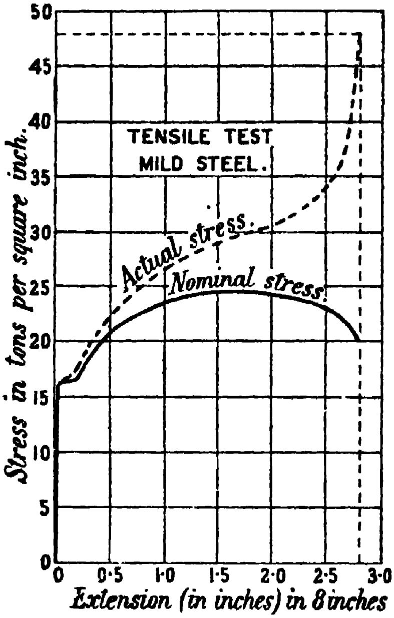


FIG. 242.

For loads within the elastic limit of the specimen the difference between the original and actual areas of the cross section is so small that it may be neglected. Beyond the elastic limit however, in the case of ductile materials, the actual area of the cross section of the specimen may differ considerably from the original area, and the actual stress may be much greater than the nominal stress when the specimen is in tension, and much less when in compression. The foregoing remarks are well illustrated by Figs. 241, 242, and 243, which show the results of two tests, one a tensile test on a specimen of mild steel, and the other a compression test on a specimen of wrought-iron. The full curves in Figs. 242 and 243 show the relation between the nominal stress and the strain produced, while the dotted curves show the relation between the actual stress and the strain produced. The specimens were originally

parallel, but they did not remain so, as the load was increased beyond the elastic limit, and in each case the actual area of the cross section is taken as the area of the smallest section. The original and final outlines of the mild steel specimen tested in tension are shown in Fig. 241. The change in the form of the wrought-iron specimen in compression as the load is increased is clearly shown in Fig. 243.

Referring to Fig. 242, it will be observed that the maximum nominal

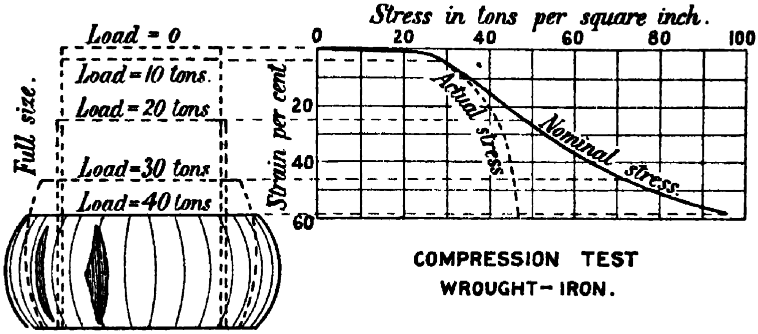


FIG. 243.

stress is greater than the nominal stress at fracture, but the actual stress at fracture is much greater than the maximum nominal stress.

In reports on tensile tests for commercial purposes, the maximum nominal stress is sometimes called the ultimate or breaking stress; this however is wrong, and it should either be called the maximum nominal stress, or the maximum stress on the original area. When the term maximum stress is used, maximum nominal stress is generally understood.

There is no definite ultimate crushing stress for ductile materials, such as mild steel and wrought-iron, but it will be seen from Fig. 243 that the curve for the actual stress gets more nearly parallel to the strain axis as the load is increased.

162. Relation of Elongation to Dimensions of Test Piece.—A study of the following results of a test of a specimen of mild steel will lead to

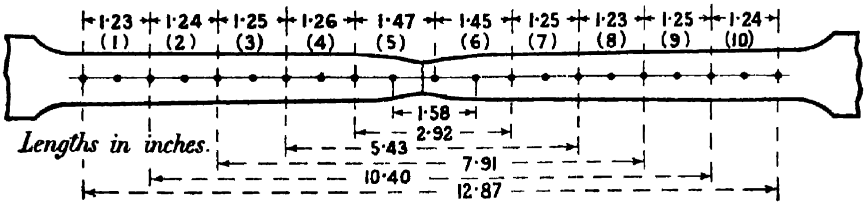


FIG. 244.

some important conclusions. Before testing, the specimen was marked off carefully in half-inch lengths, and after being broken in tension the two parts were put together, and the lengths given below the specimen in Fig. 244 were measured. The shortest of these lengths, 1·58 inches, is

the altered length of a 1 inch length on the specimen before testing, and this length contains the fracture near its centre. The other lengths, in order, are the altered lengths of 2, 4, 6, 8, and 10 inch lengths respectively on the specimen before testing, and the fracture is approximately in the middle of each of these lengths. The altered lengths of the original inch lengths numbered (1), (2), (3), etc., are given above the specimen in Fig. 244, and the elongations of these inch lengths are plotted in Fig. 245.

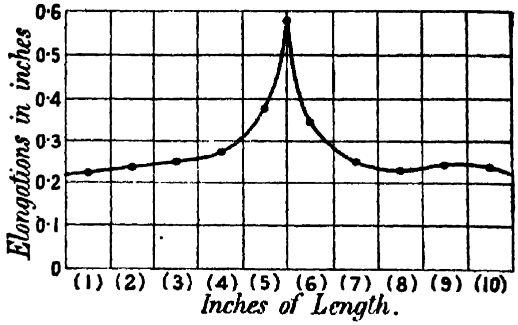


FIG. 245.

It will be seen that the elongation per inch is fairly uniform, except for about $1\frac{1}{2}$ inches on each side of the fracture, and the elongation in the immediate neighbourhood of the fracture is much greater than in any other part of the specimen.

Coming next to the actual elongations and the elongations per cent. of length in the 2, 4, 6, 8, and 10 inch lengths, which contain the fracture near their centres, the results are tabulated below.

Gauge length, inches . .	1	2	4	6	8	10
Elongation, inches . .	0·58	0·92	1·43	1·91	2·40	2·87
Elongation, per cent. . .	58·0	46·0	35·7	31·8	30·0	28·7

It will now be seen how important it is, in stating the elongation, to give the gauge length on which it is taken. It is also very important that the gauge length used in getting the elongation should contain the fracture, and if possible the gauge length should contain the fracture near its centre. For example, the elongation on the 4 inch length containing the inch lengths numbered (1) to (4) in Fig. 244 is only 24·5 per cent., while the elongation on the 4 inch length containing the fracture near its centre is 35·7 per cent.

To be able to select the gauge length so that it shall if possible contain the fracture near its centre, it is desirable that the whole length of the parallel part of the specimen be marked off in half-inch lengths.

When tests are made on specimens of the same material it is found that the elongation is influenced by the area of the cross section of the specimen as well as by the gauge length.

The elongation may be divided into two parts, one, the general, and the other, the local. The general elongation takes place over the whole length of the parallel part of the specimen, and is produced mainly before the maximum load is reached. This general elongation is practically uniform over the length, and is independent of the area of the cross section of the specimen. After the maximum load is reached, local contraction

sets in in the neighbourhood where the fracture will occur, and the subsequent elongation is mainly in this neighbourhood. But the length over which the local extension takes place is greater the larger the area of the cross section of the specimen, and is approximately proportional to the diameter, or, in the case of non-circular sections, to the square root of the area of the cross section.

Professor Unwin has shown * that within a considerable range of dimensions, the percentage elongation e, on a gauge length l, of a specimen whose cross sectional area is a, is given by the equation, $e=\frac{c\sqrt{a}}{l}+b$, where $\frac{c\sqrt{a}}{l}$ represents the local extension, and b the general extension, c and b being constants for a given material.

The following are a few values for c and b, given on the authority of Professor Unwin :—

Material.	c.	b.
Mild steel plates not very thick, average values	70	18
Gun-metal (cast)	8·3	10·6
Rolled brass	101·6	9·7
Rolled copper	84	0·8
Annealed copper	125	35

163. Position of Fracture in a Tension Test Bar.—When a test bar is gripped at the ends the outer surface of the bar at the ends first receives the tension, and this tension is transferred towards the axis of the bar by means of the longitudinal shear stresses between the different co-axial layers of material. It is therefore evident that at cross sections near the ends of the bar the tensile stress will be greatest at the outside, and that it will diminish towards the centre. But at sections further and further from the ends the distribution of the tensile stress will be more and more nearly uniform. Hence the section at which the stress is most nearly uniform will be at the centre of the length of the bar.

When a ductile material is loaded in tension it stretches, and the tendency to stretch is greatest where the stress is greatest. But where stretch occurs there must be a contraction of cross section, and the greater the tendency to stretch, the greater is the tendency to contract. Now if the stress is greater on the outside of a bar than on the inside, the tendency of the outside to contract is opposed by the inside, where the tendency to contract is less. Hence it is evident that contraction will be greatest where the stress is most nearly uniform, and this is at the centre of the length of the bar. But fracture will occur where the contraction is greatest, therefore a bar of ductile material of uniform strength should break at the centre, and this is what generally happens.

If the bar is not of uniform strength throughout it will of course tend to fracture at its weakest section, but it will not break there unless the difference between its strength at that section and its strength at the

* *Proceedings of the Institution of Civil Engineers*, vol. clv. p. 180.

centre is sufficient to counteract the tendency to break at the centre, as explained above.

In the case of brittle materials, where the contraction is negligible, the position of the fracture is either at the weakest section, or at the section on which the distribution of stress is most variable. The fracture is therefore at the weakest section, or near one end.

164. Long versus Short Tension Test Bars.—Every material is more or less variable in quality, and test pieces taken from the same piece will probably have different tenacities. In a long bar there will be one part weaker than the remainder, and if this bar is tested as a whole, it will probably fracture at the weakest part. But if a number of test bars be cut from the long bar, only one of them will contain the original weakest part. Hence the short bars will generally show a higher tenacity than long ones. A familiar illustration of the above is found in boot-laces. In tightening the lace it is more likely to break when the pull is applied at the end than when the pull is applied locally at the boot.

165. Effect of Notches and Perforations on Tenacity of Test Bars.—Reducing the cross section of a bar by notching it, as shown at (*a*) and (*b*), Fig. 246, or by perforating it, as shown at (*c*) and (*d*), will evidently fix the position of the fracture, and as the probability is in favour of this not being at the section where the material is weakest, the tenacity of the notched or perforated bar will for this reason probably be higher than that of the unaltered bar. Again, the notching or perforating will evidently disturb the distribution of the stress at the reduced section, making it less uniform, with the result that the contraction of area will be reduced. On this account, therefore, the tenacity would be increased. On the other hand, however, the notch or perforation may cause such an unequal distribution of stress that fracture may take place in consequence and the tenacity be reduced. The effect of the notch or perforation in reducing the tenacity will evidently be greater the sharper the re-entrant angle formed by the notch or perforation, and it will also be greater the more brittle the material, because a brittle material does not yield sufficiently where the stress is greatest to throw part of the stress on to the part of the bar where the stress is least.

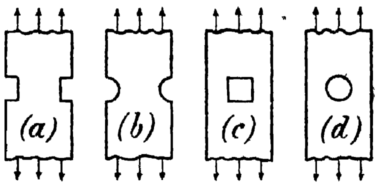


FIG. 246.

Notching or perforating a bar of mild steel raises its tenacity, but notching a piece of cast-iron lowers its tenacity very considerably.

Another important point to consider is the effect of the notch or perforation on the resilience of the bar, which is a measure of its power to resist shocks. Let l be the length of a bar of uniform cross section a. Let A be the area of the cross section of another bar of the same length, but having an indefinitely narrow notch in it, the area of the section at the bottom of the notch being a. Let f be the maximum tensile stress on each bar. Then the stress on the second bar, except at the notch, is $\frac{a}{A}f$.

The resilience of the first bar is $\frac{alf^2}{2E}$ (see Art. 88, page 69). The

resilience of the second bar is $\frac{Al}{2E}\left(\frac{a}{A}f\right)^2 = \frac{a^2lf^2}{2AE}$.

Hence $$\frac{\text{resilience of first bar}}{\text{resilience of second bar}} = \frac{A}{a}.$$

The unnotched bar has therefore a greater resilience than the notched bar when the minimum or effective cross section is the same in both.

166. Fracture by Shearing in Tension and Compression Tests.— In Art. 140, p. 138, it was shown that a piece subjected to direct tension or direct compression is also placed under shear, the shear stress being a maximum in planes inclined at 45° to the axis of the specimen. It was also shown that the intensity of the maximum shear stress is half the intensity of the tensile or compressive stress on planes perpendicular to the axis of the specimen.

Experimental evidence of the existence of this oblique shear stress in tension and compression tests is found with various materials. For example, if a test piece of mild steel be highly polished previous to testing in tension, a series of lines inclined at about 60° to the axis of

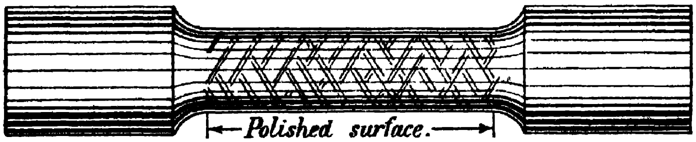


FIG. 247.

the piece are clearly seen after the yield point is reached. If the piece is cylindrical, the lines referred to form helices on the polished surface, as shown roughly in Fig. 247. These lines, sometimes called *Luders' lines*, show that the molecular slip is taking place in the direction of shear stress, and in the case of mild steel and other materials it is no doubt the resistance to pure shear which determines the yield point.

The actual fracture of a piece of mild steel in tension also shows that it really gives way by shearing, as is shown in Fig. 248, where the specimens are cylindrical, and the fractures partly conical. Fig. 249 shows the common form of fracture of a cylindrical piece of cast-iron tested in compression. The cast-iron gives way by shearing obliquely, the inclination of the fracture to the axis of the specimen being about 35°.

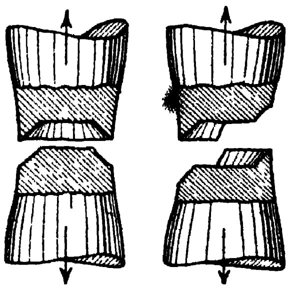


FIG. 248.

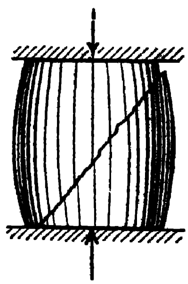


FIG. 249.

Further evidence of the shearing of a piece under a crushing load is found when testing a hard steel ball in compression between two hard flat plates, or between two other balls. Where contact takes place small circular flats are formed on the ball, and these become the bases of two cones, which form in the ball by shearing, and these cones are forced into the ball and cause it to

split into two or more pieces. The ball therefore gives way finally by tearing. Fig. 250 shows a ball which has split into two pieces by a crushing load acting in the direction of the arrows. The two cones referred to above are seen adhering, one to one half of the ball and the other to the other.

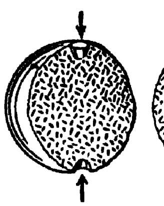


FIG. 250.

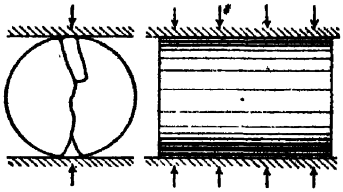


FIG. 251.

Fig. 251 shows a typical fracture of a cast-iron roller tested in compression between two hard flat plates. Here wedges form by shearing, and these finally split the roller. In the case illustrated, the splitting was probably caused by the bottom wedge.

In actual fractures by shearing in tension and compression tests the inclination of the plane of fracture to the axis is never 45°, although at that angle the shear stress produced by the external load is greatest. The reason why the fracture does not take place at 45° is that the *resistance to sliding* at an oblique section is affected by the *normal* stress on that section. The theory of the effect of the normal stress on an oblique section in altering the inclination of the shear fracture is as follows. Referring to Fig. 182, p. 138, and using the notation of Art. 140, the external load P causes a shear force $p \sin\theta\cos\theta$ along CD per unit area of CD, and also a force $p\cos^2\theta$ normal to CD per unit area of CD. If s is the normal cohesive force per unit area holding together the parts on opposite sides of CD, then the resultant normal force on CD per unit area is $s \pm p\cos^2\theta$, where the upper sign applies to a compression test, and the lower sign to a tension test. Assuming that the resistance to sliding along CD per unit area is proportional to $s \pm p\cos^2\theta$, and that it is equal to $\mu(s \pm p\cos^2\theta)$, where $\mu = \tan\phi$ is a coefficient of resistance to sliding, then $p\sin\theta\cos\theta = \mu(s \pm p\cos^2\theta)$, and this reduces to

$$p = \frac{2\mu s\cos\phi}{\sin(2\theta \mp \phi) \mp \sin\phi}.$$

Hence p will be a minimum when $\sin(2\theta \mp \phi)$ is a maximum, that is, when $2\theta \mp \phi = 90°$, or when $\cot 2\theta = \mp\mu$, where the upper sign applies to a compression test, and the lower sign to a tension test. It follows from the above that the value of θ in a compression test is the complement of its value in a tension test of the same material.

167. Fracture by Tension in Torsion Tests.—When a cylindrical specimen is subjected to simple torsion there is a pure shear stress in the material in planes perpendicular to the axis, and also in planes containing the axis, and in Art. 142, p. 140, it was shown that at any point in the material there is also a pure tensile stress equal to the shear stress at that point, the direction of this tensile stress being at 45° to the shear stresses. It may therefore be expected that when a specimen of a material whose resistance to direct tension is less than its resistance to pure shear is subjected to torsion it will give way in tension. Cast-iron is such a material, and the form of the fracture of a hollow cylindrical

specimen of this material when tested in torsion is shown in Fig. 252. It will be seen that the surface of the oblique fracture is a screw surface, the

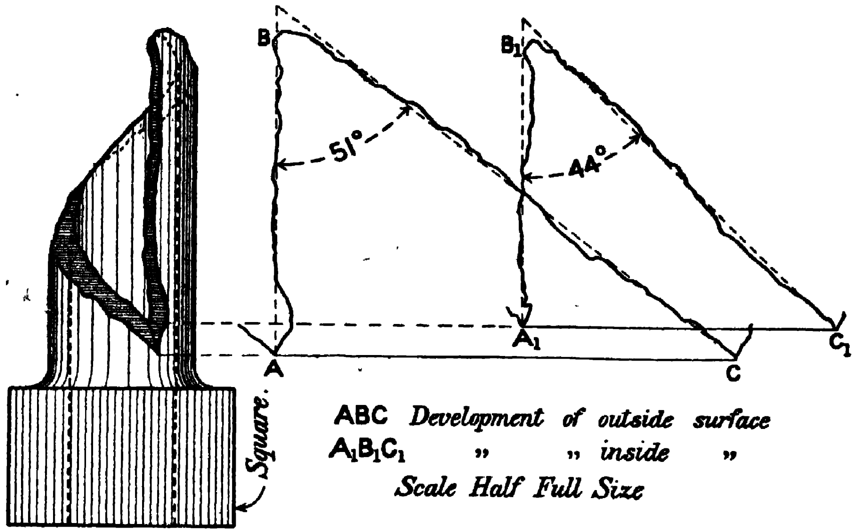


FIG. 252.

edges of which are helices. The developments of the outside and inside surfaces of the specimen after fracture are shown to the right in Fig. 252.

168. Hardening Effect of Overstraining.—When a ductile material like mild steel is loaded beyond the yield point, and therefore given a definite permanent set, and then unloaded, it is found that when the load is again applied the yield point is higher than before. This is well illustrated by Fig. 253, which is a stress strain diagram (the stress being the *nominal* stress) for a bar of mild steel tested in tension. The first loading produced a decided yield when the stress reached 16·5 tons per square inch. When the stress reached about 21·5 tons per square inch the load was almost entirely removed, and then again immediately applied, with the result that the yield point was found to be at about 22·2 tons per square inch. The load was increased until the stress was about 24·4 tons per square inch, when it was again almost entirely removed. A third application of the load showed a yield point at about 24·7 tons per square inch. The load was then continued until fracture occurred.

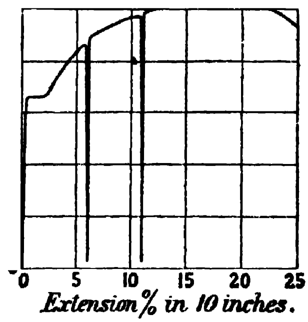


FIG. 253.

Compared with a test of the same material, in which the bar was broken with one loading, the maximum and breaking loads were found to be higher, but the final elongation was less in the bar tested with interrupted loading than in the other.

If after each unloading an interval of a few hours is allowed before the next application of the load, it is found that this has the effect of raising the yield point still higher. If a bar which has been overstrained be loaded below its yield point and the extensions measured with a delicate extensometer, it is found that the elasticity of the bar is very imperfect, but after a sufficient period of rest the elasticity is restored. The elasticity lost through overstraining a bar of mild steel may be quickly restored by immersing the bar for a few minutes in boiling water. Another effect of this heating is that the yield point is raised as much as it would be after a considerable period of rest.

For further information on the above subject the student is referred to papers by Ewing in the *Proceedings of the Royal Society*, 1880 and 1895, also a paper by Muir in the *Phil. Trans. Roy. Soc.*, 1899.

169. Effect of Fluctuating Loads.—General experience, and direct experiments, have shown that when the load on a piece is made to vary over a given range a sufficient number of times, fracture may take place at a much lower stress than the piece would have originally stood under a static load. For example, in one of Wöhler's tests on wrought-iron, the tenacity under a static load was about 23 tons per square inch, but when loaded and unloaded about 10 million times, the load in each case producing a tensile stress of 15·28 tons per square inch, the bar broke. In another test on the same material the stress was made to vary from 8·6 tons per square inch in tension to 8·6 tons per square inch in compression, and the bar broke when the number of repetitions of the load was about 19·2 millions.

Wöhler was the first to investigate in a comprehensive manner the effects of fluctuating loads on the strength of iron and steel. His researches, which were carried on for about twelve years, embraced loading and unloading, and also partial unloading in tension, repeated bending in one direction and also in opposite directions, repeated twisting in one direction and also in opposite directions.*

Further researches on this subject have been conducted by Spangenberg, Bauschinger, Sir Benjamin Baker, Dr. J. H. Smith and Professor Osborne Reynolds, Dr. Stanton, and others. The subject is still being investigated by a number of experimenters.

The general result of the numerous experiments which have been made seems to be that the maximum stress at which fracture will occur in any particular case depends to a large extent on the range of the fluctuation of stress as well as on the static strength of the material.

Various empirical formulæ have been constructed to express the relation between the maximum stress at fracture after a very large number of repetitions of the load, the static strength of the material, and the range of stress. One well-known formula is the following, given by Unwin in his "Machine Design,"

$$f_{\text{max.}} = \tfrac{1}{2}\Delta + \sqrt{f^2 - n\Delta f},$$

where $f_{\text{max.}}$ is the stress at which fracture will probably occur after a sufficiently large number of repetitions of the load, f is the static ultimate

* For details of Wöhler's tests the student may refer to Unwin's *Testing of Materials of Construction*, and also to *Engineering*, vol. xi. (1871).

stress, Δ is the range of stress, and n is a coefficient depending on the material. For ductile iron and ductile steel the average value of n is about 1·5. For the harder and more brittle qualities, n may be as high as 2·2. In estimating the range of stress, if a tension is taken as positive, then a compression must be taken as negative.

The following are examples of the application of the foregoing formula to a bar of mild steel having a static tenacity of 26 tons per square inch. $n = 1·5$.

(1) Range of stress from $f_{max.}$ to 0. $\Delta = f_{max.}$

$f_{max.} = \frac{1}{2}f_{max.} + \sqrt{26^2 - 1·5 \times 26 f_{max.}}$ Hence $f_{max} = 15·75$ tons per sq. in.

(2) Range of stress from f_{max} to $\frac{1}{2}f_{max}$ $\Delta = \frac{1}{2}f_{max}$

$f_{max.} = \frac{1}{4}f_{max} + \sqrt{26^2 - 1·5 \times 26 \times \frac{1}{2}f_{max}}$ Hence $f_{max} = 21·43$ tons per sq. in.

(3) Range of stress from $f_{max.}$ to $-f_{max}$ $\Delta = 2f_{max.}$

$f_{max.} = f_{max.} + \sqrt{26^2 - 1·5 \times 26 \times 2f_{max.}}$ Hence $f_{max} = 8·67$ tons per sq. in.

The safe working stress is obtained by dividing $f_{max.}$ by a factor of safety.

Other empirical formulæ for fluctuating load stresses are given on p. 252.

170. Fatigue of Metals.—The loss of strength which occurs when a metal is subjected to a fluctuating load for a considerable time is frequently said to be due to *fatigue*. It is necessary, however, to distinguish between deterioration of strength due to mere fluctuation of stress and deterioration due to shocks. It is well known that a crane chain, if kept in use for a long time, may fracture abruptly when carrying a load less than that which it has been in the habit of carrying This deterioration of strength is, however, probably due to a slow accumulation of permanent set produced by shocks due to the sudden starting or stopping of the load when raising or lowering (see Art. 88, p. 69, and Art. 168, p. 177), and the chain becomes in consequence less able to resist shocks. It has been found that the power of the chain to resist shocks is restored by annealing, and it is a common practice to anneal crane chains at frequent intervals, say, once or twice a year. Unwin has proposed to restrict the term fatigue to deterioration due to shocks which is removable by annealing.

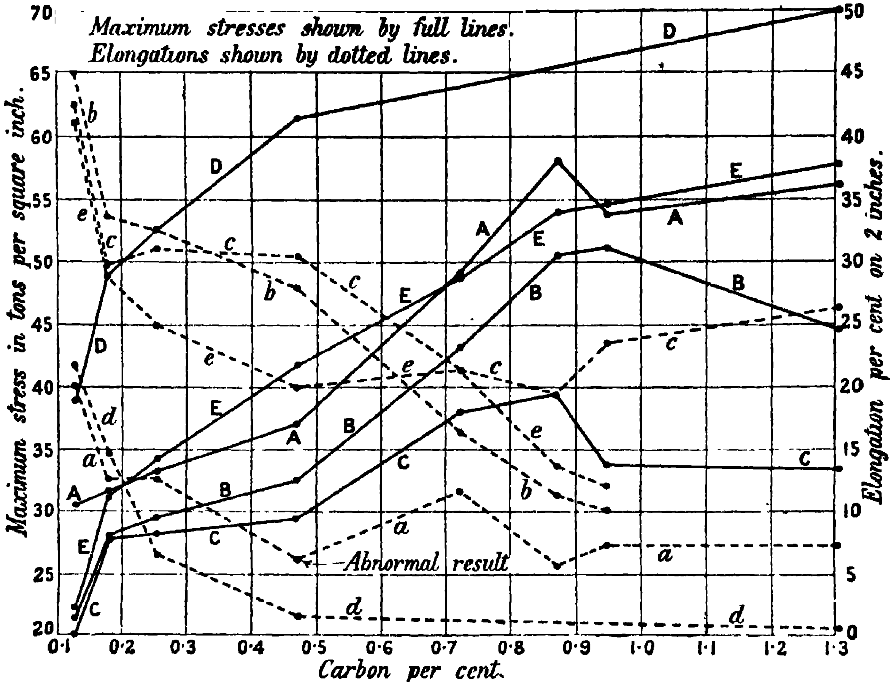
171. Mechanical Properties of Steel after Heat Treatment.—The sixth report to the alloys research committee of the Institution of Mechanical Engineers by the late Sir William C. Roberts-Austen and Professor William Gowland, relates to the heat treatment of steel.* This voluminous report contains a record of a large amount of research on the effects of various kinds of heat treatment on the mechanical properties of samples of steel containing different amounts of carbon. A few of the results will be given here.

The table on page 181 gives the maximum stress and the elongation obtained in tensile tests of the bars as received from the rolls, and also after the different kinds of heat treatment described. Additional

* *Proceedings of the Institution of Mechanical Engineers*, 1904.

results, and the results on the bars as received, are shown plotted in Fig. 254.

The maximum stress is in tons per square inch of the original area of the test pieces, and the elongation is the percentage elongation on a length of 2 inches in each case. The reduced parallel part of each specimen was $\frac{7}{16}$ inch in diameter before testing.



The full lines A, B, C, D, *and* E *show the maximum stresses.*
The dotted lines a, b, c, d, and e show the elongations.
A, *a, refer to the bars as received from the rolls.*
B, *b, refer to bars annealed at* 900° *C. for half-an-hour.*
C, *c, refer to bars soaked at* 720° *C. for twelve hours.*
D, *d, refer to bars quenched at* 800° *C. in water at* 20° *C.*
E, *e, refer to bars quenched at* 720° *C in oil at* 80° *C. and subsequently reheated to* 350° *C.*

FIG. 254.

Bars as received.—The maximum stress for the bars as received from the rolls is about normal, but the elongation is extremely low. The low elongation was found, after microscopic examination, to be due, to a certain extent, to rapid cooling near a certain critical temperature.

Annealing.—The test pieces were annealed as follows. "The bars were packed in lime in $\frac{3}{4}$-inch wrought-iron tubes, closed at each end by screwed iron caps. These tubes were then packed in a large wrought-iron tube, the ends of which were covered by wrought-iron plates. This was placed in a closed gas-muffle, and a record of the temperature was taken by means of two thermo-couples attached to an autographic recorder." After attaining the temperature desired, the contents of the

Tenacity and Elongation of Steel after Heat Treatment.

Carbon, per cent.	0·130	0·180	0·254	0·468	0·722	0·871	0·947	1·306
Bars as received from the rolls.								
Maximum stress	30·22	31·52	33·25	36·96	49·00	58·02	53·74	56·11
Elongation	20 00	12·50	12·70	6·25*	11·72	5·47	7·03	7·03
Bars annealed at 720° C. for half-an-hour.								
Maximum stress	20·16	26·52	28·80	32·82	39·96	43·98	39·30	50·58
Elongation	44·50	36·00	32·50	29·00	23·00	20·00	22·00	7·00
Bars annealed at 1100° C. for half-an-hour.								
Maximum stress	...	26·91	28·11	31·38	42·38	47·69	49·62	46·62
Elongation		39·50	33·50	27 00	14·00	11 00	7·50	6·50
Bars soaked at 620° C. for twelve hours.								
Maximum stress	19·75	26·81	28·61	32·01	40·63	45·25	45·41	46·96
Elongation	48·50	32·50	31·00	33·00	29·50	19·50	18 50	11·50
Bars soaked at 900° C. for twelve hours.								
Maximum stress	...	26·91	29·55	29 91	43 21	49·32	47·76	44·41
Elongation	...	37·00	29 00	30 00	12 50	11·00	11·90	10·00
Bars quenched at 900° C. in Water at 20° C.								
Maximum stress	32 34	49·92	70·32	55·44	61·98	26·04	21·60	...
Elongation	28 20	6·50	3·80	1·00	1·00	2 00	nil.	.
Bars quenched at 1200° C. in Water at 20° C.								
Maximum stress	45·29	72·78	77 04	31·74	13·44	5·40	4·20	3·52
Elongation	9 00	3·50	3 00	nil.	nil	nil	nil.	nil.
Bars quenched at 870° C. in Oil at 80° C., and subsequently reheated to 350° C.								
Maximum stress	24·50	36 12	41 13	54 65	78·59	90·68	102·52	90·57
Elongation	39·50	23·50	24 50	25·00	23·50	10 50	7·00	5·50
Bars quenched at 900° C. in Oil at 80° C., and subsequently reheated to 600° C.								
Maximum stress	24·55	30·13	31·24	35·71	50·44	52·68	49·10	51·32
Elongation	40·70	30·00	26·20	27·20	17·50	15·50	17·00	13·00

* Abnormal result.

muffle were kept at that temperature for the time required (half-an-hour in these tests), the gas was then turned off, and the whole allowed to cool slowly.

It will be observed that the general results of annealing are, reduction of strength and increase of elongation of the steel when tested.

Soaking.—The difference between *annealing* and *soaking* is, that in the latter operation the bars are heated for a much longer time.

Hardening.—"By heating steels to the hardening temperatures, some or all of the iron carbide is dissolved in the iron, and the latter is restrained from reverting to its soft condition by sudden cooling. Within certain limits, the more rapidly the heat is abstracted from the bar the more effective will be the hardening."

Referring to Fig. 254, and comparing the D, *d* lines with the A, *a* lines, the effect of quenching the steels at 800° C. in water is to increase the tenacity of all the steels considerably, but at the same time the elongation is diminished, except in the case of the 0·13 and 0·18 carbon steels.

Tempering.—Steel which has been hardened by quenching in water or oil may be *tempered*, that is, its hardness may be reduced to any required extent, by subsequent annealing at a temperature depending on the degree of hardness required.

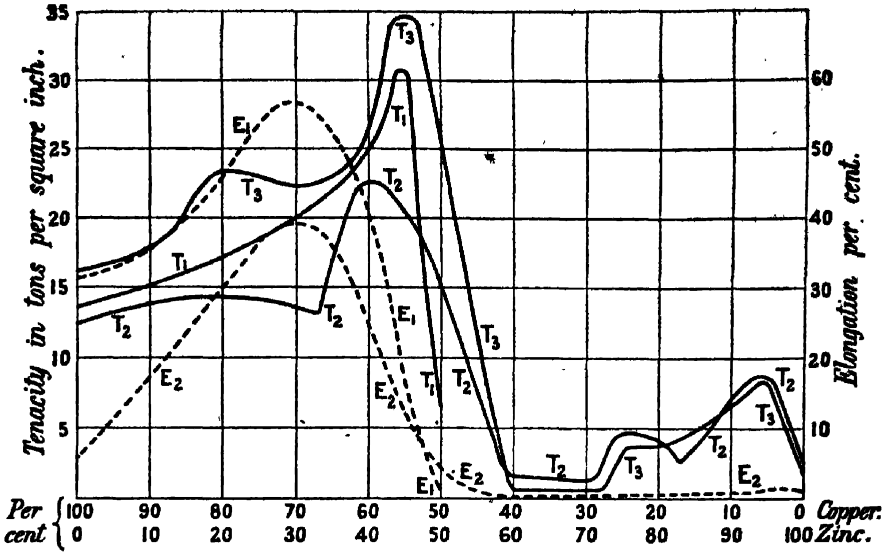
Oil hardening.—"When steel is quenched in oil at 80° C., the effect is to increase the tensile strength, but to a somewhat less extent than by quenching in water at 20° C., and also at the same time to increase the elastic limit and rather diminish the ductility."

"The most suitable temperature for quenching steel, in order to obtain the best combined results as regards tensile strength, elastic limit, and elongation, is about 900° C., and the most suitable temperature for reheating, when elongation, and consequently resistance to shock, is not of paramount importance, is about 350° C. If, however, the steel be required to withstand violent percussive action, as in a gun tube, then reheating at a higher temperature, say 600° C., will be found to be necessary, as such steel, when thermally treated in this way, although possessing a relatively high tensile strength and elastic limit, nevertheless has also a high percentage of elongation."

The student is recommended to study carefully the results given in the table on p. 181 and in Fig. 254, and to plot the results as directed in Exercise 12, p. 190.

172. Tests of Copper-Zinc Alloys (Brasses).—The curves in Fig. 255 show the tenacity and extensibility of alloys containing different proportions of copper and zinc. These curves have been reproduced, with modifications as to scales, from the fourth report to the alloys research committee of the Institution of Mechanical Engineers by Professor W. C. Roberts-Austen.*

* *Proceedings of the Institution of Mechanical Engineers*, 1897.



The full lines T_1, T_2, *and* T_3 *show tenacities.*
The dotted lines E_1 *and* E_2 *show elongations.*
The lines T_1 *and* E_1 *refer to tests by Charpy, the test pieces being completely annealed.*
The lines T_2 *and* E_2 *refer to tests by Thurston on cast brasses.*
The line T_3 *refers to tests by the alloys research committee on worked rods.*

FIG. 255.

173. Tests of Copper-Tin Alloys.—The curves in Fig. 256 show the tenacity and extensibility of alloys containing different proportions of

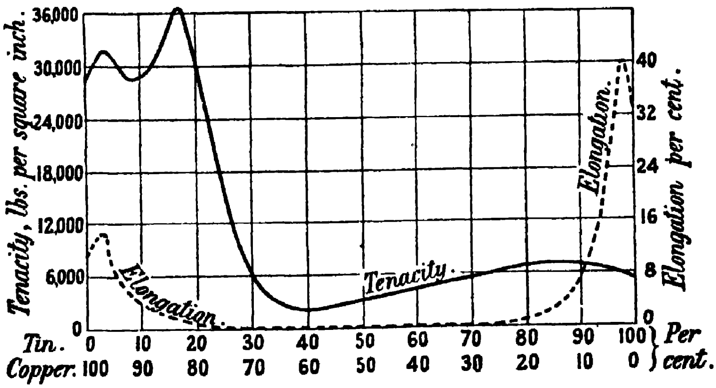


FIG. 256.

copper and tin. These curves represent the results of tests made by Professor Thurston.*

* *Proceedings of the Institution of Mechanical Engineers*, 1895, Plate 43.

174. Tests of Lead-Tin Alloys.—The results of tests on the tenacity and extensibility of alloys containing different proportions of lead and tin are shown plotted in Fig. 257.*

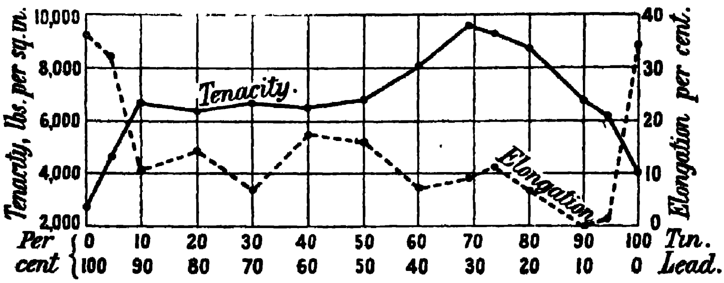


FIG. 257.

175. Tests of Alloys of Aluminium and Copper.—The eighth report to the alloys research committee of the Institution of Mechanical Engineers by Professor H. C. H. Carpenter and Mr. C. A. Edwards contains a large amount of information on the properties of alloys of aluminium and copper. Some of the results of the tests made will now be given. For further particulars the student is referred to the full report.†

The results of tensile tests of specimens cast in sand moulds, and of specimens cast in thick cast-iron moulds, are given in the following table :—

Alloy.	Aluminium.	Sand Castings			Chill Castings.		
		Yield Point Stress	Ultimate Stress.	Elongation on 2 inches.	Yield-Point Stress.	Ultimate Stress.	Elongation in 2 inches.
No.	Per cent.	Tons per sq. inch.	Tons per sq. inch.	Per cent	Tons per sq. inch.	Tons per sq. inch.	Per cent.
1	0·10	3·8	11·5	46·0	4·1	11·53	46·0
2	1·06	3·0	13·4	52·0	5·2	11·8	53·0
3	2·10	3·4	13·5	53·5	4·5	13·7	54·5
4	2·99	3·8	14·5	60·0	6·8	13·8	60·0
5	4·05	3·5	16·7	83·0	4·9	17·1	82·0
6	5·07	4·3	18·1	75·0	7·1	18·1	60·5
7	5·76	4·8	17·8	67·0	6·0	18·8	61·0
8	6·73	4·8	18·65	... ‡	. §	19·96	69·0
9	7·35	6·6	21·3	71·0	.. §	21·58	84·0
10	8·12	7·7	24·91	58·0	9·7	27·47	62·0
11	8·67	9·8	28·1	48·0	. §	30·8	55·0
12	9·38	9·7	30·38	36·2	10·5	33·98	43·5
13	9·90	11·3	31·70	21·7	12·4	36·93	30·5
14	10·78	14·1	29·52	9·0	16·9	36·73	9·0
15	11·73	14·0	25·43	5·0	14·8	30·63	6·0
16	13·02	19·75	19·75	1·0	25·06	25·06	nil.

The stresses are given on the original area.

* *Proceedings of the Institution of Mechanical Engineers*, 1897, Plate 4.
† *Ibid.*, 1907. ‡ Could not be measured. § Not taken.

The specimens were turned to the forms and dimensions shown in Figs. 258 and 259.

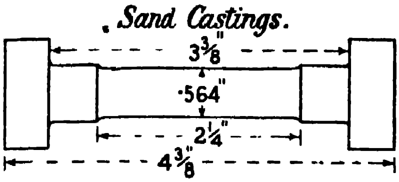


FIG. 258.

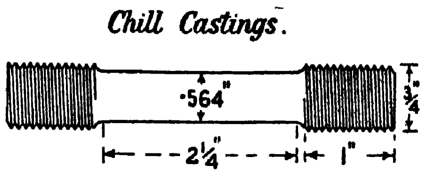


FIG. 259.

The next table shows the results of tests on specimens turned out of rolled bars $1\frac{3}{16}$ inch diameter. These rolled bars were made as follows. Billets 3 inches in diameter and about 20 inches long were cast in cast-iron moulds. These billets were first turned in a lathe, and the diameter thus reduced to $2\frac{1}{16}$ inches. They were then heated to about 800° C. (1472° F.) and rolled down, first to $1\frac{1}{4}$ inches diameter, and then to $1\frac{3}{16}$ inch diameter. The gauge length of the specimens was 2 inches.

Alloy.	Aluminium	Yield-Point Stress.	Ultimate Stress.	Elongation on 2 inches.	Reduction of Area.	Weight per Cubic Foot.
No	Per cent.	Tons per sq. inch.	Tons per sq. inch.	Per cent.	Per cent.	Lbs.
1	0·10	6·9	14·50	65·5	90·71	556
2	1·06	6·9	15·88	61·0	88·63	548
3	2·10	8·6	17·46	56 5	89·65	537
4	2·99	11·6	19·79	57 2	86 11	528
5	4·05	11·3	23·80	67 0	83·27	518
6	5·07	11·4	26·41	69 2	77·80	510
7	5·76	11·8	28·40	74·2	76 93	503
8	6 73	10·4	28·85	71 0	75·02	499
9	7·35	10·6	29 68	72·5	74·34	492
10	8·12	13·0	33·22	51·5	60·40	487
11	8·67	11·1	36·67	38·0	50·66	482
12	9·38	17·7	38·00	34·0	33·60	477
13	9·90	14 8	38·10	28·8	30·80	473
14	10·78.	15·4	38·62	14·0	18·60	466
15	11·73	12·6	33·85	8·5	15·37	460
16	13·02	.	37·14	2·0	1·87	451

Fig. 260 shows stress-strain diagrams taken from longer specimens prepared from the $1\frac{1}{16}$-inch rolled bars. The parallel portion was 10·5 inches long and 0·564 inch diameter, and the gauge length was 8 inches. The numbers on the curves are the descriptive numbers of the alloys. It will be observed that the various diagrams are displaced to the right, so that the zero points of elongation are at the points showing the percentage of aluminium in the respective alloys.

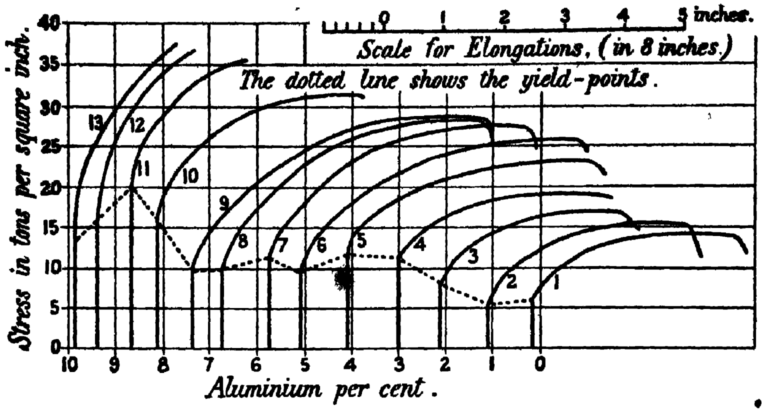


FIG. 260.

The results of torsion tests on a number of these alloys are shown plotted in Fig. 261.

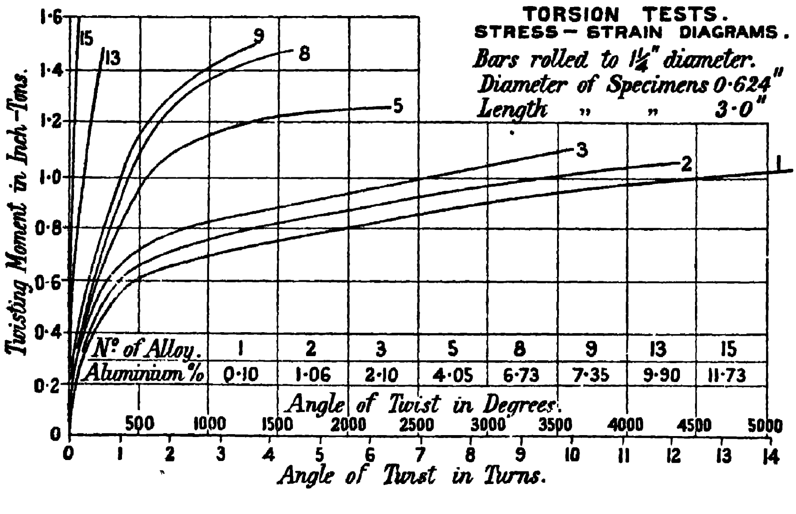


FIG 261.

176. Tables of Strength and Elasticity of Materials.—The tables which follow give *approximate average values* of the ultimate strength and modulus of elasticity of various materials.

Ultimate Crushing Strength in Tons per Square Inch.

Cast-iron	45	Granite	6
Brass, cast	5	Brick, London stock	1
Timber	3	,, Staffordshire blue	3
Sandstone	3	Concrete	0·5

Ultimate Tensile Strength in Tons per Square Inch.

Cast-iron	8	Aluminium, cast	5
Wrought-iron	23	,, rolled	8
Mild steel	28	Delta metal, cast	20
Steel castings	30	,, rolled	30
Steel wire	80	Manganese bronze, cast	25
Copper, cast	10	,, ,, rolled	30
,, rolled or forged	14	Muntz metal	22
,, wire, annealed	18	Naval brass	24
Brass, cast	11	Phosphor bronze, cast	16
,, rolled	20	Leather	2
Gun-metal or bronze	14	Timber	6

Ultimate Shearing Strength in Tons per Square Inch.

Cast-iron	12	Gun-metal	15
Wrought-iron, across fibre	19	Yellow pine, across fibre	2
,, ,, along fibre	10	,, ,, along fibre	$\frac{1}{4}$
Mild steel	22	Oak, across fibre	$2\frac{1}{2}$
Brass	10	,, along fibre	$\frac{1}{2}$

Modulus of Elasticity in Tons per Square Inch.

	Direct. (E.)	Transverse. (C.)
Cast-iron	6,700	2,600
Wrought-iron	13,000	5,200
Steel	13,400	5,500
Copper, cast	5,500	2,100
,, rolled	6,700	2,500
Brass	5,600	2,200
Gun-metal	5,600	2,200
Phosphor bronze	6,100	2,300
Timber	700	270

Exercises XI.

1. In a tensile test of a wrought-iron bar, the following observations were made: W = load in tons, x = extension in inches in a length of 8 inches, d = smallest diameter of bar in inches. 9·24 was the load at the yield point, 13·6 was the maximum load, and 12·78 was the breaking load. The second value of x (0·175) was at the end of the "yield."

W	0	9·24	11·94	13·21	13·60	12·78
x	0	0·175	0·55	0·975	1·56	2·45
d	0·881	0·872	0·854	0·833	0·809	0·628

Plot x and the nominal stress (load ÷ original area), also x and the actual stress (load ÷ actual area of least section). Scales.—Stresses, 1 inch to 5 tons per square inch; x, $2\frac{1}{2}$ times full size.

2. In a tensile test of a mild steel bar, the following observations were made: W = load in tons, x = extension in a length of 8 inches in inches, d = smallest diameter of bar in inches. 10·64 was the load at the yield point, 15·64 was the maximum load, and 12·97 was the breaking load. The second value of x (0·24) was at the end of the "yield." Plot x and the nominal stress, also x and the actual stress. Scales.—Stresses, 1 inch to 5 tons per square inch; x, $2\frac{1}{2}$ times full size.

W	0	10·64	13·81	15·07	15·64	15·60	12·97
x	0	0·24	0·60	1·02	1·57	2·12	2·79
d	0·906	0·891	0·871	0·851	0·825	0·799	0·588

3. In a tensile test of a mild steel bar, the following observations were made: Diameter of bar, unloaded, 0·748 inch, W = load in tons, x = extension, in inches, on a length of 8 inches. Load at elastic limit, 6 tons. Maximum load, 12·54 tons.

W	1	2	3	4	5	6	6·81 "yield" point.
x	0·0014	0·0027	0·0040	0·0055	0·0068	0·0082	0·18 at end of "yield."
W	7·5	9·0	10·5	12·0	12·54	12·25	10 25 Breaking load.
x	0·19	0·27	0·55	1·05	1·75	2·10	2·42 Total extension.

(a) Plot W and x up to W = 6 and x = 0·0082. Scales.—W, 1 inch to 1 ton; x, 1000 times full size.

(b) Draw the straight line which most nearly contains the points in (a), and calculate from it the modulus of elasticity in lbs. per square inch.

(c) Calculate the load, in tons, necessary to elongate the bar 0·006 inch.

(d) How many ft.-lbs. of work have been done in stretching the bar 0·0082 inch?

(e) Plot W and x from no load up to the breaking point. Scales.—W, 1 inch to 2 tons; x, $2\frac{1}{2}$ times full size.

(f) Determine the total work done, in ft.-lbs., in breaking the bar.

(g) Plot x and the nominal stress, also x and the actual stress. Scales.—Stresses, 1 inch to 5 tons per square inch; x, $2\frac{1}{2}$ times full size. Assume volume of bar constant in finding cross section up to maximum load. Assume also that the contracted section at fracture is 0·43 of the original section.

4. A cylindrical piece of mild steel was tested in compression. The load W, in tons, acted on the ends of the piece. The mean diameters of the piece at the top, middle, and bottom of its length were d_1, d_2, and d_3 inches respectively, and its length was l inches. Values of these dimensions for various values of W are given in the following table:—

W	0	5	10	15	20	25	30	35	40
d_1	0·719	0·720	0·757	0·813	0·884	0·965	1·054	1·13	1·20
d_2	0·719	0·723	0·763	0·832	0·922	1·045	1·144	1·21	1·60
d_3	0·720	0·721	0·760	0·815	0·886	1·000	1·085	1·14	1·22
l	1·624	1·589	1·452	1·236	1·025	0·804	0·690	0·61	0·54

Under the greatest load the piece was free from cracks.

Calculate the nominal and actual compressive stresses on the smallest sections, and plot the results in the manner shown in Fig. 243, p. 171. Scales.—Linear, twice full size. Stresses, 1 inch to 20 tons per square inch.

5. A test piece of steel boiler plate of rectangular section $1\frac{1}{2}$ inches wide and $\frac{5}{8}$ inch thick, when tested for elongation, gave, after fracture, the following results:—

Gauge length (l), inches	4	6	8	10	12	14
Elongation (e), per cent.	37·8	31·8	28·5	26·7	25·5	24·5

Plot on squared paper e and $\frac{\sqrt{a}}{l}$, where a is the area of the cross section of the bar in square inches. Draw the straight line which most nearly contains all the points, and find the values of the constants c and b in the equation to the line, which is $e=\frac{c\sqrt{a}}{l}+b$. Apply the equation to find the probable elongation per cent. in 8 inches of a test piece of the same material 1 inch wide and $\frac{1}{2}$ inch thick.

6. A bar of mild steel 10 inches long and $1\frac{1}{4}$ inches in diameter has a groove turned on it at the centre of its length, the groove being $\frac{1}{2}$ inch wide and $\frac{1}{8}$ inch deep. Another bar of the same material has the same length and a uniform diameter of 1 inch. Compare the resilience of the second bar with that of the first for the same maximum stress, the bars being loaded in tension.

7. The averages of the results of numerous tests of the crushing strength of hard steel balls are given in the following table :—

d . . inches	$\frac{1}{8}$	$\frac{3}{16}$	$\frac{1}{4}$	$\frac{5}{16}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{5}{8}$
W . . tons	0·77	1·60	2·88	4·07	5·82	9·50	13·04

where d is the diameter of the ball, and W the crushing load. The balls were tested between two hard steel plates. Calculate for each size of ball the stress f in the formula $W=\frac{\pi}{4}d^2f$. Plot f and d, also W and d. Scales.—For d, eight times full size; for f, 1 inch to 10 tons per square inch ; for W, 1 inch to 2 tons. Show that an expression of the form $f=a-bd$ gives approximately the relation between f and d in the above results, where a and b are constants, and find the values of these constants. Hence the relation between W and d is $W=\frac{\pi}{4}d^2(a-bd)$.

8. The following table gives the results of crushing tests on cast-iron rollers tested between steel plates :—

d . . inches	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{7}{8}$	1	$1\frac{1}{8}$	$1\frac{1}{4}$
l . . inches	$\frac{3}{4}$	1	$1\frac{1}{8}$	$1\frac{1}{4}$	$1\frac{1}{2}$	$1\frac{5}{8}$	$1\frac{7}{8}$
W . . tons	4·32	6·90	9·95	12·70	17·11	21·20	27·00

d=diameter of roller, l=length of roller, W=crushing load. Plot W and $d\times l$, and find the most approximate value of c in the expression $W=cdl$ for the above results. Scales.—For $d\times l$, 1 inch to $\frac{1}{4}$ square inch ; for W, 1 inch to 5 tons.

9. A cylindrical piece of cast-iron 0·727 inch in diameter and 2 inches long was tested in compression, the load being axial. The piece gave way by shearing in a plane inclined at 37° to the axis when the crushing load was 22·91 tons. Neglecting the alteration in the diameter of the piece, calculate the intensity of the shear stress in the plane of fracture. What is the value in this case of the coefficient μ used in Art. 166, p. 175?

10. Same as Exercise 9, except that the angle was 33° instead of 37°, and the crushing load was 19·25 tons instead of 22·91 tons.

11. The load on a certain steel tie-bar in a bridge truss varies from 14 tons to 21 tons (both tensions). If the tenacity of the material is 28 tons per square inch, and the coefficient n in the formula given on p. 178 is 1·56, what must be

the area of the cross section of the bar if the maximum stress allowed on it is one-fourth of the maximum stress due to the above fluctuating load ?

12. Plot on squared paper in the manner shown in Fig. 254, p. 180, the results in the table on p. 181 and in Fig. 254, grouping the results on separate diagrams as follows:—(1) Annealed bars; (2) soaked bars; (3) bars quenched in water; (4) bars quenched in oil. Show also on each diagram the results of the tests on the bars as received from the rolls. Scales.—Carbon, 1 inch to 0·2 per cent.; stress, 1 inch to 10 tons per square inch; elongation, 1 inch to 10 per cent. Examine all the results carefully, and discuss the effects of the different kinds of heat treatment on the different steels.

CHAPTER XII

STRÈSS DIAGRAMS

177. Stress Diagrams for Framed Structures.—It will be assumed that the framed structures considered are made up of bars which are connected by frictionless pin joints at their ends. It will also be assumed that the loads on the structure are concentrated at the joints. If a bar carries a load uniformly distributed over its length this load is divided into two equal parts, and one part is placed at each end of the bar. If a bar carries a load concentrated at an intermediate point, this load is divided into two parts, which are to one another as the distances of the load from the ends of the bar; these parts are then placed one at each end of the bar, the greater part being at that end of the bar which is nearest to the original load.

In studying the equilibrium of a structure, two kinds of fôrces have to be considered, (1) the external forces, which for the whole structure must balance one another, and (2) the internal forces. As a consequence of the two assumptions mentioned at the beginning of this Article, the bars forming the structure are subjected either to direct compression or to direct tension under the action of the external forces. It follows, therefore, that the lines of action of the internal forces are the lines which represent the bars on the diagram of the structure (called the *frame diagram*). At any joint, therefore, the forces acting are the internal forces acting along the bars meeting at that joint, and the external forces, if there are any, acting at that joint.

If a sufficient number of the forces acting at any joint are known, the polygon of forces for that joint can be drawn and the unknown forces determined.

The general method of drawing the complete stress diagram for a framed structure will be understood by reference to the example worked out in Fig. 262. A simple roof truss is shown carrying a load AB at its apex. The other external forces are the reactions BC and CA at the supports. The internal forces are the forces acting along the bars AD, BD, and CD. The lines of action of all the forces are known, but AB is the only force whose magnitude is known as yet.

At each joint there are three forces acting, and the polygon of forces for each joint is therefore a triangle. The triangle of forces for the joint 2 or for the joint 3 cannot yet be drawn, because the magnitudes of all the forces at these joints are as yet unknown, but the triangle of forces for the joint 1 may be drawn, and this is shown at (*m*). This triangle determines the magnitudes *bd* and *da* of the internal forces in the bars

BD and DA respectively. The sense of these forces is also determined, and it will be observed that the internal forces in the bars BD and DA both act towards the joint 1, therefore these bars are in compression. In drawing the triangle (m) the forces have been taken in the order in which they occur in going round the joint 1 in the watch-hand direction, beginning with the known force AB. Beginning with BA, and going round the joint in the opposite direc-

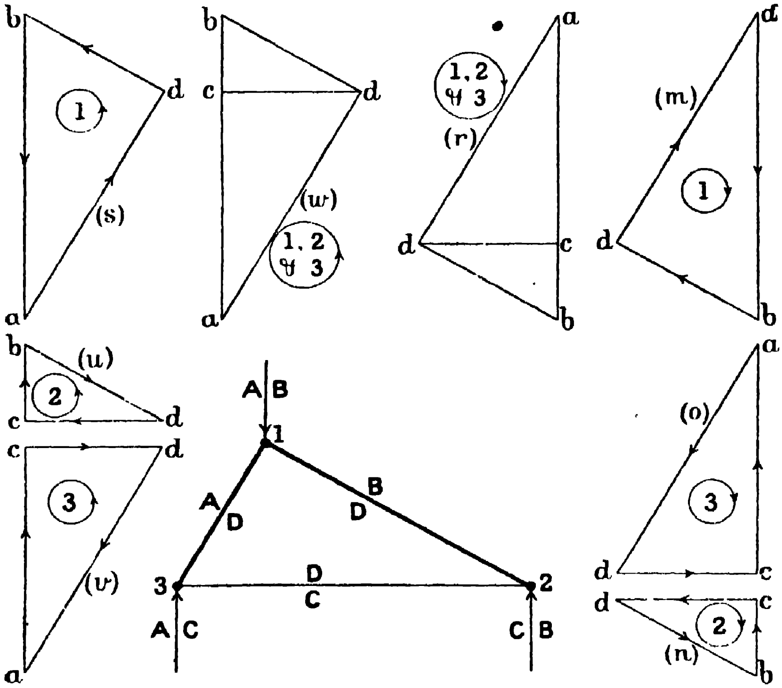


FIG. 262.

tion, the triangle (s), which is similar to (m) but differently situated, is obtained.

Passing next to the joint 2, the three forces acting there are known in direction, and the magnitude of one of them, BD, has been determined by the drawing of the triangle (m) or the triangle (s). Beginning with DB, and taking the forces in the order in which they occur in going round the joint in the watch-hand direction, the triangle of forces (n) is drawn. If the forces be taken in the order in which they occur in going round the joint in the opposite direction, beginning with BD, the triangle (u) is obtained. Proceeding next to the joint 3, the triangle (v) is obtained when the forces are taken in the watch-hand order, and the triangle (v) is obtained when the forces are taken in the opposite order.

The construction of the three triangles (m), (n), and (o), or the three triangles (s), (u), and (v), determines the magnitude and sense of each of the three internal forces, and also the magnitudes and sense of the external forces BC and CA.

It is obvious that the triangles (n) and (o) may be applied to the triangle (m) so as to form the figure (r), and this figure gives all the results which were found from the separate triangles (m), (n), and (o), and this figure (r) is the complete stress diagram for the given framed structure. The figure (r) may of course be drawn at once without drawing the triangles (m), (n), and (o). It should, however, be noticed that in order that the force polygons for the different joints may be combined into one diagram, these polygons must be drawn by taking the forces in the order in which they occur in going round each joint in the *same direction.* (r) is the form of the stress diagram when the forces are taken in the order in which they occur when going round each joint in the watch-hand direction, and (w) is the form of the diagram when the order is reversed.

178. Example.—A roof truss carrying a load at each joint is shown in Fig. 263. The loads are in pounds. The total load is 10,000 lbs., and since the loads are placed symmetrically about the centre of the truss, it is obvious that the reaction at each support is 5000 lbs. In cases where the loading is not symmetrical, the reactions at the supports may be determined by means of a funicular polygon.

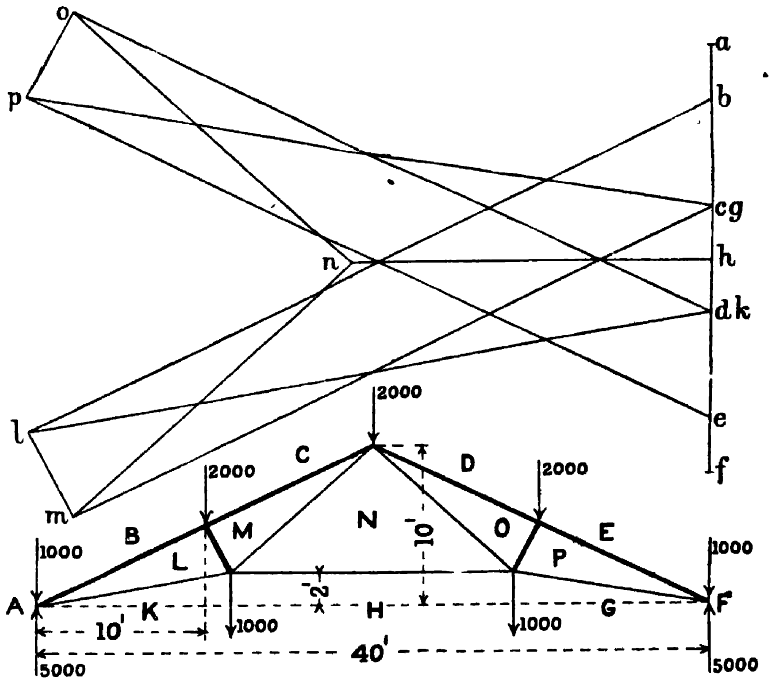


FIG. 263.

The line of loads *abcdefghka* is first drawn. Starting with the joint ABLKA at the left-hand support, the polygon of forces *ablka* is drawn. Proceeding next to the joint BCMLB; the polygon of forces *bcmlb* is drawn. The polygon of forces *lmnhkl* for the joint LMNHKL may now be drawn, and for practical purposes no more of the stress diagram need

be drawn, since the truss is symmetrical, and symmetrically loaded. The complete stress diagram for the whole truss is, however, shown in Fig. 263. The results are tabulated under the heading "dead load" in the table on p. 196.

179. Wind Pressure.—The force exerted by the wind on a plane surface at right angles to the direction of the wind may amount to about 50 lbs. per square foot of surface. When the surface is inclined at an angle θ to the direction of the wind, the normal pressure on the surface is usually determined by Hutton's formula, which is

$$\frac{p}{\mathrm{P}} = (\sin\,\theta)^{1\cdot84\,\cos\theta - 1}$$

$$\text{or} \qquad \log\frac{p}{\mathrm{P}} = (1\cdot84\,\cos\,\theta - 1)\,\log\,\sin\,\theta,$$

where p is the normal pressure, and P is the pressure on a plane at right angles to the direction of the wind. Values of $p \div \mathrm{P}$ for various values of θ are given in the following table:—

θ	10°	15°	20°	25°	30°	35°
p/P	0·241	0·350	0·457	0·563	0·663	0·754
θ	40°	45°	50°	60°	70°	80°
p/P	0·834	0·901	0·952	1·012	1·023	1·010

When $\theta = 90°$, $p = \mathrm{P}$.

180. Stress Diagrams for Wind Pressure.—It is usual to assume that the direction of the wind is horizontal, and that its maximum pressure on a plane at right angles to its direction is 50 lbs. per square foot. The inclination of a roof being known, the normal pressure of the wind on it may be determined by the formula given in the preceding Article. It is assumed that the wind acts on one side of the structure only at one time. The total load due to the wind pressure is divided up into parts, which are placed at the joints, as explained in Art. 177.

Figs. 264 and 265 show the stress diagrams for the wind pressure on the roof truss whose dimensions are given in Fig. 263. The truss is assumed to be fixed rigidly at the right-hand end and simply supported at the other end, so that it may expand and contract freely with changes of temperature. The reaction at the left-hand end must therefore be vertical, and the line of the reaction at the right-hand end must therefore pass through the point where the resultant of the wind pressure cuts the line of the reaction at the left-hand end.

The directions of the reactions having been fixed, the load polygon *abcde* can be drawn, and upon this the stress diagram is built, as in Fig. 263, which shows the stress diagram for the same truss under the dead load.

The stresses in the various bars of the truss due to the dead load and

wind pressure are tabulated on p. 196. In the last column, the maximum stresses which the bars will be subjected to are given.

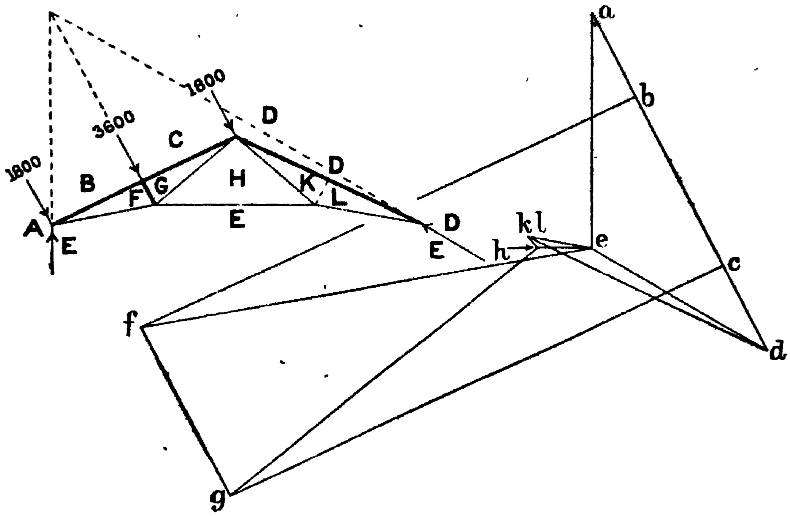
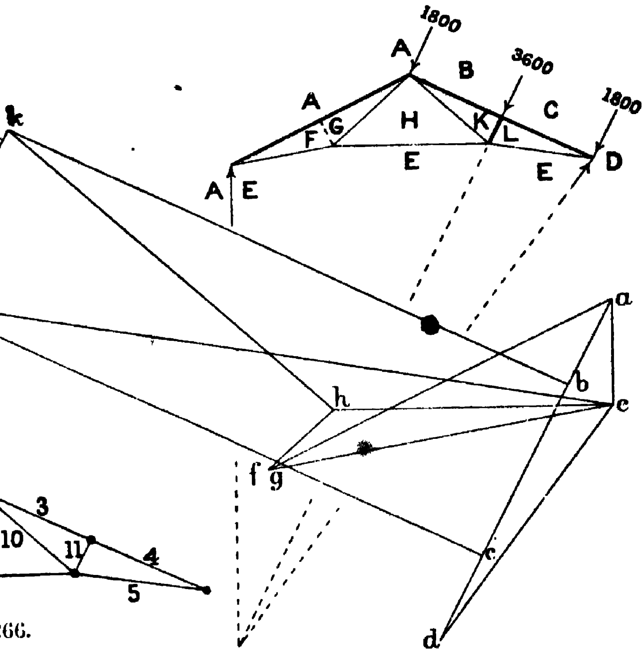


FIG. 264.



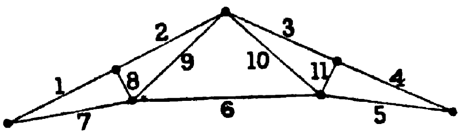


FIG. 266.

FIG. 265.

Compressive stresses are underlined with a thick line, and tensile stresses by a thin line.

No. of Bar (Fig. 266).	Dead Load. W (Fig. 263)	Wind on Left. P (Fig. 264)	Wind on Right. Q (Fig. 265)	Maximum Stress.	
1	13715	9180	6900	W + P	22895
2	12820	9180	6900	W + P	22000
3	12820	4980	11100	W + Q	23920
4	13715	4980	11100	W + Q	24815
5	12450	1253	12528	W + Q	24978
6	6438	1006	5031	W + Q	11469
7	12450	7517	6264	W + P	19967
8	1789	3600	0	W + P	5389
9	6906	6577	1566	W + P	13483
10	6906	313	7830	W + Q	14736
11	1789	0	3600	W + Q	5389

181. The Method of Sections.—Conceive that a framed structure is divided into two parts by cutting three bars A, B, and C. Next suppose that one of these parts is removed, and that external forces P, Q, and S are applied to the bars A, B, and C respectively, so as to balance the internal forces in these bars, then the part of the structure which remains (Fig. 267) will evidently be in equilibrium.

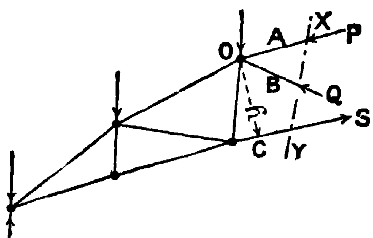


FIG. 267

If moments of all the external forces acting on the part of the structure under consideration be taken about the point O, where the bars A and B intersect, then the moment of S will balance the resultant moment of all the remaining external forces; and since the moments of P and Q are zero, and the other forces are known, their resultant moment can be determined, as in Art. 60, p. 43, and therefore the moment of S is found. Again, since y, the perpendicular distance of S from O, is known, therefore S can be found. If in constructing the resultant moment of the known external forces the pole distance be made equal to y, then the line which (measured with the force scale), multiplied by the pole distance, gives the resultant moment, will, when measured with the force scale, give the magnitude of S.

Having found S, the force P may be found in like manner by taking moments about another point in the bar B (say at the intersection of B and C). Lastly, the force Q may be determined by taking moments about a point outside the bar B.

The forces P and Q may, however, be determined by the polygon of external forces after S has been found.

Another method of finding S is by means of the polygon of forces

and a funicular polygon, the latter having one angular point at O, as explained in the latter part of Art. 57, p. 38.

182. The Three-Hinged Arch.—If the ends of a roof or bridge truss are secured to foundations by hinged joints, and there is another hinged joint at an intermediate point, say, at the middle of the truss, such a truss is known as a *three-hinged arch*, and it is said to be constructed on the *three-hinged system*. The determination of the stresses in the various bars of such a truss may be proceeded with as in an ordinary truss as soon as the reactions at the hinges are determined.

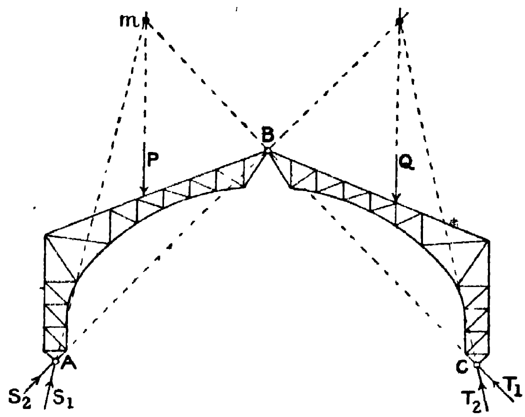


FIG. 268.

One method of finding the reactions at the hinges is as follows. Fig. 268 shows a truss hinged at A, B, and C. The resultant load on the part AB is the force P, and the resultant load on the part BC is the force Q. First neglect the load acting on the part BC. The part BC is then under the action of two forces only, viz. the reactions at B and C, and these forces must balance one another, and will therefore act in opposite directions along the straight line BC. The truss as a whole is now under the action of three forces, viz the force P, the reaction T_1 at C, which acts along CB, and the reaction S_1 at A. Since these three forces are in equilibrium, and since the lines of action of two of them, T_1 and P, meet at *m*, therefore the line of action of the third one, S_1, must be A*m*. By means of the triangle of forces the magnitudes of S_1 and T_1 can be determined.

Next neglect the load on the part AB, and consider the load Q on the part BC. This load Q will cause reactions S_2 and T_2 at A and B respectively, and these reactions may be found in the same way as S_1 and T_1 were found.

When both loads P and Q act, it is evident that the reaction at A will be the resultant of S_1 and S_2, and the reaction at B will be the resultant of T_1 and T_2.

The reaction of the part AB on the part CB at B will be the force which will balance the force Q and the reaction at C, and the reaction of the part CB on the part AB at B will be the force which will balance the force P and the reaction at A. These two reactions will, of course, be equal and opposite.

When the truss is symmetrical about a vertical centre line, and is symmetrically loaded, the reactions at B will be horizontal, and the line of action of the reaction at A will be the line joining A with the point of intersection of the line of action of the resultant load on the half truss AB with the horizontal line through B. The direction of the reaction at C is found in like manner.

Exercises XII.

1–8. Draw the stress diagrams for the structures shown in Fig. 269. Measure the results, and either tabulate them or mark them on the frame diagram. Indi-

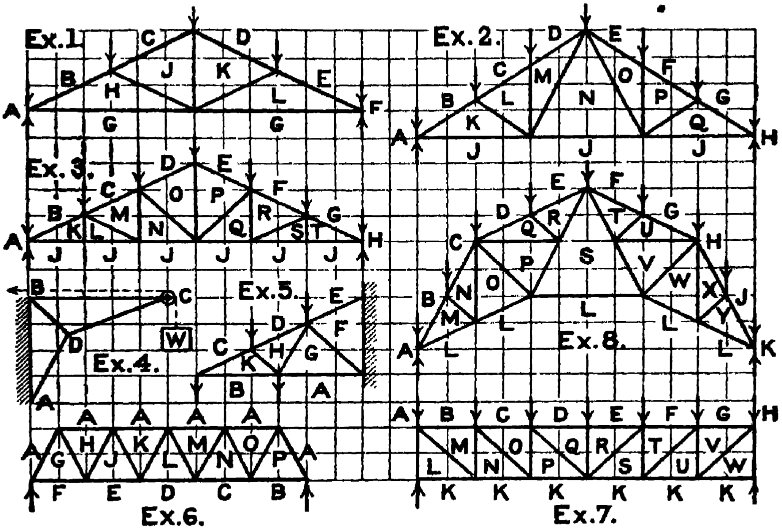


Fig. 269.

cate on the frame diagram the members which are in compression by lining them in with thick lines. In addition to determining the results graphically, they should also be found by calculation.

1. Span = 24 feet. The loads are as follows: AB = EF = 250 lbs., BC = CD = DE = 500 lbs.

2. Span = 30 feet. The dead loads are as follows: AB = GH = 400 lbs., BC = CD = DE = EF = FG = 800 lbs. The wind pressure is to be taken at 6000 lbs., acting at right angles to, and distributed over, the sloping surface as follows: 2000 lbs. at each of the intermediate joints, and 1000 lbs. at the top and bottom joints. The reactions at the supports due to the wind pressure are to be assumed to be parallel. Tabulate the stresses due to (1) dead load, (2) wind on left, (3) wind on right, and state also the maximum stress in each bar.

3. Span = 48 feet. The loads are as follows: AB = GH = 500 lbs., each of the other loads = 1000 lbs.

4. A wall crane. The bar BC is horizontal, and 10 feet long. The bar BD bisects the angle ABC, and is 3 feet 9 inches long. The distance AB is 8 feet. The chain passes over a pulley at C, as shown, and supports a weight W of 1000 lbs.

5. Pent roof truss projecting 18 feet from the wall. Each load shown is 1000 lbs.

6. Warren girder of 40 feet span. Case (*a*). There is a load of 10 tons at the joint EJKLD. Case (*b*). There is a load of 8 tons at the joint EJKLD, and a load of 6 tons at the joint CNOPB. Case (*c*). There is a load of 6 tons at each of the joints in the bottom boom.

7. Span, 48 feet. Load AB = load GH = 3 tons. Each of the other loads shown = 6 tons.

8. Curb roof truss of 48 feet span. The loads AB and JK are each 500 lbs., and each of the other loads shown is 1000 lbs. Case (*a*). Take the truss as shown. Case (*b*). Suppose the bar LS to be removed, and that the truss is converted into a three-hinged arch, as explained in Art. 182. Determine the reactions at the hinges, and the stresses in the various members.

9. A jointed frame, shown in Fig. 270, is subjected to six forces, acting one at each joint. The directions of the forces bisect the angles marked, and the magnitudes of three of them are given in the sketch. Find, graphically or other-

wise, the magnitudes of the unknown forces X, Y, Z, and deduce the force in the member AB. [U.L.]

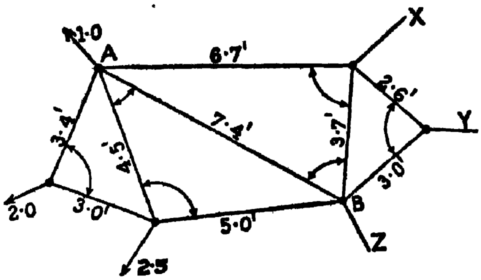


FIG. 270.

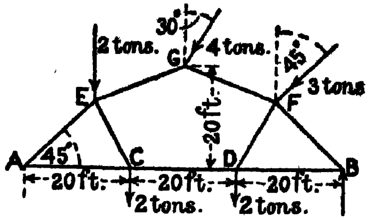


FIG. 271.

10. In the truss shown (Fig. 271), AB is a beam member. The links AE, EG, GF, and FB are equal, and the links EC and FD connect the joints E and F to the points C and D of the beam member. The reaction at B is vertical. Draw to scale, diagrams of bending moment and shear force for the beam member AB. [U.L.]

11. The span of the frame given in Fig. 272 is 57 feet, the depth LM is 7 feet, and the distance NO is 1·8 feet. There are loads of 2 tons at P, 4 tons at O, and 2 tons at M. Find the stresses in the members. [U.L.]

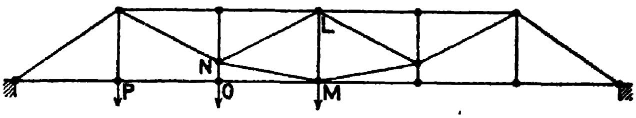


FIG. 272.

12. A truss is as shown in Fig. 273. Each top joint carries a load W, and the top boom is divided into equal segments. The length of the horizontal tie-rod is half the span. Determine the ratio of depth to span so that the tension in all the tie-rods shall be the same, and verify your result by drawing a force diagram. [U.L.]

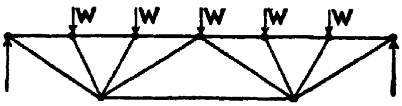


FIG. 273.

13. The sketch (Fig. 274) shows the frame of a tandem bicycle. Calculate

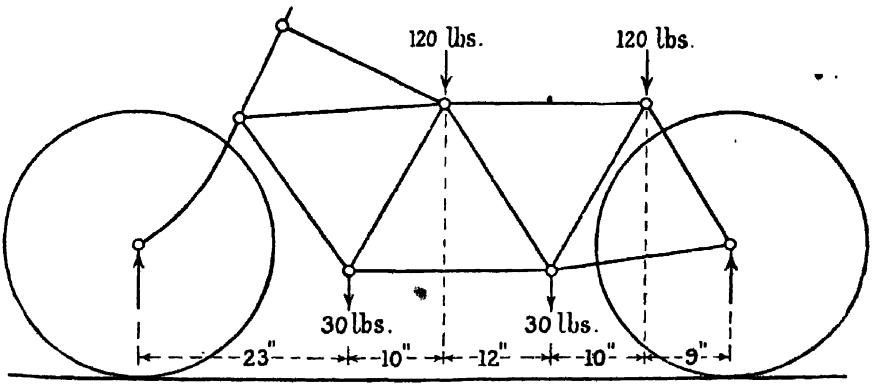


FIG. 274.

and draw bending moment and shear diagrams for this frame, when each of the two riders weighs 150 lbs., 30 lbs. of the weight of each rider being assumed to be borne at the driving axle centres. Regarding the frame as a plane structure, draw a complete force diagram for it. [B.E.]

14. A *Bollman truss* is shown in Fig. 275. The vertical struts divide the span into six equal parts. The truss carries a uniformly distributed load of 1 ton per foot run, and a single load of 10 tons at 20 feet from the left-hand support. Find the forces, in tons, in the various members. Note that the sloping members have joints at their ends only. The top member is really a beam, but in working this problem the beam may be assumed to have pin joints at the junctions with the vertical struts.

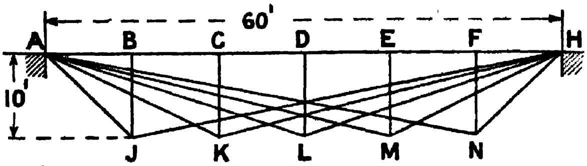


FIG. 275.

15. Show that for a Bollman truss, if n = number of equal panels in the truss, d = depth of the truss, l = span, and w = intensity of the uniformly distributed load, then the horizontal stress in the top member is $\frac{wl^2}{6n^2d}(n^2 - 1)$. [U.L.]

16. A *Fink truss* is shown in Fig. 276. The vertical struts divide the span into eight equal parts. There is a load of 8 tons at the top of each vertical strut. Find

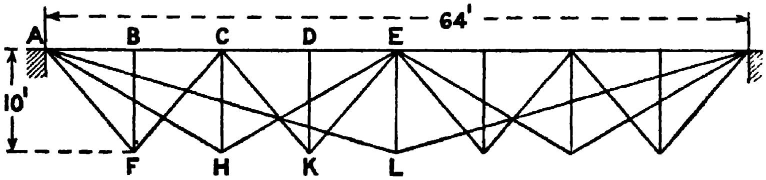


FIG. 276.

the forces, in tons, in the various members. Note that the sloping members have joints at their ends only. The top member is really a beam, but in working this problem the beam may be assumed to have pin joints at the junctions with the vertical struts.

CHAPTER XIII

DESIGN OF STRUCTURES—ROOFS

183. Roofs and Roof Trusses.—The function of a roof or upper covering to a building is to protect the interior from wind and weather. It consists of a weather-proof covering supported on a suitable framework. This covering may be made either flat, sloping, or curved, the pitch or slope depending largely upon the nature of the material of which it is composed. The framework consists of (1) the *roof trusses*, or *principals*, which span from support to support and carry the roof structure; (2) the *purlins*, longitudinal beams which run from truss to truss along the roof; (3) the *rafters*, *sash bars*, etc., which rest upon the purlins, and to which the covering proper is fixed; (4) the *wind ties*, which prevent longitudinal distortion of the roof by the wind.

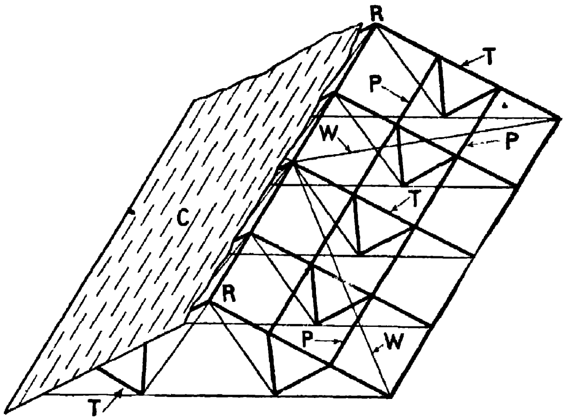


FIG. 277.

In Fig. 277, which shows, in oblique projection, the framing of a roof, TT . . . are the main trusses or principals, PP . . . the purlins, WW the wind ties, and C the roof covering. RR is called the *ridge* of the roof, and the lower longitudinal edges are called the *eaves*.

184. Iron Roof Trusses.—Roof trusses may be made of wood, but iron or steel principals are superior in nearly every respect for spans of any considerable size. Figs. 278–290 represent the more common types of iron or steel roof trusses. They are all composed of the following members or parts: (1) the *principal rafters*, which are the members, either straight or curved, running from the ridge to the abutments or supports, and carrying the purlins; (2) the *tie rod*, which may be straight or cambered, whose function it is to tie the two feet of the principal rafters together, and thus relieve the abutments of the horizontal thrust, which would otherwise come upon them; (3) *secondary bracing*, which divides the principal rafter into panels, and thus supports it both as a strut and as a beam. The upper ends of these secondary members should come directly under the purlins, and thus relieve the principal rafter of transverse bending actions. The axes of any three adjacent

members of a truss should either meet at a point or form a triangle. All members of the truss should either be simple ties or struts. Struts should be as short as possible, and as many members as possible should be in tension.

The feet of the principal rafters rest in *shoes*, which rest in turn upon *wall plates*, bolted to the walls or other abutments.

185. Forms of Roof Trusses.—Figs. 278–283 represent six of the commoner forms of "King-rod" and "Queen-rod" trusses. The member depending from the apex at the junction of the principal rafters is called the *king-rod*, while the other vertical suspension rods are called *queen-rods*.

FIG. 278.

FIG. 279.

Fig. 278 shows the simplest form of iron roof truss. The principal rafters are only supported at their ends, and a single king-rod with the tie rod complete the framing. This design may be used for spans up to 15 feet. In Fig. 279 each principal rafter is divided into two equal panels by a secondary brace (a strut), and the span may be increased to about 25 feet.

The design shown in Fig. 280 may be used for spans up to 30 feet.

By dividing each principal rafter into three equal panels, and adding

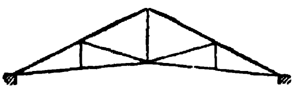


FIG. 280.

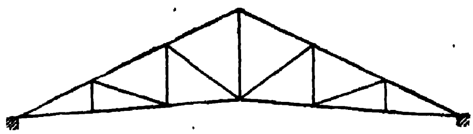


FIG. 281.

two queen-rods in addition to the king-rod, as shown in Fig. 281, the span may be from 35 feet to 45 feet.

The design shown in Fig. 282 is sometimes called an *English truss.* The principal rafters are each divided into four equal panels. This truss may be used for spans up to 60 feet.

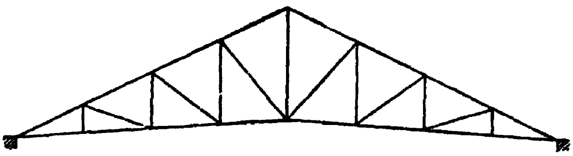


FIG. 282.

The *saw-tooth* or *workshop truss* is shown in Fig. 283. This form of truss is extensively used for the roofs of weaving sheds and the like. The slopes of the rafters are unequal, the covering on the lesser slope being slates or tiles, while that on the greater slope is glass to light the interior. The truss shown may be used for spans of from 20 feet to 35 feet.

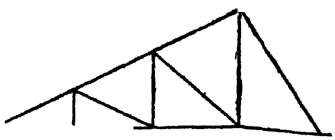


FIG. 283.

King and queen-rod roof trusses, having vertical members, are very suitable for hipped roofs. They have the disadvantage that the long braces are struts and the short ones ties. This is sometimes obviated by sloping the diagonals in the

other direction, as shown in Fig. 284. The verticals, except the centre one, then become struts, and the diagonals ties.

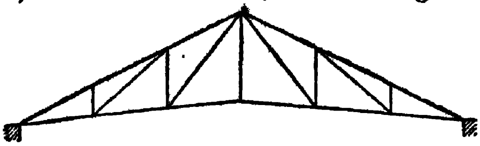


FIG. 284.

FIG. 285.

Figs. 285–288 show the more frequent types of *trussed rafter* roofs. The principal rafters are supported by trusses, consisting of struts and

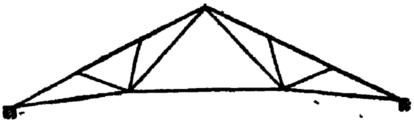


FIG. 286.

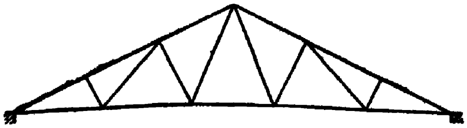


FIG. 287.

ties. The upper ends of the struts divide the rafters into panels, and the lower ends are supported by the tie rods. The trusses supporting the two opposite rafters are held together by the main tie rod. This type of truss has the advantage that the struts, which are usually perpendicular to the rafters, are short. For a given span and a given system of loads, this type of truss probably makes a lighter roof than any other type of principal.

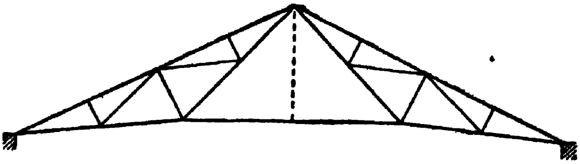


FIG. 288.

The design shown in Fig. 285 is used for small spans up to, say, 20 or 30 feet. The design shown in Fig. 286 may be used for spans up to 40 feet, and that in Fig. 287 up to 45 feet. The truss shown in Fig. 288 is known as the *French*, *Belgian*, or *Fink truss*. It is a very common design for trusses of from 40 to 60 feet span. A light suspension rod,

FIG. 289.

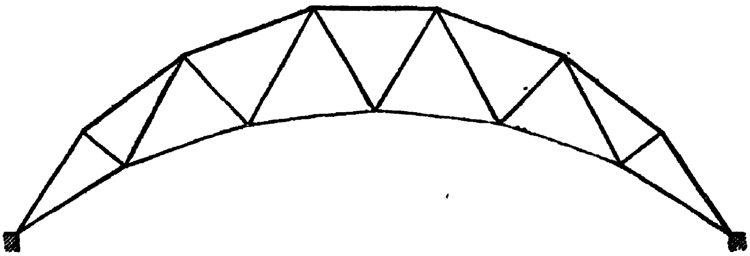


FIG. 290.

shown dotted, is often introduced to give support to the main tie rod.

Two types of curved roof truss are shown in Figs. 289 and 290; that

shown in Fig. 289 has curved rafters, and may have a span of from 20 to 40 feet. The sickle-shaped truss, shown in Fig. 290, is of a type suitable for large spans of, say, from 40 to 100 feet.

For very large spans, arched principals are used. The abutments must then be made strong enough to carry the thrust of the arch. The span may, however, be divided, and combinations of simple roof trusses used.

186. Roof Coverings.—*Zinc*, in sheets, about $\frac{1}{32}$ inch thick, laid upon boarding with wooden rolls, forms a light covering. The joints must be arranged to allow free expansion and contraction of the metal with changes of temperature, while still remaining water-tight. The sheets are 7 to 8 feet long and about 3 feet wide.

Corrugated iron is much used as a roof covering. For small spans, not exceeding 10 feet, it may be simply arched, and used without any main trusses, the free ends being held in at intervals by tie rods, or simply screwed to timber wall plates. On larger spans curved angle or tee-irons are introduced as rafters to give support and stiffness. If, however, the span exceeds 15 feet, a properly designed roof truss should be used; the sheets of corrugated iron are then laid upon purlins. The sheets of corrugated iron vary from 6 to 8 feet in length, 2 to 3 feet in width, and from 24 to 16 I.S.W.G. thick (0·022 to 0·064 inch). The corrugations vary in width from 3 to 6 inches centre to centre, the depth being about one-fourth of the width.

The span of the sheets depends upon the depth of the corrugations, the thickness of the metal, and the weight per square foot to be carried. The following formula may be used. $L=12\sqrt{\frac{td}{w}}$, where L = span in feet, t = thickness of metal in inches, d = depth of corrugations in inches, and w = weight per square foot to be supported.

The sheets may be laid directly on the purlins, the corrugations following the slope of the roof. The lap of the horizontal joints should not be less than 6 inches, and these joints should come directly over the purlins. Where two sheets join along their sides, at least one complete corrugation should overlap, and the two sheets should be fastened together by screw bolts or rivets pitched about 9 inches apart.

If the purlins are of wood, the corrugated iron may be fastened to them directly by means of screws or by stirrup bolts, as shown at (*b*), Fig. 291. If angle iron purlins are employed, the sheets are best fixed by means of hook bolts, as shown at (*a*), Fig. 291. These hook bolts are commonly about $\frac{7}{16}$ inch in diameter. All bolts and rivets should pass through the ridges of the corrugations, and should be provided with washers to prevent leakage. A flat bar, or sometimes an angle bar, is often introduced at the eaves, running along the length of the roof, and held down by the lowest row of bolts. This prevents the wind from tearing the corrugated iron from the roof should it get beneath the sheets. This wind tie is shown at (*b*), Fig. 291.

Large *slates* (Duchess, 24 inches × 12 inches, or other sizes) are often used for the covering of iron roofs. These may be laid upon boarding and nailed in the usual way, or light angle iron purlins may be fixed to the rafters, and on these the slates are laid and then wired on, as shown at (*e*), Fig. 291. These angle iron purlins may be about $1\frac{1}{2}$ in. × $1\frac{1}{2}$ in. × $\frac{1}{4}$ in.

when the distance between the rafters does not exceed 8 feet, or 2 in. × 2 in. × $\frac{5}{16}$ in. when that distance is increased to 10 feet.

Tiles of great variety are used for roof coverings. They are supported in the same way as slates. Tiles are heavy, and they require the roof to be of high pitch.

Glass is largely used as a roof covering. Many roofs have glass skylights, while some are entirely covered by glass. The shorter slope of a saw-tooth roof and many railway station roofs have glass coverings. The

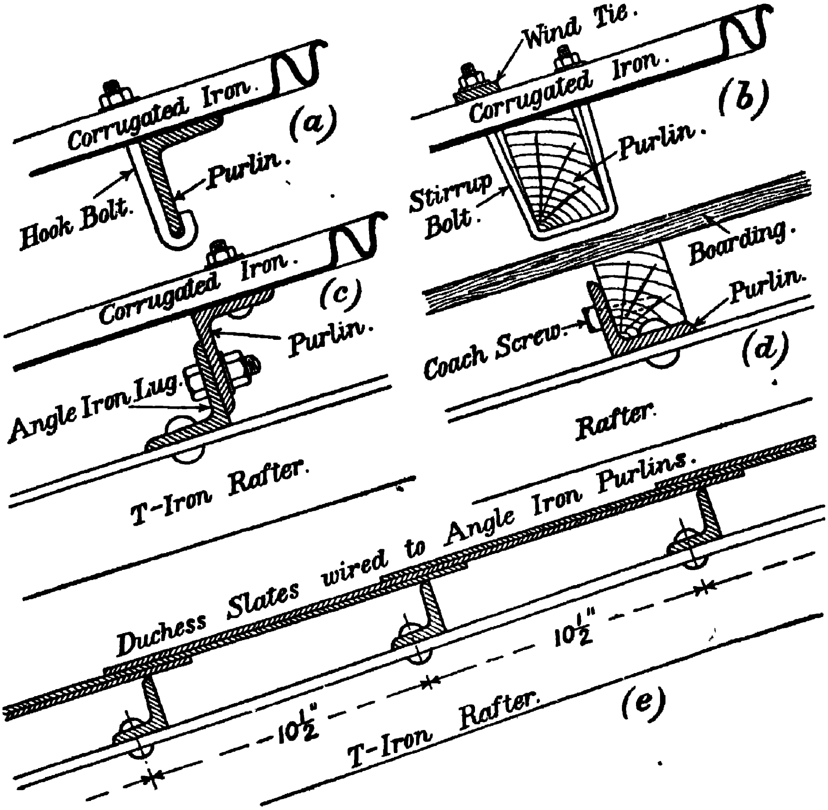


FIG. 291.

glass may be laid in sash bars of wood or tee-iron, with putty. Iron sash bars, however, expand and contract more than the glass with changes of temperature, and the putty is liable to crack. Hence many systems of glazing without putty have been introduced.

Glass sheets suitable for roofing vary in width from 12 to 20 inches, and in thickness from $\frac{1}{8}$ inch to $\frac{1}{4}$ inch. They are made in lengths up to 6 feet. About 3 inches of lap should be allowed between two sheets. Tee-iron sash bars vary from 1 inch to 2 inches in depth, and from $\frac{1}{4}$ to $\frac{5}{16}$ inch in thickness.

187. Details of Roof Trusses.—The various parts of roof trusses should be made as plain and simple as possible. Forging and welding

should be avoided, being expensive, and welding is not always reliable. Riveted and bolted joints form the best connections. Rivets and bolts are best placed in direct shear, but if such fastenings have to be in tension, bolts should be used in preference to rivets. Bolts are also generally preferred when the diameter exceeds 1 inch. Gussets, in many cases, form a convenient means of attachment, particularly where many members meet. Care should be taken that at a joint the axes of the

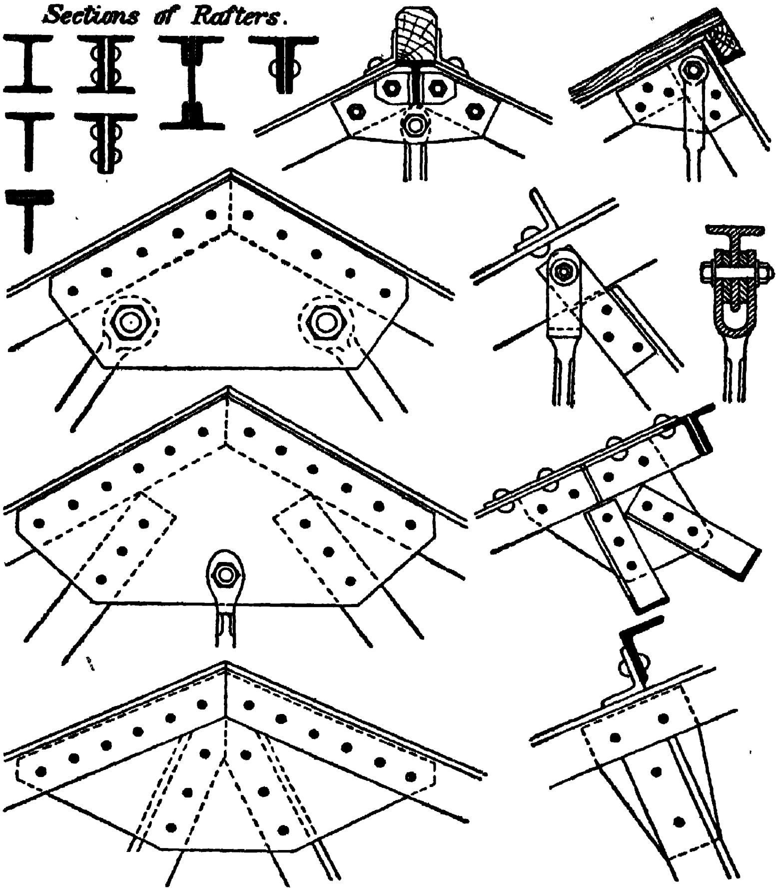
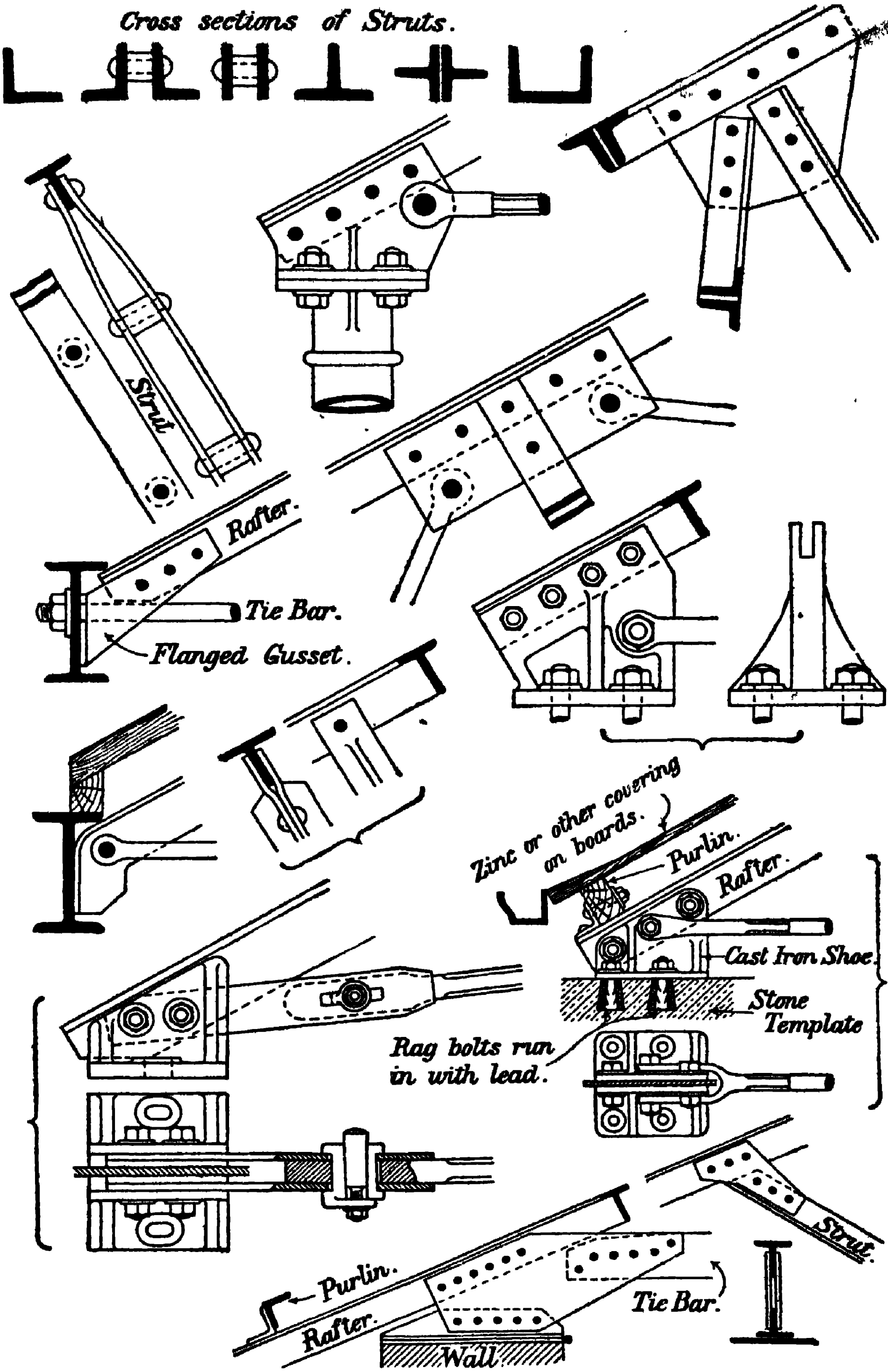


FIG. 292.

various members intersect at a point. This is a condition which is too often neglected in practice.

Sections of rafters are shown in the upper part of Fig. 292. A tee section is the most common form for small roofs. Double angles and built up sections are used for larger spans. A rafter must be of a form enabling it to act as a strut, and at the same time affording convenient attachment for the secondary members and purlins.

At the ridge or apex the rafters are united by double gussets, which also form the fastening for the braces which are attached there. The trusses should be tied together at the apex, either by the purlins placed



Cross sections of Struts.
Strut
Rafter.
Tie Bar.
Flanged Gusset.
Zinc or other covering on boards.
Purlin.
Rafter.
Cast Iron Shoe.
Stone Template
Rag bolts run in with lead.
Strut.
Purlin.
Rafter.
Tie Bar.
Wall

FIG. 293.

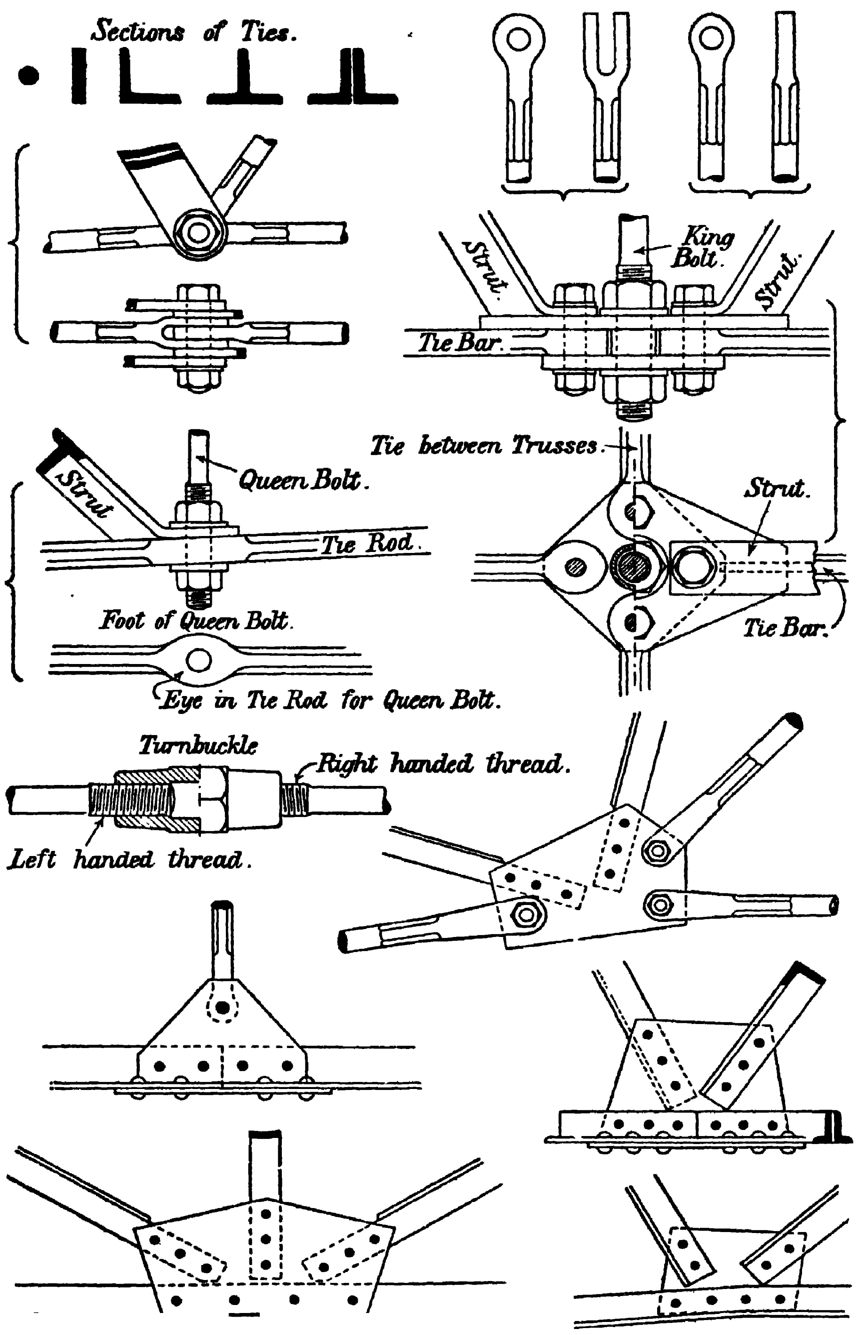

Sections of Ties.
King Bolt.
Strut.
Strut.
Tie Bar.
Tie between Trusses.
Queen Bolt.
Strut
Tie Rod.
Strut.
Foot of Queen Bolt.
Tie Bar.
Eye in Tie Rod for Queen Bolt.
Turnbuckle
Right handed thread.
Left handed thread.

FIG. 294.

there, or by a special ridge member. Examples of ridge joints and methods of connecting the secondary braces to the rafters are shown in Fig. 292.

Further detail illustrations, mainly relating to struts and shoes, are given in Fig. 293. Struts are usually angles or tees, or combinations of them. A simple and efficient strut is formed of two flat bars, or two bars of other suitable section, held apart by distance pieces suitably spaced. Such a strut must be arranged to carry the total load upon it while acting as a whole, and must also be strong enough between the distance pieces to resist local buckling. The attachment of struts to the rafters is usually by means of gussets, but they may also be attached directly. Angles, tees, etc., are sometimes joggled at the ends to suit the rafter.

Shoes may be made of cast-iron, but built up shoes from rolled sections are common. They must hold the end of the rafter firmly, allow convenient attachment for the tie bar, and afford suitable bearing for the truss. The axes of the tie bar and rafter should meet at a point on the line of the vertical reaction from the wall or support. When the

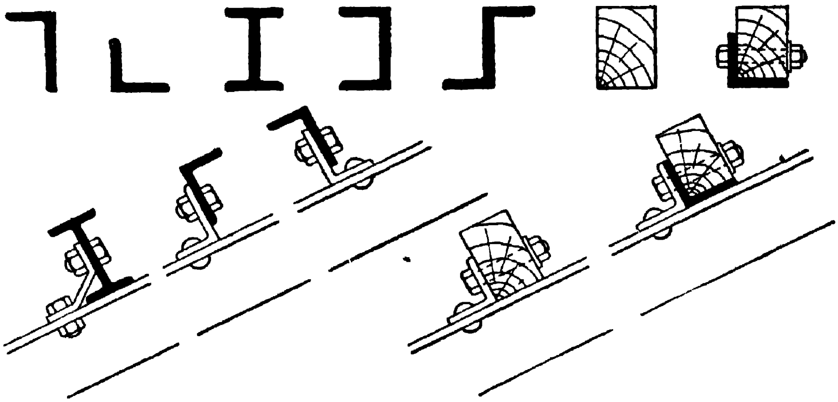


FIG. 295.

truss rests upon a wall a stone templet is provided for the shoe to rest upon; Lewis or rag bolts let in with lead form a suitable holding. When the shoe is to slide, slotted holes are provided for the bolts. Sliding shoes often rest upon sole plates.

Ties and their connections are shown in Fig. 294. Flat bars placed with their widths in the plane of the truss form good ties. Round bars may look neater, but they are more costly than flat bars as ties, especially for large sizes. Angle and tee bars work in well for ties of large section. With flats, angles, etc., the joints are usually made by means of gussets with rivets or bolts. In the case of round tie bars, forked ends or eyes may be forged on them to make pin joints. Another practice is to screw the ends of round tie bars, and the ends are sometimes staved up before screwing in order to save weight. Adjustment may be obtained by making a cottered joint, or by cutting the rod and introducing a turnbuckle. With good workmanship such adjustment should be unnecessary. At the foot of the king-rod a tie between the trusses is often introduced; this prevents lateral movement of the tie rods.

Fig. 295 shows cross sections and details of fixings for purlins.

Several forms of joints for purlins are shown in Fig. 296.

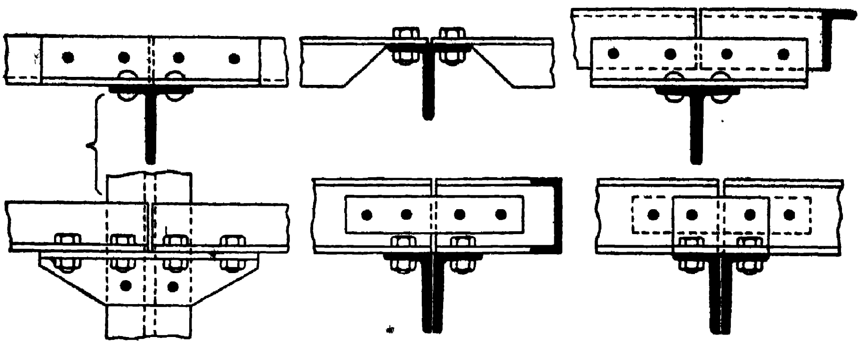


FIG. 296.

Two methods of introducing longitudinal wind bracing are shown in Fig. 297.

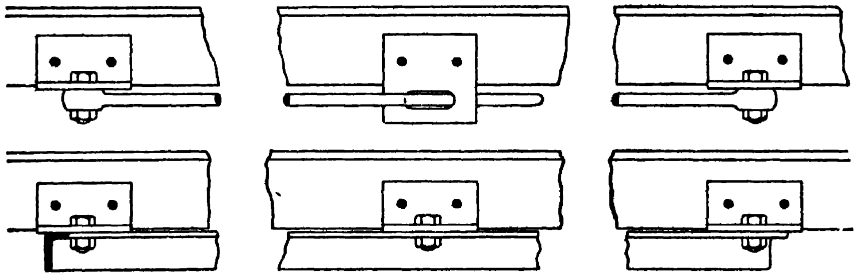


FIG. 297.

188. Weight of Roof Coverings.—For the purpose of estimating approximately the weight to be carried by a roof truss, the particulars given in the following table may be used. The weights given are in pounds per square foot of covered area.

Sheet zinc	$1\frac{1}{2}$	Slates, large	10
Corrugated iron	$3\frac{1}{2}$	„ medium	7
Tiles, plain		„ small	$5\frac{1}{2}$
Pantiles	12	Boarding, 1 inch thick	$3\frac{1}{2}$

189. Pitch and Slope of a Roof.—The ratio of the rise to the span is called the *pitch* of a roof. If the roof is symmetrical and the slope or inclination is denoted by θ, then $\tan\theta = \text{rise} \div \text{half the span}$. The minimum slope for a roof depends on the nature of the covering, and is roughly 5° for zinc, 11° for corrugated iron, 22° for large slates, 26° for pantiles and medium sized slates, 30° for small slates, and 45° for plain tiles.

190. Procedure in Designing a Roof.—The method of procedure in getting out the designs for a roof may now be briefly given.

Decide upon the type of truss. This will depend upon the various

conditions which the roof has to satisfy, the type of building it has to cover, the span, whether the ends are to be hipped or not, etc.

Settle the type of roof covering, and arrange for a suitable support for it, seeing that the proposed distance apart of the purlins works in with the secondary bracing of the truss chosen.

The pitch should next be fixed. This depends largely on the type of covering to be used.

Decide whether the tie rod is to be cambered or not. This depends upon the conditions of the case, whether the roof is to support a ceiling, whether head room is a necessity, etc. The advantages of a camber are, shorter struts, greater head room, and better appearance. Speaking generally, it is better to give the tie rod a small camber if possible.

Draw an outline diagram of the truss. The proportions should please the eye.

Fix the distance apart of the principals. This will depend to some extent on the type of roof covering, purlins, etc. Usually it may be made from one-eighth to one-fourth of the span. The larger the interval chosen, the larger the ratio of the least lateral dimension of the struts to their length, and they will therefore be lighter in proportion to their strength. Too large a pitch of principals involves heavy purlins and increases the cost.

Find the loads upon the truss. These are: (1) The weight of the principal. (2) The weight of the covering. (3) The weight of snow upon the roof. (4) The weight of the ceiling, if any, carried by the trusses. (1) is often neglected, but in large roofs it should be allowed for. (2) can be approximately estimated (see table, p. 210). Allowance should also be made for the weight of the purlins, rafters, etc. (3) can be taken at about 6 lbs. per square foot of area covered in the British Isles. (4) must be estimated approximately, the weight being carried from the lower joints of the roof. For wind pressure, see Arts. 179 and 180, p. 194.

The loads should now be divided up, and the resulting forces at each of the joints found. It is well to keep the loads at the joints due to the wind pressure separate from the others.

Choose next the methods of support for the ends of the trusses. Usually one end is left free, to allow the principal to expand and contract with changes of temperature. The reaction at this end is then assumed to be vertical.

Find the stresses in the members, either graphically or analytically, or preferably by both methods. The dead load stresses should be found, the stresses with the wind pressure on one side and then on the other, and the three sets of figures should be combined, as shown on p. 196.

The sections of the various members can now be ascertained by the ordinary rules. It is safer to assume that all the struts are hinged at the ends. See also that members in which the stress reverses are capable of withstanding the reversed load, although it may be smaller than the normal load. Use a low working stress for these members. Use also a low stress for members which are welded.

Design the joints. Arrange sufficient rivets, bolts, or pins to take the stress from the bars on to the gussets. Where a gusset connects one or more ties or struts to a rafter, bear in mind that the total shearing

force on the rivets connecting the gusset to the rafter is the resultant of the forces acting along the various members, other than the rafter itself, connected to the gusset.

Design the shoe and whatever other details of the truss have not already been arranged for.

Fix the size of the purlins, treating them as beams carrying their load from rafter to rafter, and arrange for suitable joints in them.

Settle any further details of the covering which may be necessary.

Arrange for suitable wind ties if it is deemed prudent to fit these.

Make complete working drawings of the whole roof, seeing that all the parts go together properly, can be easily made, and are in every way suitable for the functions which they have to perform.

Exercises XIII.

In the following exercises the various members must be proportioned according to the loads which they have to carry, and working drawings of the various details should be made.

1. Design for a king-rod roof truss (Fig. 298). Span, 20 feet. Rise at centre, 5 feet. Distance between principals, 6 feet. Assume that the truss has to support a total load of 2 tons per "square" acting vertically. (A "square" is 100 square feet of area covered.) The following sections are to be used.—Rafters, tee; struts, angle; ties, flat. Material to be mild steel, and the joints to be riveted.

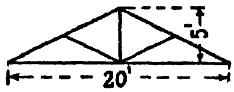


FIG. 298.

2. Design a roof suitable for covering a shed with open ends. The span between the supports is 35 feet, and the trusses are to be placed 8 feet apart. The principal rafters have a rise of 10 feet, and the tie bar has a camber of 2 feet. The form of truss to be used is shown in Fig 299. The covering is to be corrugated iron on angle iron purlins. The dead weight upon the roof may be assumed in the first instance to be 10 lbs. per square foot. Snow, 6 lbs. per square foot, and horizontal wind pressure 50 lbs. per square foot. Lateral wind bracing is to be provided. Rolled steel sections and riveted joints are to be used.

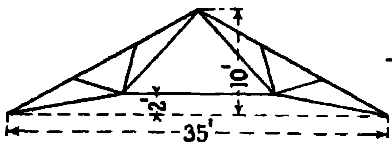


FIG. 299.

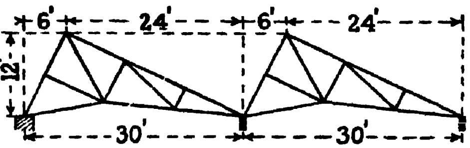


FIG. 300.

3. A design is required for a trussed roof of the saw-tooth pattern (Fig. 300). There are several similar spans. The trusses over the first span are bolted to a wall at one side, and all the other supports are columns, as shown. The distance apart of the principals longitudinally is 7 feet 6 inches. The steep slope is to be covered with glass, the other with slates on boarding. Take the dead weight of the roof as 18 lbs. per square foot, the weight of the snow as 6 lbs. per square foot, and the horizontal wind pressure as 50 lbs. per square foot. Round bars may be used for the ties, angles and tees for the other members.

4. Design for a slated roof. The trusses to be of the "French" pattern. Pitch, $\frac{1}{4}$. Camber of tie rod, $\frac{1}{40}$. Span, 50 feet. Distance apart of principals, 10 feet. The covering to be Duchess slates laid upon 2 inch boarding supported by angle purlins. The roof is estimated to weigh as follows: Slates, 9 lbs. per square foot. Boarding, 7 lbs. per square foot. Purlins, 3 lbs. per square foot. One truss, $\frac{3}{4}$ ton. Snow, 6 lbs. per square foot, and the *normal* wind pressure

28 lbs. per square foot. One end of the truss is to be firmly bolted down, and the other to be capable of sliding. Rolled steel sections only to be used, with riveted joints.

5. Design a queen-rod roof truss with vertical struts of the form and dimensions shown in Fig. 301. Distance apart of principals, 10 feet. Take the total dead weight of the roof as 20 lbs. per square foot, horizontal wind pressure as 50 lbs. per square foot, and snow as 6 lbs. per square foot. One end of the truss to be free to slide. The struts to be formed of flats, with distance pieces. The ties also to be formed of flats.

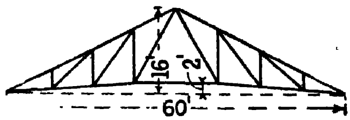


FIG. 301.

6. Design for a sickle-shaped roof truss of the form shown in Fig. 290, p. 203. The span is 100 feet. The total rise is 25 feet, and the bottom chord has a rise of 10 feet. There are eight equal segments in the top chord, and seven equal segments in the bottom chord. The trusses are 20 feet apart. Take the dead load as 18 lbs. per square foot. Snow, 6 lbs. per square foot. Horizontal wind pressure, 50 lbs. per square foot. One end of the truss is bolted down, and the other slides. Use rolled steel sections only.

7. The principals of a steel roof for a dock shed are of the form sketched in Fig. 302. The rafters are equally divided at the joints, and a vertical load of $1\frac{1}{4}$ tons acts at each top joint. The principals are supported on girders 8 inches wide. Draw the force diagram for the roof, and tabulate your results, distinguishing between ties and struts. Design also the joints at A and B. Choose your own stresses, and draw the details one-quarter full size. [U.L.]

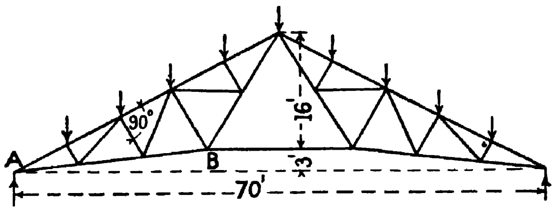


FIG. 302.

8. The tie rod of a roof truss is connected to the foot by two clip plates, and by a cotter joint with two gibs. The diameter of the tie rod is $1\frac{3}{4}$ inches. Design this joint for equal strength throughout. The type of joint is indicated in the sketch (Fig. 303). Draw, full size, plan and elevation, and any necessary sections. The drawings must be fully dimensioned, and finished off neatly in pencil. All calculations must be fully worked out, and must be handed in with your drawings. [U.L.]

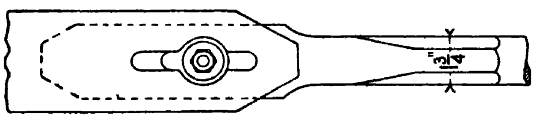


FIG. 303.

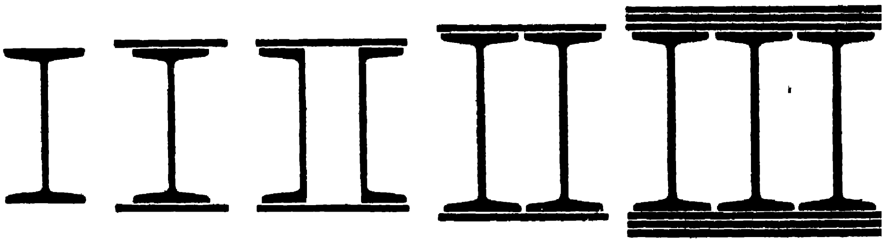
CHAPTER XIV

DESIGN OF STRUCTURES—PLATE GIRDERS

191. Beams and Girders.—In Chapter VII. it has been shown that the straining actions at any cross section of a beam, due to any system of vertical loading upon it, may be resolved into two distinct effects, namely, a bending action and a shearing action. It has also been shown that, for economy, beams are made with a cross section shaped like the letter **I**, in which case the top and bottom flanges may, for practical purposes, be assumed to resist the bending moment, whilst the web takes the shearing force.

The concentration of the material into a web and flanges may be obtained by using a rolled steel joist or channel, and these may be combined with plates for stronger sections. Again, separate plates may be used for the web and flanges, which are united by angles. Another form is obtained by substituting diagonal bracing or lattice work of bars for the plate web.

192. Beams of Rolled Joists or Channels and Plates.—The most common form of beam in use for short spans is the rolled steel joist



(*a*) (*b*) (*c*) (*d*) (*e*)

FIG. 304.

shown at (*a*), Fig. 304. The standard sections for rolled steel joists are numerous, and range from 3 inches deep by $1\frac{1}{2}$ inches wide, weighing 4 lbs. per foot of length, to 24 inches deep by $7\frac{1}{2}$ inches wide, weighing 100 lbs. per foot of length. These joists can generally be obtained from stock in lengths of every foot from 10 feet to 40 feet for ordinary sections. For convenience in rolling, the flanges are tapered in section, as shown, the angle between the inside of the flange and the web being 98°. Should the load require it, two or more of these joists may be placed side by side, and they may also be strengthened by having plates riveted to their flanges, as shown at (*b*), (*d*), and (*e*), Fig. 304. A beam, formed of two channels connected by plates, is shown at (*c*), Fig. 304.

193. Connections between Rolled Joists.—When two or more joists are used side by side, without connecting flange plates, cast-iron *separators*,

or distance pieces, are placed between them, as shown in Fig. 305. Separators should be placed at intervals of about 5 feet, and also where a concentrated load occurs on the beam.

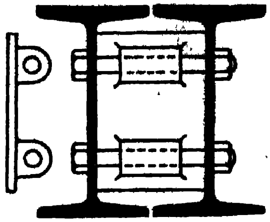


FIG. 305.

A joint between two lengths of joists is made by means of fish plates, as shown in Fig. 306, and if there is any bending moment where this joint is made, cover straps on the flanges should be added, as shown by the dotted lines.

Angle connections between horizontal joists at right angles to one another are shown in Fig. 307. Angle connections between horizontal joists and joists used as columns are shown in Figs. 308 and 309. In these various connections, where the load on one beam is transmitted to another, or to a column, through rivets or bolts,

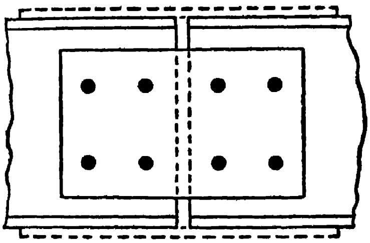


FIG. 306.

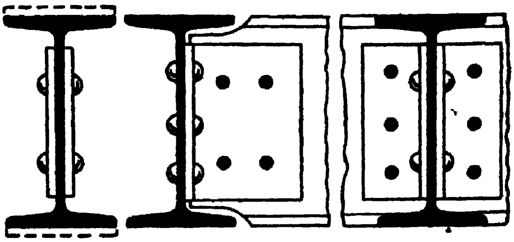


FIG. 307.

care must be taken that the rivet or bolt section is sufficient to transmit the load. Many of these details are, however, standardised by the manu-

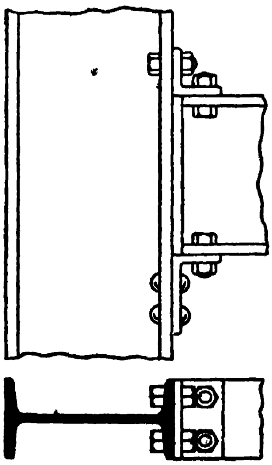


FIG. 308.

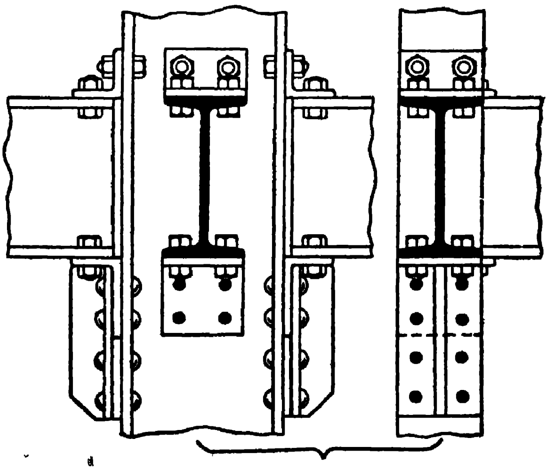


FIG. 309.

facturers, and provided that the standard connections are capable of carrying the loads which will come upon them, they should be used in preference to specially designed ones.

194. Parallel Girders and Girders of Variable Depth.—*Parallel girders*, as their name implies, have their flanges parallel to one another,

and they are therefore of constant depth throughout their length. *Hog-backed girders* have a curved top boom, and *fish-bellied girders* have a curved bottom boom, as shown in Fig. 310. The effect of curving one boom is to increase the depth of the girder towards the centre, where the bending moment is greatest. This permits of the cross section of the

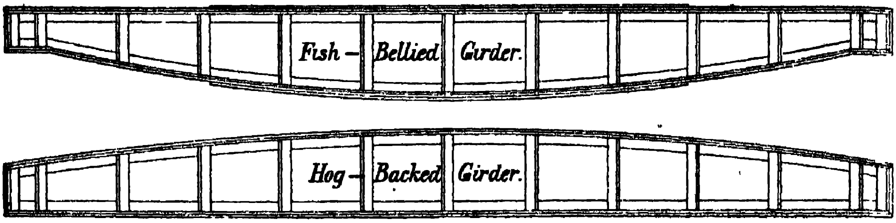


FIG. 310.

booms being kept more nearly constant. Except under special circumstances, it is generally better and cheaper to use a parallel girder than one of variable depth, the cross section of the flanges being varied to approximately suit the bending moment. Fish-bellied girders are usually adopted for overhead travellers of large span. Hog-backed girders are frequently used for large span railway bridges.

195. Plate Girders.—When the depth of a girder exceeds a foot, but is less than the limiting depth for a rolled joist, it is frequently more economical to build it up of plates and angles rather than use a rolled joist, and when the depth exceeds the limiting depth for rolled joists, the built up girder must be used. Types of built up plate girders are shown in Figs. 311 and 312. For smaller spans and lighter loads, one web plate and one or two flange plates may be sufficient, while for larger spans and heavier loads two, or even three, web plates and many flange plates may be required. When more than one web plate is used, as in Fig. 312, the girder is called a *box girder*. The box type is more suitable for large than for small girders, and it is better only to employ this type when there is sufficient room inside for the girder to be properly painted, and so protected from corrosion. Care must also be taken that the girder can be properly riveted up, a not altogether unnecessary caution.

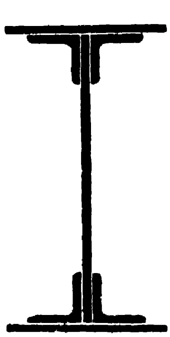


FIG. 311.

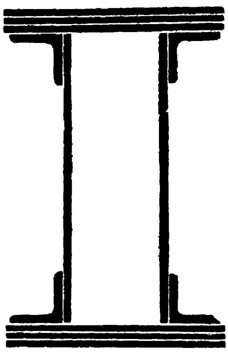


FIG. 312.

The depth of the girder must never be less than 1-20th of the span. For economy, the depth should be 1-12th to 1-10th of the span. The breadth varies from 1-20th to 1-50th of the span, depending on the amount of lateral support the girder gets. If there is no lateral support, the breadth should not be less than 1-20th of the span, whilst if it is well supported laterally, say by closely spaced cross girders, this dimension might be diminished to 1-40th or 1-50th of the span.

196. Booms or Flanges.—The booms, or flanges, of built up girders are almost invariably made up of flats* or plates. These are united to

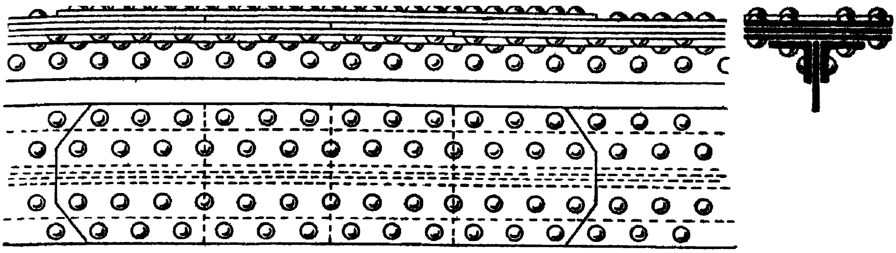


FIG. 313.

the web plates by angles, which of course act with the plates in resisting the bending moments.

The boom plates are not all of the same length, but are curtailed as the bending moment falls off. The usual graphical method for determining the length of the flange plates is shown in Fig. 331, p. 227. Care should be taken that the angles and plates are of convenient lengths. When there are many plates in a boom, the joints in them should be grouped, where possible, under one cover, as shown in Fig. 313. It is, however, sometimes convenient to make one flange plate form the cover for the joint of another, as shown in Fig. 314.

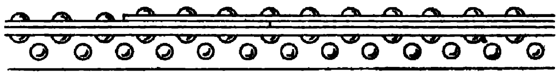


FIG. 314.

Joints in the flange angles are made with *round back covers,* and are arranged as shown in Fig. 315.

In the type of grouped joint shown in Fig. 313, a single cover is used and is placed on the outside, hence the rivets in the joint, although they

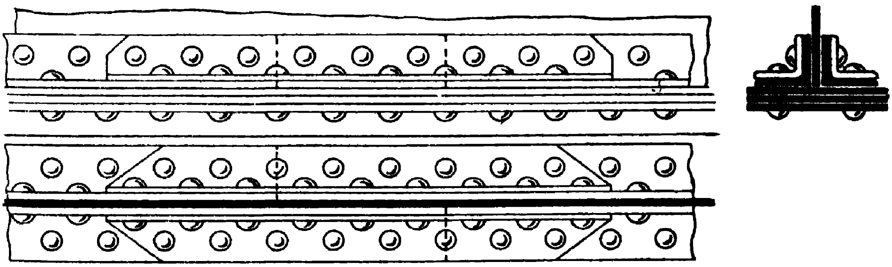


FIG. 315.

pass through several plates, are only in single shear. An underneath flange plate or angle must not be regarded as forming a cover to the joint in a plate above it, for it has its own load to carry, and cannot act as a flange plate and also as a cover at the same time.

* *Flats* are narrow plates, rolled to definite widths, usually not exceeding 12 inches.

Various means are adopted to place the rivets in a flange joint in double shear. The rivets which do not pass through the flange angles may be placed in double shear by the addition of covers underneath the flange plates, as shown in Fig. 316. The addition of round back covers to the angles, as shown in Fig. 317, will place the remainder of the rivets in double shear, but these angle covers cannot act as just stated

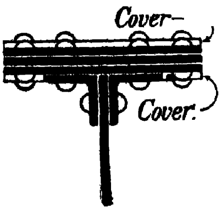


FIG. 316.

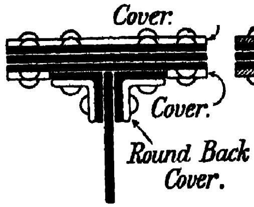


FIG. 317.

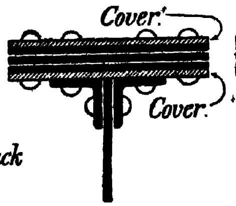


FIG. 318.

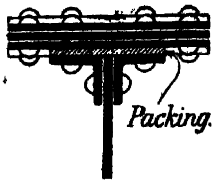


FIG. 319.

and also act as covers to a joint in the flange angles at the same time. The underneath cover may extend right across the flange, as shown in Fig. 318, which is a section of the flange in the neighbourhood of the joint. This, however, prevents the flange angles being placed directly on the flange plates, and where the covers do not occur, packing pieces have to be introduced, as shown in Fig. 319. These packing pieces cannot, however, be counted as forming part of the flange section, at any rate in the neighbourhood of a joint.

The thickness of each flange plate should not be less than $\frac{3}{8}$ inch or greater than $\frac{3}{4}$ inch. Four $\frac{3}{8}$ inch plates, or three $\frac{1}{2}$ inch plates, make a much better flange than two $\frac{3}{4}$ inch plates, supposing the required flange thickness to be $1\frac{1}{2}$ inches.

197. Web Plates.—The web in small plate girders consists of a single plate suitably stiffened to resist the shearing forces. Except for very small and unimportant girders, it is not desirable to make the web plate less than $\frac{3}{8}$ inch thick. On the other hand, the thickness of a single web plate should in general not exceed $\frac{3}{4}$ inch.

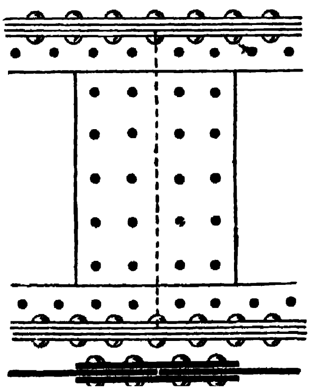


FIG. 320.

FIG. 321.

When more than one plate is required to form the web, the different plates are united by butt joints with double cover straps, as in Fig. 320, which shows a vertical joint in a web. Sometimes stiffeners are utilised to do duty as covers, as shown in Fig. 321. This figure also shows how a change in the thickness of the web plate may be effected. Such changes, made with the idea of proportioning the thickness of the web to the shear stresses, are only advisable in large and important girders, or when

many of a type are required. It is often more economical to have the same web thickness throughout, especially in small spans, where the web plate can be obtained in one piece, than to use plates of different thickness and special joints.

198. Web Stiffeners.—To prevent buckling and twisting it is necessary to give web plates lateral support. This is done by riveting to them at intervals angle- or tee-section bars placed vertically. Fig. 322

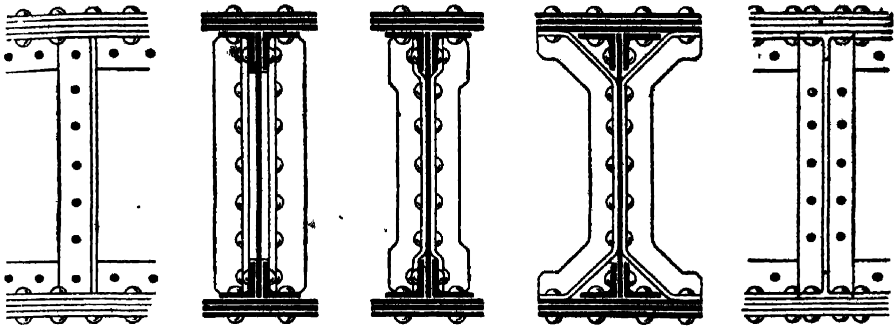


FIG. 322.

shows examples of stiffeners applied to single web girders. When the stiffeners are straight and not set in to meet the web plate, intermediate packing pieces are introduced. Stiffeners formed of plates and angles are shown on the main girders in Figs. 328 and 329, p. 223.

Box girders should have diaphragm plates fitted between the webs at intervals, as shown in Fig. 323. This ensures that the cross section of the girder remains rectangular, and that all the parts bend together. Manholes must be provided so that the space enclosed may be got at.

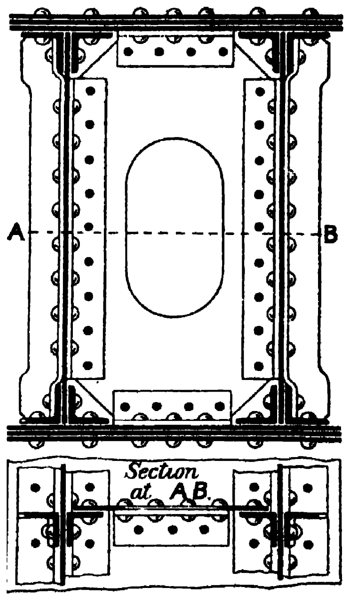


FIG. 323.

The distance apart of the stiffeners is determined by the shearing force upon the web. It is necessary, however, to place a stiffener wherever a local load occurs upon the girder.

No satisfactory theory for the spacing of the web stiffeners has yet been formulated, and the rules which will be given presently are almost entirely empirical. In Chapter IX. it was shown that a shear stress in one direction must be accompanied by another of the same intensity, but in a direction at right angles to that of the first. Also, it was shown that these two shear stresses are equivalent to tensile and compressive stresses of the same intensity as the shear stresses and at right angles to one another, but in directions making angles of 45° with the directions of the shear stresses. A square panel of the web plate is therefore

in compression along one diagonal, and in tension along the other. Since the thin plate is much less able to withstand the crumpling tendency of the compression than the direct tension, it is usual to consider the web as if composed of a number of parallel strips inclined at 45°, terminated either by the stiffeners or by the flange angles, and acting as struts. This consideration establishes a relation between the thickness of the web and its unsupported length. The shear force at any one section is assumed to be uniformly distributed over the depth of the web, which is very nearly correct (see Fig. 203, p. 150). This determines the shearing stress and the diagonal compressive stress, which is equal to it, and hence the load upon the strut. The foregoing reasoning leads to the construction of formulæ such as are given below. It will be observed that these formulæ are of the form of the Rankine-Gordon formula for struts.

S = safe shearing force in tons per inch of depth at any section, found by dividing the total shearing force, in tons, at the section, by the over-all depth of the web plate there in inches.

t = thickness of web plate in inches

d = horizontal distance between centre lines of stiffeners, or vertical distance between centre lines of rivets in the boom angles, in inches, whichever is least.

$$S = \frac{8t}{1 + \frac{3}{4000} \times \frac{d^2}{t^2}}$$

Another rule, due to Mr. Theodore Cooper, reduces to the following,

$$S = \frac{5 \cdot 36t}{1 + \frac{d^2}{3000t^2}}$$

In any case, S must not exceed $4t$.

Preferably proceed graphically, as shown in Fig. 330, p. 226. Draw the shear per inch of depth diagram found as above. Plot on this lines parallel to the datum line representing the possible shear per inch of depth of $\frac{3}{8}$ inch, $\frac{7}{16}$ inch, $\frac{1}{2}$ inch, etc., plates, corresponding to the proposed spacing of the stiffeners, as found by one of the formulæ given above. An examination of such a diagram will show, either the limits between which a given thickness of web plate may be used with a given spacing of the stiffeners, or the limits between which a given spacing of the stiffeners may be used for a given thickness of web plate. This matter is further considered in connection with the worked example, Art. 204, p. 223.

199. Riveting of Plate Girders.—For ordinary everyday work punched holes, $\frac{1}{16}$ inch greater in diameter than the rivets, are usually specified. In first-class work the holes are punched $\frac{1}{16}$ inch to $\frac{1}{8}$ inch smaller than the rivets, and reamered to size after the work is bolted together. The bolts are then removed, and the burrs formed by the reamer taken off, after which the work is riveted up.

The riveting in the joints of the plates in the booms must be designed to carry the tension or compression which exists in the plates they unite. The riveting through the angles connecting the boom plates to the web

plate is determined by the shearing forces tending to slide the boom over the web. Where the shearing force is small, the pitch may be large, but near the ends of the girder, or where the shearing force is large, closer riveting must be adopted. It is sometimes even necessary to adopt zig-zag riveting, or larger rivets at places where the shearing force becomes very great. It is not economical, however, to make many changes; two different diameters or two different pitches may be regarded as the limit.

The riveting in the vertical joints of the web itself must be made capable of withstanding not only the shearing forces in the web, but also the stress in the web plate, due to the bending moment, for although the boom may be considered as carrying the bending moment, there is also a bending stress in the web. In fact, the stress in the outer fibres of the web is the same as that in the boom.

Roughly, the size of the rivets may be as follows. For plates under $\frac{3}{8}$ inch thick, $\frac{5}{8}$ inch rivets. For plates from $\frac{3}{8}$ inch to $\frac{1}{2}$ inch thick, $\frac{3}{4}$ inch rivets. For plates from $\frac{1}{2}$ inch to $\frac{5}{8}$ inch thick, $\frac{7}{8}$ inch rivets. In each case the hole is $\frac{1}{16}$ inch larger than the rivet. These are about the usual proportions for punched work, and will serve as a guide. When many plates are to be united, larger rivets should be used.

The pitch of the rivets should not be less than three diameters, or greater than sixteen times the thickness of the thinnest outside plate. Unless it is absolutely impossible, simple pitches 3, $3\frac{1}{2}$, 4, $4\frac{1}{2}$, 5, or 6 inches should be adopted. It is not advisable to go above 6 inches if the work is exposed to the weather.

The longitudinal pitch is easiest determined graphically. Since the shear stress in the web is uniformly distributed, or practically so, over its depth, and the shear in two directions at right angles to one another is the same, the shearing force per inch-run, which the longitudinal rows of rivets must carry, is equal at any point to the shearing force per inch of depth there. The shear per inch of depth diagram, already referred to (Fig. 330, p. 226), can therefore be used to determine the pitch of the longitudinal riveting. Let P be the safe load on a single rivet, and p the pitch of the row, then P/p is the shear per inch of depth or length it will safely carry. Set this up on the diagram as a line parallel to the base line for a number of different pitches. The points of intersection of these horizontal lines with the shear per inch of depth diagram determine the points to which each pitch must extend. This question is further considered in connection with the worked example, Article 204, p. 223.

200. Ends of Girders—Bearings for Girders.—The ends of girders are specially formed to carry the reactions. Special web stiffening is provided to spread the load over the depth of the web plate. Examples are shown in Figs. 324 and 325. When the end of a girder is carried on a wall, a stone templet is built into the wall to give a strong support for the girder. Between the stone templet and the girder a hair felt or sheet lead packing is placed, in order that the pressure between the girder and the stone may be properly distributed. It is better to limit the length of the bearing surface by riveting a piece of plate, called a *bolster plate*, to the under side of the bottom flange, as shown in Figs. 324 and 325.

The safe bearing pressure between the girder and its supports will

depend on the nature of the material on which the girder rests, and is usually limited in the case of stone to from 12 to 20 tons per square foot. The safe bearing pressure between the stone and brickwork set in

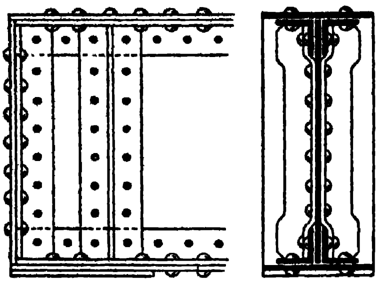


FIG. 324.

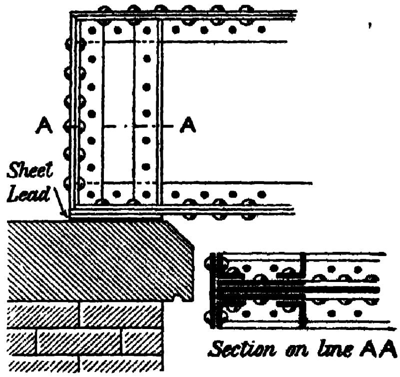


FIG. 325.

cement may be taken at from 6 to 10 tons per square foot, and between stone and brickwork in mortar at from 4 to 5 tons per square foot.

One end, and sometimes both ends, of a girder are left free to slide, so that a certain amount of expansion or contraction can take place with changes of temperature.

For large spans, say, of 50 feet and upwards, cast-iron bed plates are provided, on which the ends slide. These bed plates are usually sunk into the stone templet a small distance, as shown in Fig. 326. These

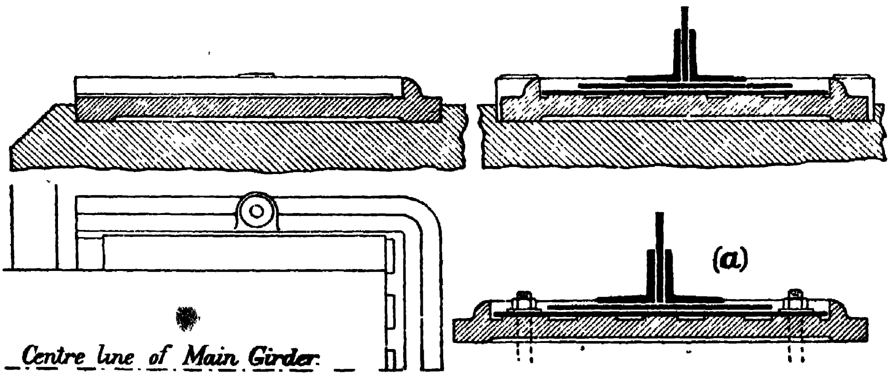


FIG. 326.

bed plates are bolted down to the stone templet. At (*a*), in Fig. 326, the holding down bolts for the bed plate are shown passing through the bolster plate, the bolt holes in the bolster plate being elongated to permit of the girder sliding a small amount.

For spans of over 80 feet, bearings similar to those shown in Figs. 368 and 369 would be used.

201. Connection of Cross Girders to Main Girders.—Figs. 327, 328, and 329 show methods of attachment of cross girders to main

girders. It will be observed that while the flange of the main girder

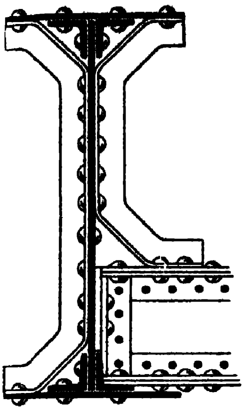


FIG. 327.

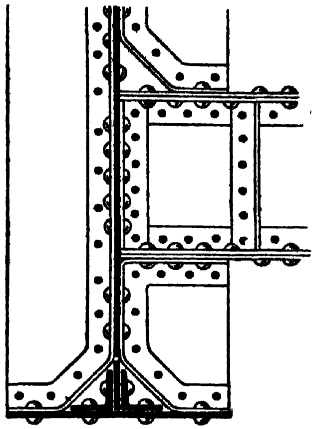


FIG. 328.

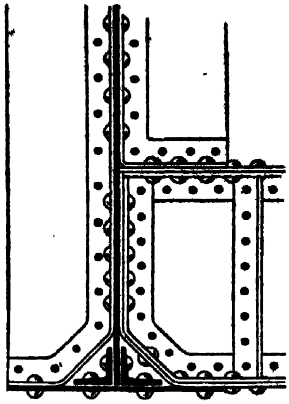


FIG. 329.

gives some support to the cross girder, the end of the latter is securely riveted to the web of the former.

202. Weight of Plate Girders.—An estimate of the probable weight of a plate girder may be made by means of Unwin's formula.

W = total external distributed weight in tons (exclusive of girder).
w = weight of girder itself in tons.
l = actual span in feet.
f = stress in booms in tons per square inch.
r = ratio of span to depth.
c = coefficient varying from 1400 to 1500 for small plate girders, and varying from 1500 to 1800 for large plate girders.

$$w = \frac{Wlr}{cf - lr}.$$

As a check, the following rough rule is given, $w = \frac{Wl}{500}$.

203. Camber and Deflection.—Girders are usually given a slight camber while being built, so that they just become straight when in place and loaded. A common allowance for camber is $\frac{3}{8}$ inch to $\frac{1}{2}$ inch per 10 feet of span. In calculating the deflection, take E, the modulus of elasticity, equal to 9000 tons per square inch, for riveted structures.

204. Plate Girder—Worked Example.—The method of procedure in designing a plate girder will be shown by working out a practical example.

It is required to design a plate girder such as might be used to carry a heavy floor, the clear span being 36 feet. It is to carry twelve loads of 6 tons each, spaced 3 feet centre to centre. An inexpensive design is required, and it is not desirable to take up much head room.

Type.—Parallel flanges. Single web plate. Punched holes. Material, mild steel. This will make the cheapest design.

Actual Span.—The clear span is 36 feet. Each end reaction will be about 40 tons. Two square feet of bearing area at least will be required

at each end, supposing the girder to rest on stone templets. Wall plates 1 foot 6 inches square would give a bearing area of $2\frac{1}{4}$ square feet at each end, and the actual span or the distance between the reactions would then be about 37 feet 6 inches.

Depth and Width.—The minimum depth would be $\frac{1}{20}$ of 37 feet 6 inches, say 22 inches. It would be desirable for economy to go to $\frac{1}{12}$ of 37 feet 6 inches, say 37 inches. Since head room is valuable, a compromise of about 30 inches will be tried, say 24 inches between the centre lines of rivets in the flange angles. The girder is, it may be supposed, fairly well supported laterally by the cross girders which bring on the loads, and a width of $\frac{1}{30}$ to $\frac{1}{40}$ of the span may be taken, say a flange width of not less than 12 inches.

Weight.—Using Unwin's formula (p. 223). $W=72$ tons. $l=37\frac{1}{2}$ ft. $r=37\frac{1}{2}/2\frac{1}{2}=15$. $c=1500$. $f=7$ tons per square inch.

$$w=\frac{72\times 37\frac{1}{2}\times 15}{1500\times 7-37\frac{1}{2}\times 15}=4\cdot 07 \text{ tons.}$$

As a round figure, the weight of the girder will be taken as 4 tons.

End Bearings and Exact Span.—The total weight is $72+4=76$ tons. Each end reaction will be 38 tons. If wall plates 18 inches by 18 inches be used, the bearing pressure on each stone templet will be practically 17 tons per square foot. A hard stone will safely carry this. The actual span may therefore be taken as $36+1\frac{1}{2}=37\frac{1}{2}$ feet.

Bending Moment and Shearing Force Diagrams.—These can now be drawn, and are shown in Fig. 331, p. 227.

The maximum bending moment is at the centre, and is 4440 inch-tons, and the maximum shearing force is at the ends, and is 38 tons.

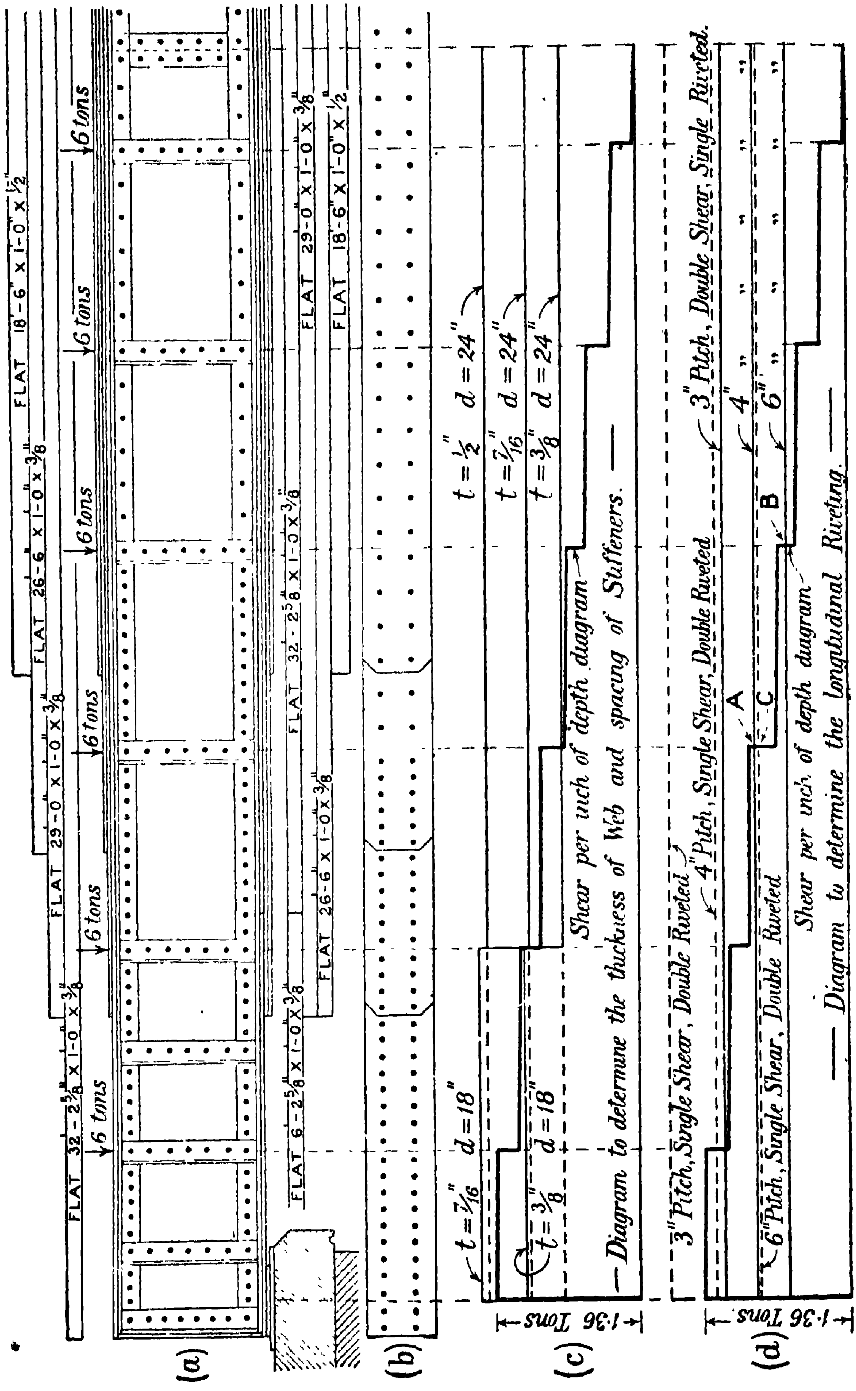
Thickness and Stiffening of Web.—The depth of the girder between the centre lines of the flange angles is 24 inches. The depth of the web plate may therefore be taken at about 28 inches, and this is constant throughout the span. The shear per inch of depth diagram is at once set out (Fig. 330, p. 226). Its value at the extreme end is $38/28=1\cdot 36$ tons. As the loads brought on by the cross girders are at intervals of 3 feet, this decides that stiffeners must be placed at 3 feet intervals under the loads, and it remains to be seen whether further stiffeners will be required. Considering the panels between the stiffeners under the loads, the dimension d in the formulæ on p. 220 is 24 inches, the vertical distance between the centre lines of the rows of rivets in the boom angles and the web. Putting $d=24$ inches in the first instance, and giving t, the web thickness, the values $\frac{3}{8}$ inch, $\frac{7}{16}$ inch, and $\frac{1}{2}$ inch, S from the formula given is 0·73, 1·07, and 1·47 tons respectively, and these are plotted on the shear per inch of depth diagram (Fig. 330). It is now evident that except at the extreme ends a $\frac{7}{16}$ inch plate is of ample strength. Near the centre a $\frac{3}{8}$ inch plate would suffice. A change of thickness however, entailing, as it would, two web joints, would probably cost more than the metal saved, unless of course many similar girders are required. If a $\frac{7}{16}$ inch plate is to be adopted, extra stiffening must be used near the ends. The most convenient way to carry this out is to put intermediate stiffeners between those under the loads, reducing d to 18 inches. A $\frac{3}{8}$ inch plate would then stand 1·1 tons per inch of depth, and a $\frac{7}{16}$ inch plate 1·54 tons. A $\frac{7}{16}$ inch plate will therefore serve, a $\frac{3}{8}$ inch plate being too weak.

Extra stiffening to make a $\frac{3}{8}$ inch plate suitable is not advisable for three reasons—the necessary stiffeners would be awkward to get in, the stress per square inch would approach very near to the limit, and, what is even more important, the web riveting in the $\frac{3}{8}$ inch plate would be very difficult to design. On the other hand, it need hardly be pointed out that the extra stiffeners required by the $\frac{7}{16}$ inch plate will be much cheaper than if a $\frac{1}{2}$ inch plate were used with no extra stiffening. A $\frac{7}{16}$ inch web plate will therefore be used, stiffened as shown.

Longitudinal Riveting in Web and Flanges.—$\frac{3}{4}$ inch rivets in $\frac{13}{16}$ inch holes will be adopted. The value of a rivet in single shear at 5 tons per square inch shearing stress is 2·59 tons. A rivet in double shear bearing in a $\frac{7}{16}$ inch plate will carry 3·55 tons, the bearing stress being limited to 10 tons per square inch. Dealing first with the single row in double shear through the web plate, a 3 inch pitch represents a shear per inch of depth (or length) of 1·18 tons, a 4 inch pitch 0·89 tons, and a 6 inch pitch 0·59 tons. A 5 inch pitch will not work in between the stiffeners, and need not be further considered. The above values are set up in the diagram as thin full lines. It is now seen that a 3 inch pitch must extend from the end of the girder to A, a 4 inch pitch from A to B, and a 6 inch pitch from B to the centre. Since too many changes are not desirable, a 3 inch pitch will be adopted extending to B, and a 6 inch pitch from B to the centre. Over the last two panels, near the end, a 3 inch pitch with the same size of rivets is inadequate. A closer pitch means zig-zagging the rivets and a deeper flange angle, and thickening the web plate is not desirable, as has already been seen. The third alternative is to use larger rivets. There are only a few larger rivets required, and probably the cheapest way out of the difficulty will be to punch all the holes alike and then reamer the few holes at the ends out to $\frac{7}{8}$ inch diameter. The load upon one of these $\frac{7}{8}$ inch rivets at 3 inches pitch is $3 \times 1{\cdot}36 = 4{\cdot}1$ tons. Its bearing area is 0·383 square inch, and the bearing stress will therefore be 10·7 tons per square inch, but since in these reamered holes the rivets will be in much better condition than in the ordinary punched holes, this may be allowed.

The riveting in the flanges joining the flange plates and angles is determined in exactly the same manner as for the web, except that there are two rows of rivets in single shear, instead of one row in double shear. The possible pitches are shown on the same diagram as dotted lines. It must be remembered when choosing the pitch that the rivets should zig-zag with those in the web. 3 inch and 6 inch pitches only are admissible. The 3 inch pitch will extend from the end to C, and the 6 inch from C to the centre. This will necessitate one odd $4\frac{1}{2}$ inch pitch, shown in the plan of the girder. No larger rivets are necessary, the 3 inch pitch giving ample strength.

Boom Section.—The distance between the centres of gravity of the flanges may be taken roughly as that over the backs of the angles. If angles 4 inches × 4 inches × $\frac{1}{2}$ inch be used, punched $2\frac{1}{4}$ inches from the back, this will be $24 + 4\frac{1}{2} = 28\frac{1}{2}$ inches. Using this figure, draw the diagram (Fig. 331), showing the force in the boom everywhere (got by dividing the ordinates of the bending moment diagram by $28\frac{1}{2}$ inches). The force in a boom at the centre is roughly 160 tons. The stress being



FLAT 18'-6" x 1'-0" x ½"
FLAT 26'-6" x 1'-0" x ⅜"
FLAT 29'-0" x 1'-0" x ⅜"
FLAT 32'-2⅝" x 1'-0" x ⅜"
6 tons
FLAT 6'-2⅝" x 1'-0" x ⅜"
FLAT 26'-6" x 1'-0" x ⅜"
FLAT 32'-2⅝" x 1'-0" x ⅜"
FLAT 29'-0" x 1'-0" x ⅜"
FLAT 18'-6" x 1'-0" x ½"
t = ½" d = 24"
t = 7/16" d = 24"
t = ⅜" d = 24"
t = 7/16" d = 18"
t = ⅜" d = 18"
Shear per inch of depth diagram
— Diagram to determine the thickness of Web and spacing of Stiffeners. —
1·36 Tons
3" Pitch, Single Shear, Double Riveted
4" Pitch, Single Shear, Double Riveted
3" Pitch, Double Shear, Single Riveted.
4"
6"
A
B
C
6" Pitch, Single Shear, Double Riveted
Shear per inch of depth diagram
— Diagram to determine the longitudinal Riveting. —
1·36 Tons
(a)
(b)
(c)
(d)

FIG. 330.

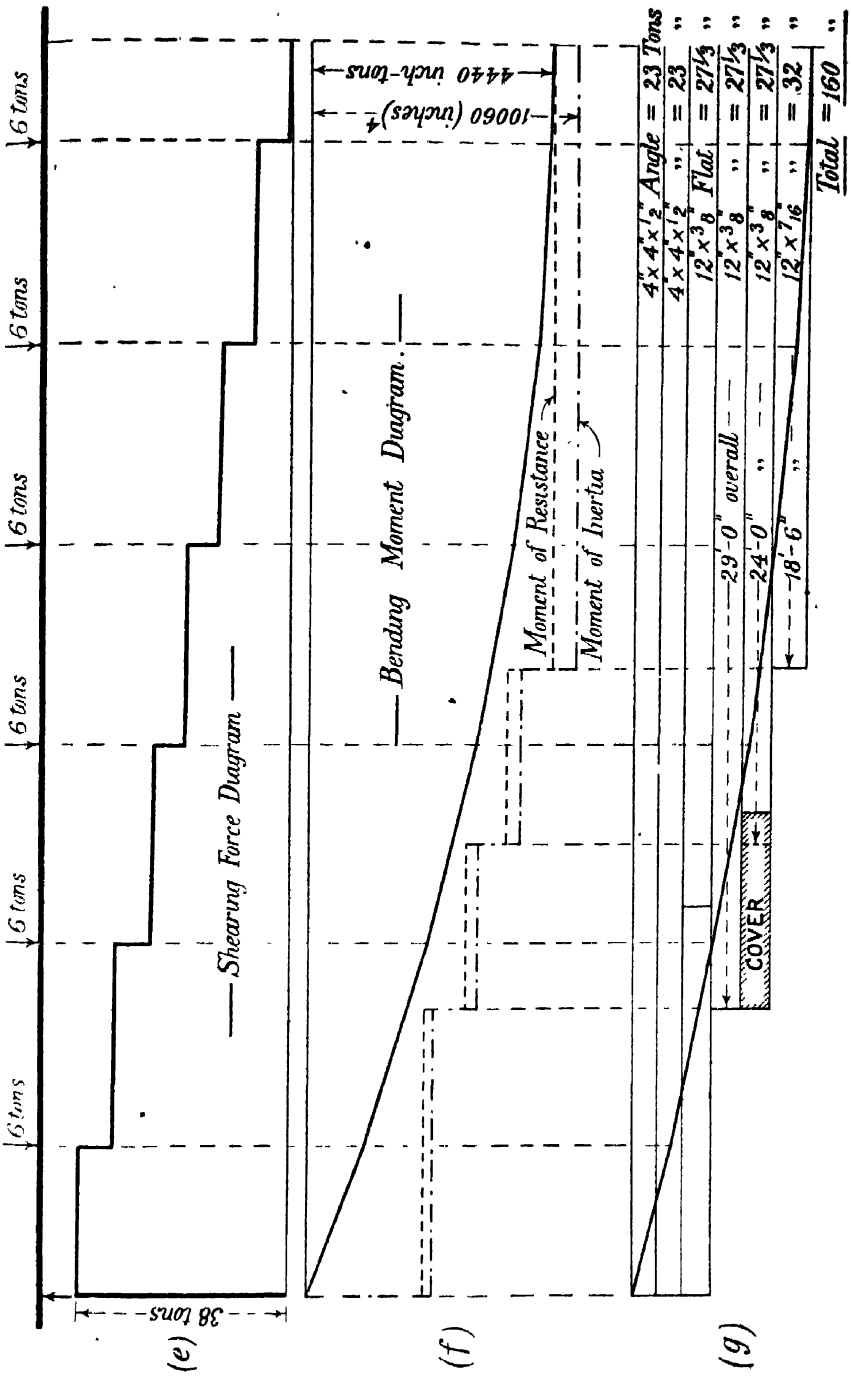

6 tons
6 tons
6 tons
6 tons
6 tons
6 tons
38 tons
Shearing Force Diagram
(e)
Bending Moment Diagram.
4440 inch-tons
10060 (inches)⁴
Moment of Resistance
Moment of Inertia
(f)
4"×4"×½" Angle = 23 Tons
4"×4"×½" " = 23 "
12"×⅜" Flat = 27⅓ "
12"×⅜" " = 27⅓ "
12"×⅜" " = 27⅓ "
12"×7/16" " = 32 "
Total = 160 "
29'-0" overall
24'-0" "
18'-6" "
COVER
(g)

FIG. 331.

limited to 7 tons per square inch, the best section, found after a number of trials, is as follows:—

Two angles 4 inches × 4 inches × $\frac{1}{2}$ inch, less one $\frac{13}{16}$ inch rivet hole in each	= 46 tons
Three flats 12 inches × $\frac{3}{8}$ inch, less two $\frac{13}{16}$ inch rivet holes in each	= 82 tons
One flat 12 inches × $\frac{7}{16}$ inch, less two $\frac{13}{16}$ inch rivet holes	= 32 tons
Total . .	160 tons

Setting these off upon the diagram, the necessary length of each plate is at once apparent. The $\frac{7}{16}$ inch plate placed outside must be 18 feet 6 inches long, the next two $\frac{3}{8}$ inch plates 24 feet and 29 feet long, while the inside plate and the angles will be carried the full length of the girder. It will be observed that an extra rivet pitch or two have been allowed in the various lengths.

In this design the top boom will be made of exactly the same pattern, length of plates, etc., as the bottom boom; that is to say, the rivet holes will be subtracted from the area of the section both for compression and tension. If it is thought desirable to take the area of the compression boom as the gross area, not subtracting the rivet holes, another diagram of the same type for the compression boom will be necessary. It is doubtful in the present case if such alteration would save money.

Set out the Girder.—Start with two horizontal centre lines 24 inches apart. Erect the centre lines of the stiffeners. Next put in the rivets, those in the flanges being staggered with regard to those in the web, due regard being paid to local conditions, taking on of cross girders, etc., remembering that a simple uniform pitch is to be aimed at. Next, on this skeleton outline, put in the outlines of the plates and angles.

Joints.—The longest boom plate in the design has a length of 38 feet $5\frac{1}{4}$ inches. It will be advisable to make a joint in this, although it might possibly be obtained in one piece. This joint will be placed so that the outer $\frac{3}{8}$ inch plate produced will form a cover. The cover being single, the rivets are in single shear, each worth 2·59 tons, and since the cut plate was worth $27\frac{1}{3}$ tons, 12 rivets will be required through each half of the cover, as shown.

If the joint occurs in the bottom boom to the left of the centre of the girder, it may be placed to the right in the top flange.

The web plate will also be made in two pieces, and the joint placed at the centre, where the shear is least. If the joint be designed to carry the shearing force only, the shear per inch of depth diagram which determined the longitudinal riveting will determine that in the transverse seam also. It will be seen that a 6 inch pitch would be more than sufficient at the joint under consideration. Actually a single riveted butt joint with double cover straps and rivets of 4 inches pitch will be used as shown.

Stiffening at Ends.—The reaction at each end is 38 tons. A bolster plate 1 foot square will be riveted to the bottom of the girder at each end to limit the span, and between this and the wall plate sheet lead is placed. The reaction must be distributed over the depth of the web plate through the vertical stiffening at the end. There are 8 rivets in the end angles, and 7 in the first stiffener. The load on each of these rivets is

$38 \div 15 = 2{\cdot}53$ tons; they are in double shear, and, as has been shown, each is worth 3·55 tons. But the load is not equally divided over the rivets, and therefore a margin is desirable. If the area of the cross section of the stiffening be worked out, it will be found that the direct stress is small. Taking only the end plate 12 inches $\times \frac{3}{8}$ inch, and the two end angles 4 inches $\times$ 4 inches $\times \frac{1}{2}$ inch into account, it is 3·2 tons per square inch.

Each stiffener under a load must distribute a load of 6 tons over the depth of the web. Through each passes 7 rivets, and if a section 4 inches $\times$ 3 inches $\times \frac{1}{2}$ inch be used, the direct stress will be less than 1 ton per square inch.

Moment of Inertia and Moment of Resistance of Cross Section.—Subtracting the rivet holes from both flanges and allowing for the middle 1 foot 8 inches of the web only at 33 per cent. efficiency, that is, the worth of the riveted joint (the joint is weakest in compression between the rivets and plates), the moment of inertia of the central cross section is found to be 9740 (inches)^4. Since the distance of the extreme fibres from the neutral axis is 15·81 inches, the modulus of the section is $9740 \div 15{\cdot}81 = 616$ (inches)^3. Hence the maximum stress at the central cross section is $4440 \div 616 = 7{\cdot}2$ tons per square inch, which exceeds the limit. The explanation of this is, that in so deep a flange the variation of stress between the outer and inner fibres is considerable, a point often overlooked when the approximate method is applied, and this shows the necessity of calculating the moment of resistance everywhere. It will be found necessary to increase the thickness of the outer plate from $\frac{7}{16}$ inch to $\frac{1}{2}$ inch. The moment of inertia of the central cross section will then become 10,060 (inches)^4, and the moment of resistance will increase to 634 (inches)^3, which will reduce the maximum stress to 7 tons per square inch.

The moment of resistance diagram is plotted on the base line of the bending moment diagram (Fig. 331), and it will be seen that the former lies entirely outside the latter.

The moment of inertia diagram is also plotted on the base line of the bending moment diagram (Fig. 331).

Deflection and Camber.—Assuming that $M \div I$ is constant throughout the girder (a rough approximation) and equal to $\frac{4440}{10060}$, then,

$$\text{deflection} = \frac{ML^2}{8EI} = \frac{4440 \times 450^2}{8 \times 9000 \times 10060} = 1\tfrac{1}{4} \text{ inches.}$$

This deflection is $\frac{1}{360}$ of the span, which may be considered as reasonable.

If $\frac{3}{8}$ inch of camber be allowed per 10 feet of span the total camber required is $1\frac{3}{8}$ inches, agreeing fairly well with the estimated deflection.

Actual Weight of Girder.—The weight of the girder as designed is 4 tons 0 cwt 2 qrs. 13 lbs., showing that the estimated weight is sufficiently accurate.

Exercises XIV.

1. A steel plate web girder with parallel booms, 100 feet long, is to support a dead load of $\frac{3}{4}$ ton, and a rolling load of $1\frac{1}{2}$ tons per foot-run. Select a suitable depth, and assuming suitable working stresses, design the centre section and the longitudinal section of the booms, taking 30 feet as the maximum length of

plate. Design also the cover plate for one of the boom joints, and show the general arrangement of stiffeners. Scale for section, 1 inch to 1 foot. Horizontal scale for booms, 1 inch to 5 feet. [U.L.]

2. A built-up steel plate girder has the following cross sectional dimensions:—The flanges consist of three plates, each $\frac{1}{2}$ inch thick and 16 inches wide; the web consists of one plate, 45 inches deep and $\frac{5}{8}$ inch thick; the web and the flanges are connected together by angles 4 inches by 4 inches by $\frac{1}{2}$ inch. If the external shear force at a particular vertical section of this girder is 112 tons, determine (*a*) The intensity of the shear stress in the horizontal plane of the section in which the web plate meets the flange plates. (*b*) The proper pitch to adopt for the $1\frac{1}{8}$ inch rivets used to connect together the web and the flanges, if the intensity of the shear stress in them is not to exceed 4 tons per square inch. [U.L.]

3. Draw the M/I diagram for the plate girder in the worked example (Art. 204). Find graphically from this the actual deflection curve of the girder. Measure the maximum deflection. Show by how much the deflection curve differs from an arc of a circle.

4. Plot the shear distribution curve for an end cross section of the plate girder in the worked example (Art. 204). What is the ratio of the mean to maximum shear stress? Compare each with the shear per inch of depth assumed.

5. Design the grouped joint for three $\frac{1}{2}$ inch steel plates 16 inches wide. Diameter of rivets, $\frac{7}{8}$ inch. Holes punched and reamered. A single outside cover to be employed. The holes in the flanges to be staggered. Calculate the various efficiencies of the joint. What saving of metal is there over three separate joints?

6. A rolled steel joist is continuous over three spans. One extreme end is built firmly into an abutment, while the other may be taken as freely supported. The load carried is 1 ton per foot-run. The two outer spans are each 10 feet, and the centre span is 12 feet. What are the loads on the piers? Draw the bending moment and shearing force diagrams. Design the beam.

7. Design a plate web girder of the fish-bellied type suitable for an overhead traveller of 50 feet span. There are two such girders upon which the traversing carriage runs. The maximum weight to be lifted is 40 tons, and the weight of the traverser may be taken as 4 tons.

8. A three-girder bridge, to carry a double line of rails, has a clear span of 36 feet, and the girders have a length of bearing at each end of 2 feet 6 inches. The girders are to be of the plate web type The flooring is to be trough form, weighing about 7 cwt. per foot-run of the whole width of the bridge. The permanent way, including rails, sleepers, etc., may be taken as equal to 160 pounds per foot-run for each line of rails. Estimate in any way you please the approximate weight of the main girders, and determine the maximum bending moments and shear on each of the side girders and on the central girder for the above dead loads, and for a live load of 40 cwt. per foot-run per single line of rails. Choose your own working stresses, and design a suitable cross section for the centres and ends of the central girder and for one of the side girders. Determine the necessary pitch of rivets in both cases. [U L.]

9. Design for a double track railway bridge. Span between bearings, 60 feet. There are to be two main girders, spaced 26 feet apart, centre to centre. The deck of the bridge is carried by cross girders placed at about 7 feet to 8 feet pitch, and consists of trough flooring running longitudinally. The sleepers are laid transversely. The dead weight of the floor may be taken as $1\frac{1}{2}$ cwt. per square foot, and the equivalent uniform live load at 2 tons per foot-run for each line of way. The maximum load on one axle may be assumed as 20 tons.

CHAPTER XV

DESIGN OF STRUCTURES—BRACED GIRDERS

205. Open Web or Braced Girders.—*Open web girders* include all those in which the web is constructed of separate bars or members instead of a continuous plate. In such girders the tensile and compressive stresses to which the shear has been shown to be equivalent are carried by ties and struts specially designed to take them.

Open web girders are lighter than corresponding plate web girders. The metal in the open web is better disposed, and the girder presents less surface to the force of the wind.

Above 60 to 80 feet span open web girders are preferable in most cases to plate web girders, and in very large spans they are a necessity. Very light girders also are often made of the open web type.

Open web girders are, however, more costly per ton than plate web girders, and the latter are therefore less expensive for small spans carrying heavy loads.

It is sometimes convenient to construct the web of a girder partly as a plate web and partly as an open web. If the shear is very large, say, at the ends, the bracing and connections are sometimes very difficult to design. In such cases it may be more convenient to use a plate web. Near the centre, however, or where the shear is small, it may be more economical to carry it by means of ties and struts. Such a girder is termed a *semi-plate web girder*. This form is, however, not much used, except in special cases.

206. Types of Open Web Girders.—The web bracing takes many diverse forms, from which the various types mainly take their names. Examples are shown in Figs. 332 to 347.

In the type known as the *Warren girder* (Figs. 332 to 335), the web braces form the sides of isosceles triangles, whose bases are parts of the booms. The web members are inclined at 60° to the booms in Figs. 332

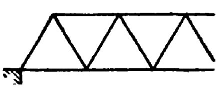


FIG. 332.

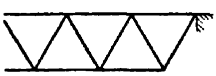


FIG. 333.

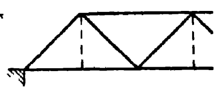


FIG. 334.

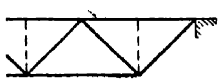


FIG. 335.

and 333, and at 45° in Figs. 334 and 335. Vertical members shown dotted in Figs. 334 and 335 are introduced to add further support to the roadway. In Figs. 332 and 334 the floor or deck of the bridge is at the bottom, and the traffic would pass between the main girders. In Figs. 333 and 335 the deck is on the top.

A *Pratt* or *Whipple-Murphy truss* is shown in Fig. 336. This is sometimes called an **N** truss. The web bracing is composed of vertical

and diagonal members alternately. The diagonals are usually, though not necessarily, placed at 45°. The shorter vertical members are struts, and the longer diagonals ties. The truss is shown inverted in Fig. 337 to get the deck on the top.

A modification of the Pratt truss, with the diagonals sloping the other way, and known as the *Howe truss*, is used in America. It is usually constructed mainly of timber. The verticals, which are now ties, are wrought-iron or steel bolts, and the diagonals, which are now struts, are of wood.

For long and heavy spans, duplicate systems of web bracing are used.

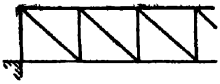


FIG. 336.

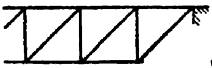


FIG. 337.

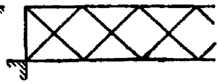


FIG. 338.

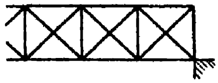


FIG. 339.

If two Warren girders of the form shown in Fig. 334 be taken, and one is inverted and superposed on the other, a *Lattice girder* (Fig. 338) is formed. If two **N** trusses of the form shown in Fig. 336 be similarly treated, the lattice girder shown in Fig. 339 is obtained, which is the girder of Fig. 338 with the verticals of Fig. 334 left in. The function of these verticals is to equalise the load between the two systems. In the type shown in Fig. 338, two diagonals in the same bay do not carry the same stress; in the type shown in Fig. 339, they should. In actual bridges it is doubtful if these verticals really act as they are supposed to do.

If two **N** trusses be superposed, one being moved half a bay along relative to the other, a *Linville truss* (Fig. 340) is obtained. Fig. 341 shows the same truss with a slightly different end post. Fig. 342 shows the Linville truss inverted to get the deck on the top.

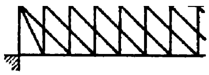


FIG. 340.

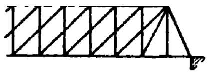


FIG. 341.

FIG. 342.

In designing girders with duplicate web systems it is usual to separate the two systems, and to design each on the assumption that it carries one half of the load, though this assumption is not strictly correct. The two are then again superposed, and the stresses combined in those members which are made to coincide.

Sometimes two girders of the type shown in Fig. 338 are combined, one being moved half a bay along relative to the other. Such a combination is termed a *double lattice girder*.

In small girders the web is often composed of a number of diagonal bars lattice braced, as shown in Fig. 343. Such a web may be looked upon as a multiple lattice girder. The bracing usually consists of flat bars, which are made wider and thicker toward the points of support. The usual assumption when designing such a web is to make the diagonals cut by any vertical section of such size that the vertical shear force at the section will be

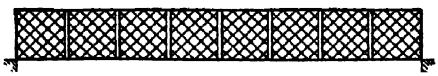


FIG. 343.

equal to the vertical component of the sum of the safe loads in all the diagonals cut. Vertical stiffeners similar to those used in plate girders are usually introduced, and for similar reasons.

So far the types of open web girders referred to have parallel booms. For very large spans it may be more economical to curve one, or even both, of the booms. Girders with curved booms are more expensive per ton than corresponding ones with straight booms, and should not be employed when parallel booms are suitable. Most of the types of web bracing already referred to can be used in girders with curved booms.

A girder with the top boom of parabolic form and the other straight (Fig. 344) is termed a *bow-string girder*, from its similarity to a bow and string. Such a girder, carrying only a uniform dead load, would, theoretically, require no diagonal bracing in the web. Since all bridges have to support both non-uniform and rolling loads, in practice diagonalisation becomes necessary, as shown in Fig. 344. This form may be inverted, when the "bow" becomes a suspension chain.

The type sometimes called the *bow and chain girder* is shown in Fig. 345. It is the bow and inverted bow or suspension chain girder types combined, the object being to neutralise the thrust of the arched bow by

FIG. 344.

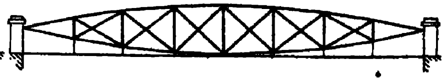


FIG. 345.

the tension in the chain. Practically, it is a girder with two curved booms. The web bracing is usually of the type shown. The bridge floor is carried by suspension rods, as shown.

Fig. 346 illustrates a type of truss common in America for large

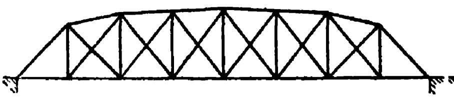


FIG. 346.

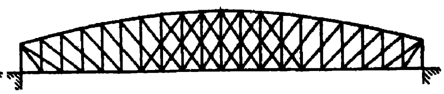


FIG. 347.

spans. The web bracing is of the lattice type. Fig. 347 shows the Linville type of truss applied to a large span.

A type of girder common in English railway practice is illustrated in the worked example, Art. 227, pp. 248–258.

207. Counterbracing.—The function of the ties and struts which form the bracing of an open web girder being to take the shear stress, it follows that if this shear stress be reversed in direction at any part of the girder, the ties become struts and the struts become ties at that part. Now, it was shown in Art. 107, p. 100, that a travelling load added to the dead load will have the effect of reversing the direction of the shear stress over a portion of the girder. In a plate web this reversal of stress is of little or no consequence, but in an open web it is obvious that provision must be made for it. The struts will as a rule act well as ties, but either the ties must be designed to carry the compressive stress, that is, to act as struts, or else special members must be introduced to carry the reversed stress. These special members, which are diagonal ties

sloping the other way to the ordinary diagonal ties, are called *counterbraces*. The dotted lines in Fig. 344 represent counterbraces.

In duplicate systems of web bracing a member of the second system will carry the reversed stress. In Linville trusses the bracing near the centre takes the form shown in Fig. 347.

208. Booms of Open Web Girders.—The booms or flanges of open web girders are usually somewhat similar in form to those of plate web girders, being made up of a number of horizontal plates. They differ, however, in having one or more vertical plates, called *stringer* or *curtain plates*, which are connected to the flange plates by angles, and which form convenient attachments for the web bracing. The boom, in fact, is usually of a T or ⊔ section, as shown in Figs. 348 and 349. In the

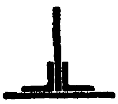


FIG. 348.

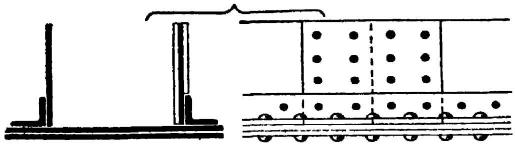


FIG. 349.

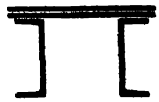


FIG. 350.

compression boom the lower edges of the stringer plates are often stiffened by angles, as shown in Fig. 351, to prevent it from buckling. Occasionally channels are used instead of these plates and angles, as shown in Fig. 350. To prevent distortion ⊔ sections may be fitted with diaphragm plates, as shown in Fig. 352. A type of boom sometimes adopted is shown in Figs. 353, 354, and 365. Instead of placing the flange plates horizontally they are placed vertically. Combinations with angles and channels are also used. In American practice these vertical plates become eye-bars in the tension flange, as shown in Fig. 375, p. 245. This type possesses the advantage that it is a

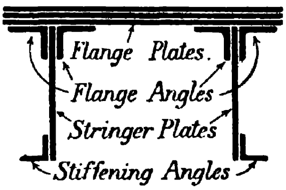


FIG. 351.

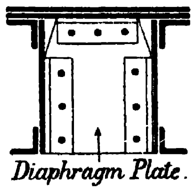


FIG. 352.

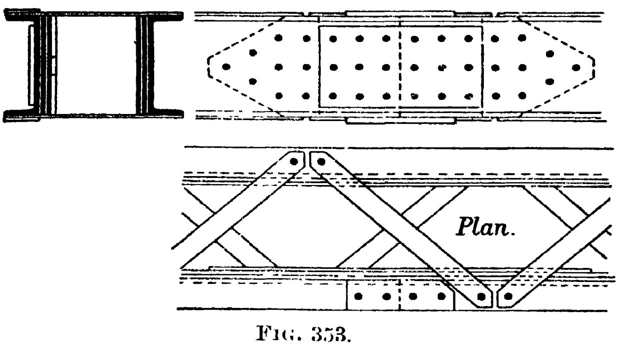


FIG. 353.

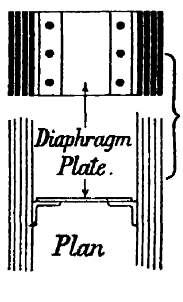


FIG. 354.

most convenient form to get a really good connection with the web bracing and for the attachment of the cross girders. Also, it does not

hold the rain, as does the ⊔ form, unless specially drained. The two halves of this form of boom are usually connected together by light secondary bracing, as shown in Figs. 353 and 365.

Fig. 355 shows a convenient form of boom for a light girder. It consists of four angles held apart by short plates at intervals.

The cross section of the boom is proportioned to the stress which it has to carry, exactly as in a plate web girder. The graphical method shown in Fig. 331, p. 227, may be used to obtain the length of the flange plates, angles, etc.

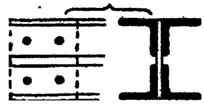


FIG. 355.

If the lateral dimensions of the compression boom are small compared with its length, it should be examined as a strut hinged at the panel points. Generally, this is unnecessary.

209. Joints in Boom Plates.—The joints in the horizontal flange plates are formed exactly as in the case of plate web girders, and similar calculations are necessary. The flange angle joints are also similar, being constructed with round back covers. The joints in the stringer plates are usually butt joints with double covers, as shown in Fig. 349, sufficient rivet section being used to develop the full net strength of the plate. A grouped joint for a boom with vertical flange plates is shown in Fig. 353.

210. Riveting in the Booms.—The riveting in the booms should be of a regular uniform pitch. As far as possible the rivets should be arranged to break pitch across the width, particularly in the tension boom, so as not to weaken the boom more than is unavoidable. A 4 inch pitch is the most common, but the pitch should not exceed 6 inches where there is a likelihood of water getting between the plates, nor in any case should the pitch be more than sixteen times the thickness of the outside plate in the compression boom.

Since the stress in the boom is transferred from the web bracing on to the stringer plate, and thence through the flange angles to the flange plates, sufficient rivets must be placed through the flange angles to transfer this stress on to the flange plates within a reasonable distance along the length of the flange.

211. Web Bracing.—The web bracing is constructed of the ordinary rolled sections, used singly or in combination. For ties, flats are most commonly used. If, however, lateral stiffness is desired, or if the stress is likely to be reversed, a channel or other suitable section may be employed. If the reversed stress is small in amount, two flat bars connected by secondary bracing, as shown in Fig. 357, may be used. Each of the flat bars must, however, be capable of carrying one-half of the load when considered as a column bending between the points of secondary support. This condition usually determines the spacing of the cast-iron distance pieces.

For a light tie a single flat bar is best, and for a light strut a single angle or tee bar is most suitable (Fig. 356).

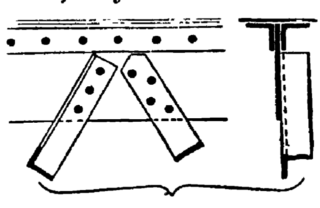


FIG. 356.

The usual sections suitable for heavier struts are shown in Figs. 358, 359, 360, 362, 364, and 365. A strong and light strut is formed by connecting together two or more simple struts by secondary bracing, as

shown in Figs. 358 and 359. Two channels braced together make a favourite form of strut.

If a strut is equally free to bend in two directions at right angles to one another, its component parts should be chosen and arranged so that

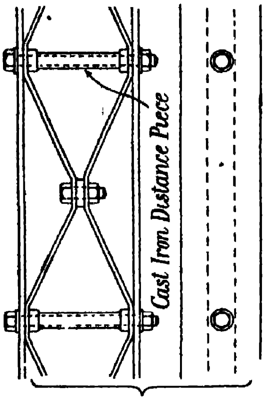


FIG. 357.

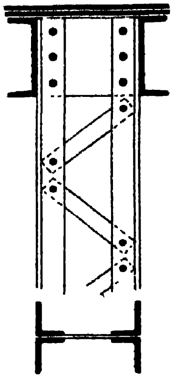


FIG. 358.

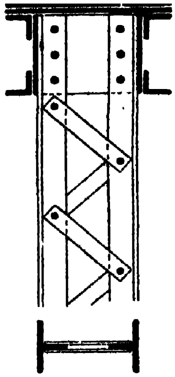


FIG. 359.

the radii of gyration of the section about axes perpendicular to these directions are as nearly as possible equal.

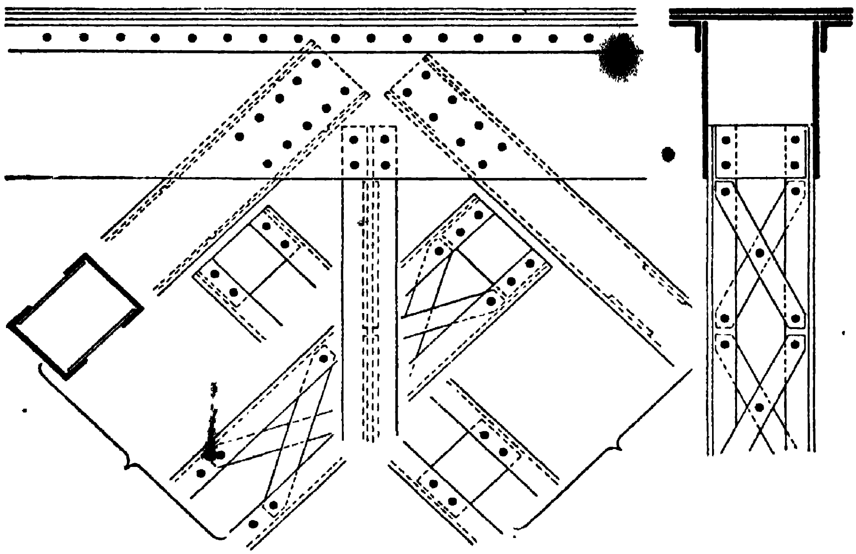


FIG. 360.

The parts of a built up strut between the points of attachment of the secondary bracing should be examined as separate short columns hinged at their ends.

Secondary bracing usually consists of light flat bars, 2 inches to $2\frac{1}{2}$ inches in width, and $\frac{1}{4}$ inch to $\frac{3}{8}$ inch in thickness. Light angles about

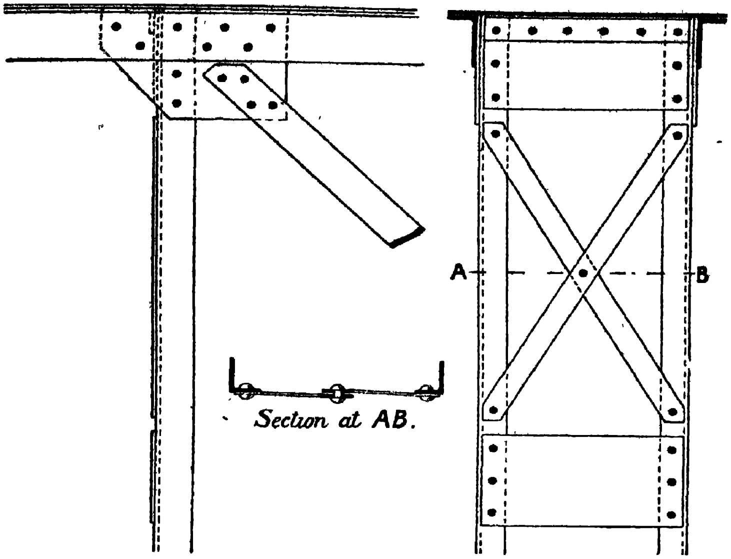


FIG. 361.

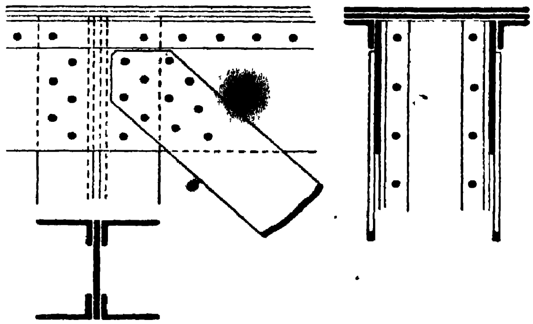


FIG. 362.

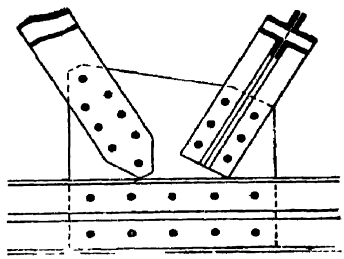


FIG. 363.

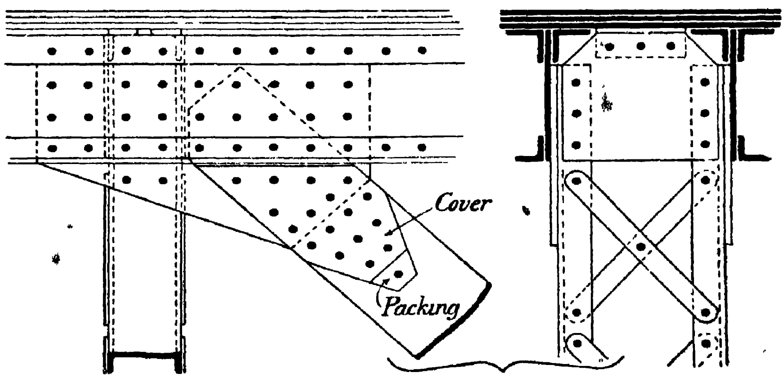


FIG. 364.

$2\frac{1}{2}$ inches × $2\frac{1}{2}$ inches are also used. Various forms of secondary bracing are shown in the illustrations of this section.

The stresses in the web members are found from the stress diagrams (see Chapter XII.), and their cross sections determined by the rules for ties and struts.

Where the connections between a strut and the booms are stiff riveted joints, and the boom sections are also stiff and capable of with-

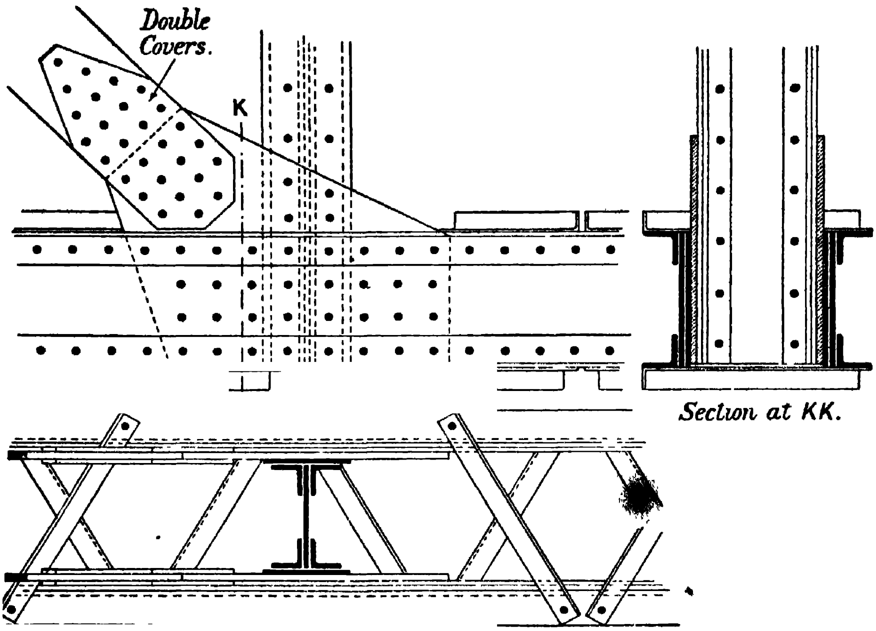


FIG. 365.

standing considerable bending and distorting moments, the strut may be considered as fixed at its ends, or nearly so. It is safer, however, in order to allow for any imperfection in the manner of fixing to take the effective length of the strut as, say, $1\frac{1}{4}$ times its real length.

212. Connection of Web Bracing to Booms.—In English practice riveted joints are almost invariably used for connecting the web braces to the booms. Examples are shown in preceding illustrations. In America, pin joints, such as shown in Fig. 375, are common.

If the number of rivets required in the ends of the web members is not too large, these members may be attached directly to the stringer plates, as shown in Figs. 356, 360, and 362. Often this is not possible, and gussets are introduced, as shown in Figs. 363, 364, and 365. In Figs. 364 and 365 the rivets in the ties are placed in double shear by the use of cover plates.

The following conditions should be observed when designing joint connections:—

(*a*) Sufficient rivet section should be provided in each member to take the load on it. If gussets are used, sufficient rivets must pass

through the gusset and stringer plate to take the load from the gusset and transfer it to the boom.

(*b*) The axes of each member (boom included) should meet at a point.

(*c*) The rivets in each member should be symmetrically grouped about its centre line.

(*d*) A tension member should not be weakened to a greater extent than one rivet hole.

(*e*) The rivets should be spaced at a convenient and uniform pitch. Those in the web members should not be permitted to upset the uniformity of pitch of the boom riveting.

(*f*) The centres of the rivets should not be less than three diameters apart, or closer to the edge of the plate than $1\frac{1}{2}$ diameters.

Too often in actual practice the above conditions are not all complied with. Sometimes a compromise has to be made, but with a little care and ingenuity much may be done towards satisfying all the conditions.

213. End Posts.—The end struts of bridge trusses are termed *end posts*. They have to carry the whole reaction due to the load on the girder, and are therefore of more massive construction than the ordinary

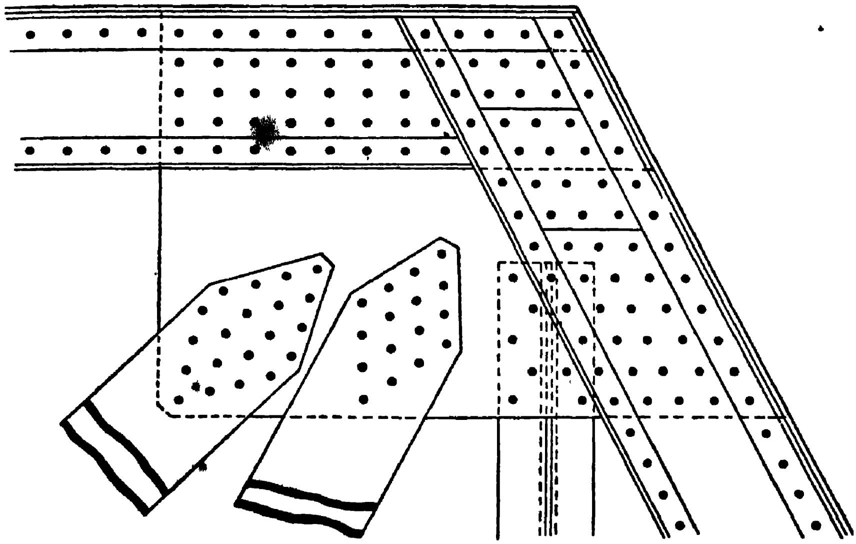


FIG. 366.

struts; in fact, they are frequently of similar cross section to the compression boom.

Details of two inclined posts are shown in Figs. 366, 367, and 368. A vertical end post is shown in Fig. 384, p. 254. In some inverted trusses, however, the end member is a tie, which may be of the usual form.

214. Bearings.—Beneath the feet of the end posts are placed the bearings. An example of a roller bearing is shown in Fig. 369, which

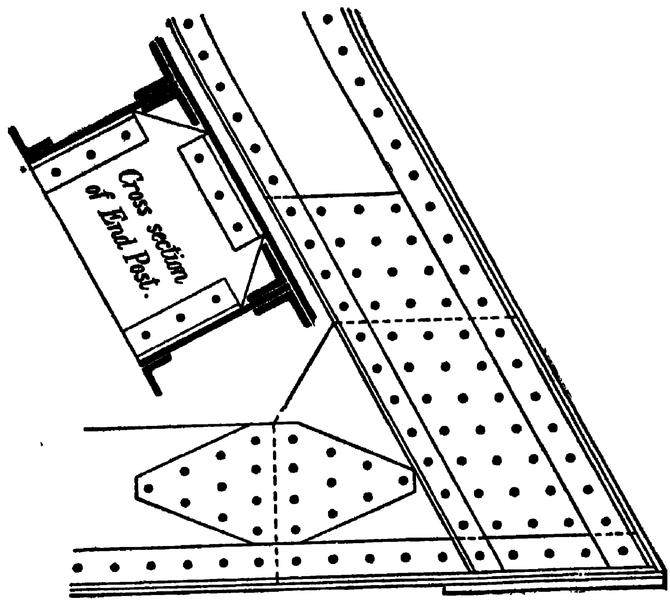


FIG. 367.

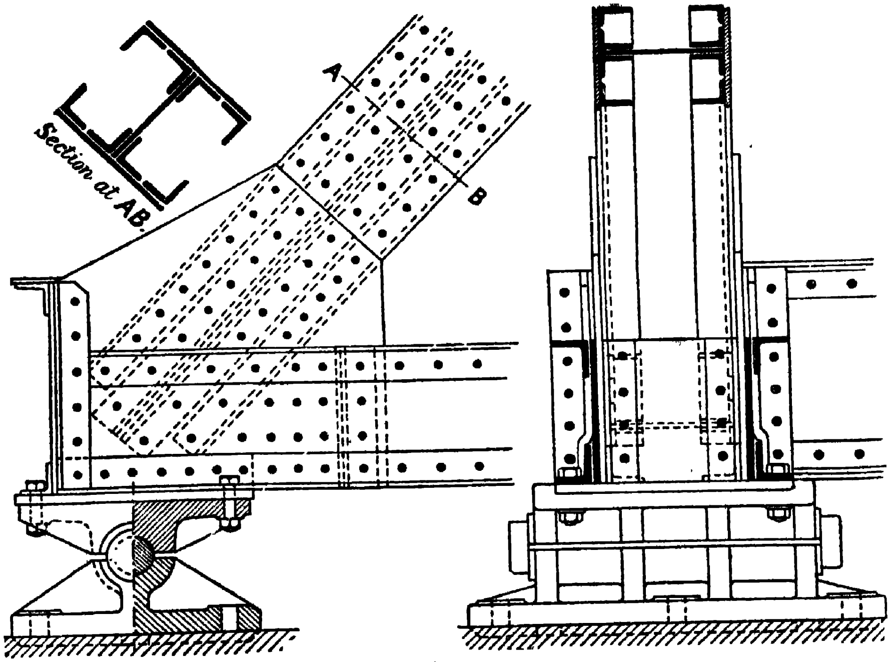


FIG. 368.

represents the standard practice of Mr. George A. Morrison, the celebrated American bridge engineer. This bearing is fully illustrated and

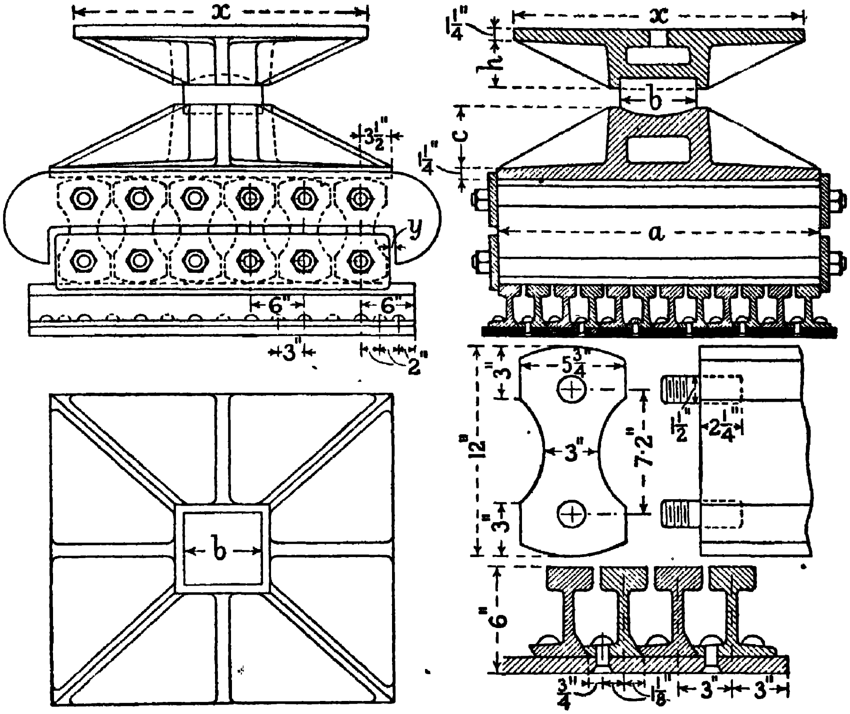


FIG. 369.

described in the *Transactions of the American Society of Mechanical Engineers for* 1893. The following particulars and table of dimensions are taken from Mr. Morrison's paper :—

Number of Rollers.	Number of Rails.	Length of Rollers (a). Inches.	Side of Rocker Plate (b). Inches.	Total Bearing. Inches.	Safe Load at 3000 lbs. per Linear Inch. Lbs.
3	6	17·5	4	45	135,000
4	8	23·5	5	80	240,000
5	10	29·5	6	125	375,000
6	12	35·5	8	180	540,000
7	14	41·5	9	245	735,000
8	16	47·5	10	320	960,000
9	18	53·5	11	405	1,215,000
10	20	59·5	12	500	1,500,000
11	22	65·5	13	605	1,815,000
12	24	71 5	14	720	2,160,000

The rail plate rests on a cast-iron bed plate, not shown in Fig. 369.

Above the rollers is the top bearing, which is a steel casting carrying the rocker plate, which carries the top plate. The bolster, or the bottom chord, is placed on the top plate, to which it is bolted rigidly. The rocker plate is square in plan, its bearing surfaces being cylindrical, of radii equal to the side of the plate. A rocking motion is possible in any direction, and the bearing may be depended on to distribute the weight not only uniformly over the several rollers, but uniformly over the length of each roller

$$c = \frac{a - (b + 1\cdot5)}{4}. \quad h = \frac{x - (b + 1\cdot5)}{4}. \quad y = \frac{\text{Span}}{3000}.$$

The roller bearing provides for the longitudinal expansion and contraction of the structure due to variations of temperature. Such provision is, however, only necessary at one end of the truss. At the other end a fixed bearing, or one which provides for rocking motion only, is provided. A form similar to that shown in Fig. 369, but without the rollers, may be used for the fixed end. Another form is shown in Fig. 368.

215. Bridge Floors.—The floor of a bridge may be carried on the top of the main girders, or it may be attached to the bottom flanges of these girders. In the former case the traffic passes over the main girders, and the bridge is called a "*deck*" *bridge*; in the latter case the traffic passes between the main girders, and the bridge is then called a "*through*" *bridge*.

216. Railway Bridge Floors—Cross Girders.—The weight of the bridge platform and the rolling train load is transmitted to the main girders by cross girders, which are usually shallow, plate web girders spaced at intervals along the main girders, and placed transversely to them. Figs. 327, 328, and 329, p. 223, show the common means of attachment if the main girders are of the plate web type.

Common methods of attaching cross girders to main girders of the open web type are shown in Figs. 370, 371, and 372. Figs. 370 and 371 apply to through bridges, and Fig. 372 to deck bridges. In Fig. 370 the cross girder rests on the flange of the main girder directly, while in Fig. 371 it is slung below. The latter method has the advantage that the load is transmitted directly to the centre of the main truss, and does not tend to twist the flange.

The minimum spacing of the cross girders should be from 7 to 8 feet, that is, not less than the distance apart of the driving axles of the heaviest locomotive crossing the bridge. This spacing, however, may be much increased in large spans. In any case, cross girders may only be attached to the main girders at panel points.

Each cross girder must be capable of carrying its share of the dead load of the bridge platform, together with the heaviest live axle load which may come upon it. Cross girders are usually assumed to be freely supported at the ends.

217. Rail Bearers.—Spanning between the cross girders, and placed directly beneath the rails, are longitudinal girders called *rail bearers* or *stringers*. For a 4 feet $8\frac{1}{2}$ inches gauge these rail bearers would be spaced about 5 feet, centre to centre. The rail bearers carry the weight of the platform and the axle loads on to the cross girders, to which they are

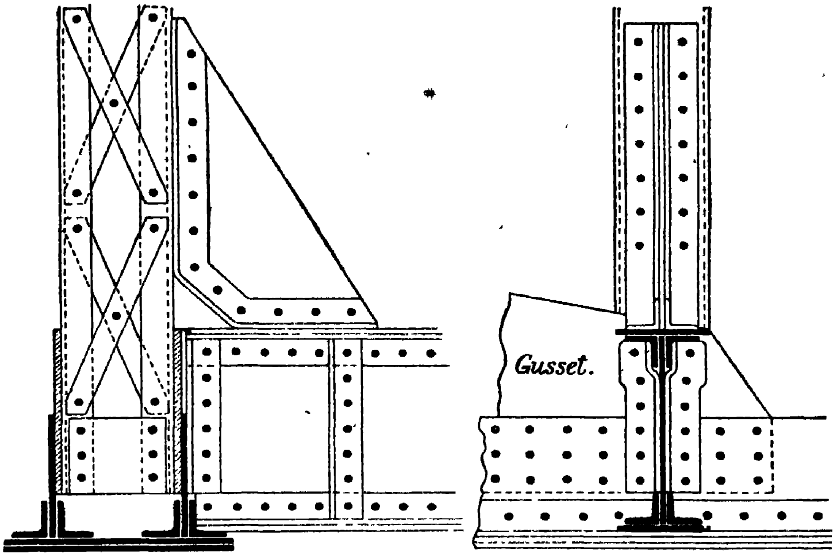


FIG. 370.

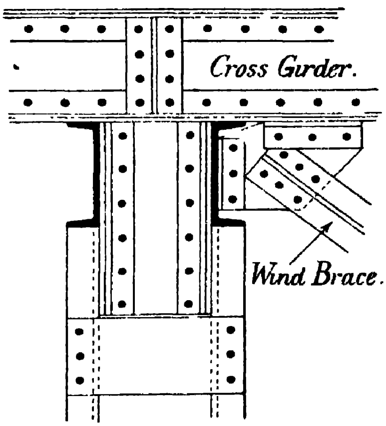


FIG. 372.

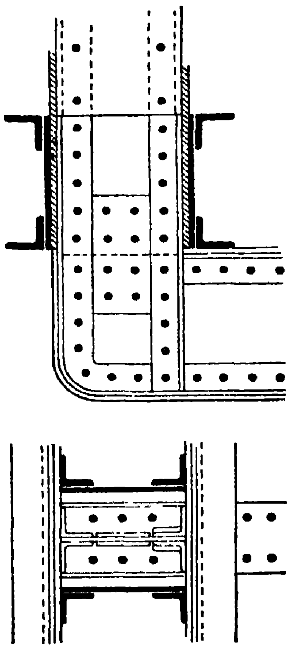


FIG. 371.

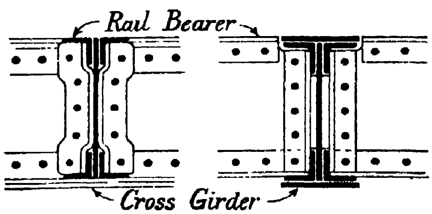


FIG. 373.

attached. Fig. 373 shows the method of connecting the rail bearers to the cross girders. For a further illustration of rail bearers, see Fig. 386, p. 256. Rail bearers are designed in a similar manner to the cross girders.

218. Floor Plating.—Upon the double system of girders, made up of the cross girders and rail bearers, is placed the deck proper, which consists either of flat or buckled floor plates, and upon which the ballast rests. The thickness of these plates varies with their area and the weight supported, but is usually about $\frac{3}{8}$ inch.

219. Ballast and Sleepers.—Ballast consisting of broken stone asphalt is spread over the floor plating to a depth of 3 inches. Above this and under the sleepers at least 4 inches of hard ballast is placed.

The sleepers are of pine, 9 feet long, 10 inches wide, and 5 inches thick, spaced at about 3 feet centre to centre. If head room is limited, the sleepers may be placed longitudinally and bolted down directly to the floor plating above the rail bearers. This method has the disadvantage that it breaks the continuity of the permanent way system.

To prevent the ballast spreading, vertical plates, called *ballast guards*, are fitted (see Fig. 386, p. 256). Provision must also be made to confine the ballast at the end of the bridge. Suitable drainage arrangements to carry off water from the bridge floor are also necessary.

220. Trough Floors.—Instead of flat or buckled plates, trough sections of various forms may be employed. Fig. 374 shows a common form. In this case rail bearers are unnecessary, the troughs running longitudinally, and resting upon the cross girders, to which they transmit the load. With plate web girders, the cross girders themselves may also be dispensed with, and the troughs are then laid transversely and attached directly to the main girders.

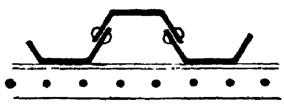


FIG. 374.

The troughs are designed as beams, the total moment of resistance of those which actually bear the load being equated to the bending moment upon them. Dimensions and moments of resistance of trough flooring are given in the various makers' catalogues.

221. Widths of Railway Bridges.—With a single line of rails, the clear width between the parapets should be 15 feet. A double line running between two main girders requires 26 feet, the distance between the roads being 6 feet.

222. Road Bridges.—The flooring takes much the same form as that of railway bridges. It must, however, be capable of carrying a live load, consisting of heavy traction engines and other vehicles, anywhere upon the surface of the road. The various forms of trough flooring are very suitable. Very small bridges may even be made without main girders, longitudinal troughs carrying the load from abutment to abutment.

The widths of road bridges correspond to those of the roads which they serve.

223. Example of American Practice.—Fig. 375 illustrates the details of a type of bridge common in America. The line diagram near the top of the figure shows a portion of the truss, and details of the joints at A, C, D, and E are shown to a larger scale. It will be observed that the tension members are eye-bars, which are connected to the other members by pin joints.

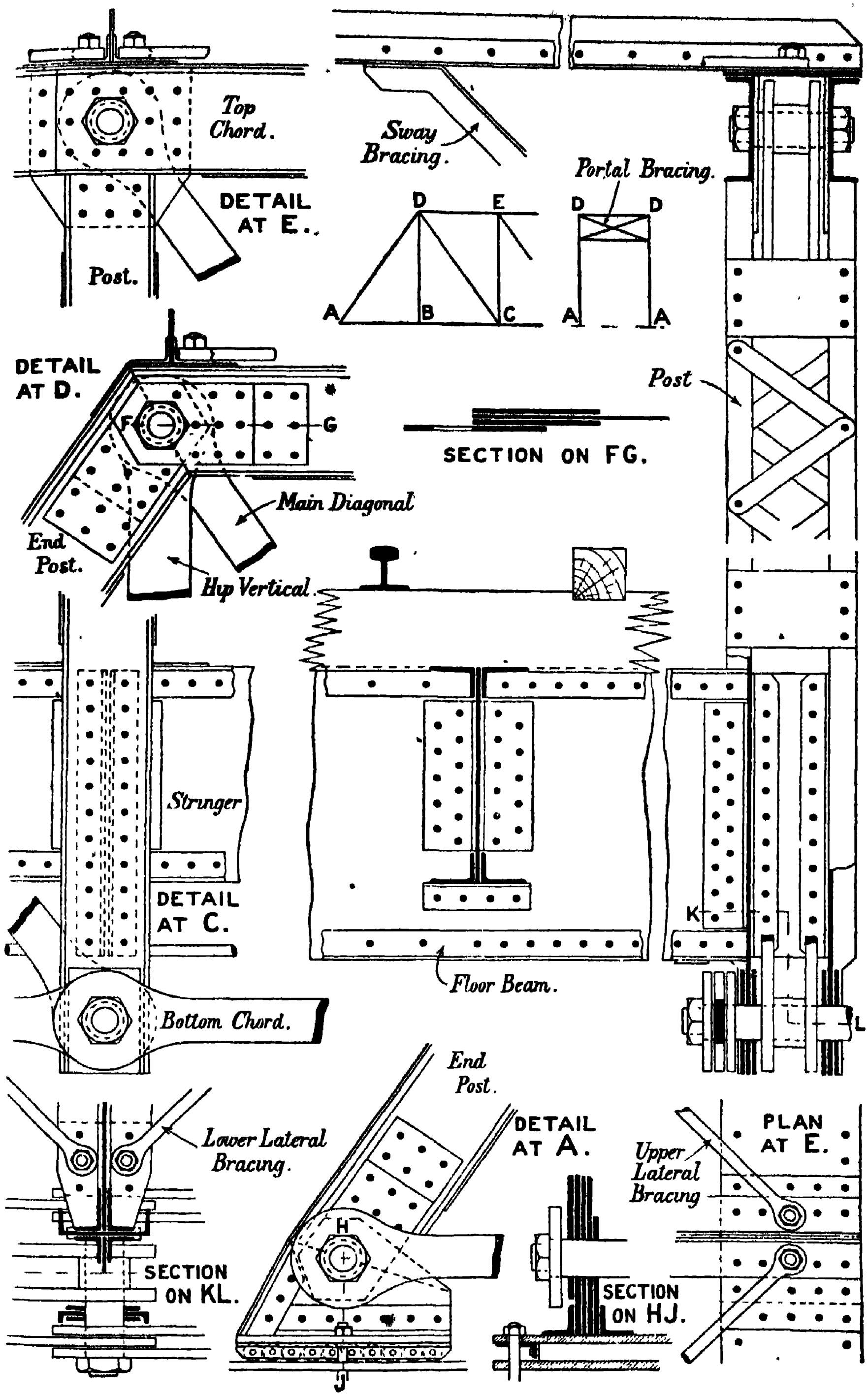

Top Chord.
DETAIL AT E.
Post.
Sway Bracing.
Portal Bracing.
D E D D
A B C A A
Post
DETAIL AT D.
F G
SECTION ON FG.
Main Diagonal
End Post.
Hip Vertical
Stringer
DETAIL AT C.
Bottom Chord.
Floor Beam.
K L
Lower Lateral Bracing.
SECTION ON KL.
End Post.
DETAIL AT A.
H J
SECTION ON HJ.
Upper Lateral Bracing
PLAN AT E.

FIG. 375.

It may be noted here that in American practice the ratio of depth to span commonly adopted is larger than is usual in British practice.

224. Wind Pressure.—In what follows, the direction of the wind is supposed to be horizontal. In estimating the effect of wind pressure on a bridge, two alternative cases should be considered. (*a*) When the bridge is unloaded, and a wind pressure of 56 lbs. per square foot is acting on it. (*b*) When a train is crossing, and a wind pressure of **30 lbs.** per square foot is acting on both the bridge and train. The wind pressure on the moving train forms a travelling load acting laterally on the structure.

The area upon which the wind acts may be estimated as follows. For a single flat bar or a solid body like the floor system, the face area presented to the wind may be taken. When two bars lie, the one directly behind the other, but from two to three diameters apart, the combined area may be taken as one and a half times that of a single bar. If, however, the distance between them is relatively great, the combined area presented is twice that of one. For example, the area presented by two ties, one behind the other, in the same panel of a lattice truss, would be one and a half times the face area of one, but the total wind pressure on the two girders of the bridge would be twice that on one. In a plate girder bridge, however, the windward girder may be assumed to shield the leeward girder to an extent depending on their distance apart. The train surface may be taken as 10 square feet per foot-run, and the travelling wind load is then 300 lbs. per foot-run.

225. Wind Girder.—The lateral wind load is supported by a girder formed by bracing together two of the main booms in a horizontal plane, usually that of the bridge floor. Sometimes the other two booms are also similarly braced together, then the two girders so formed share the load.

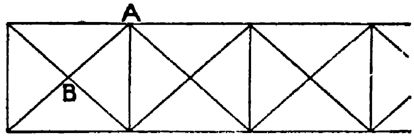


FIG. 376

If the bridge floor consists of continuous plating, this may be looked

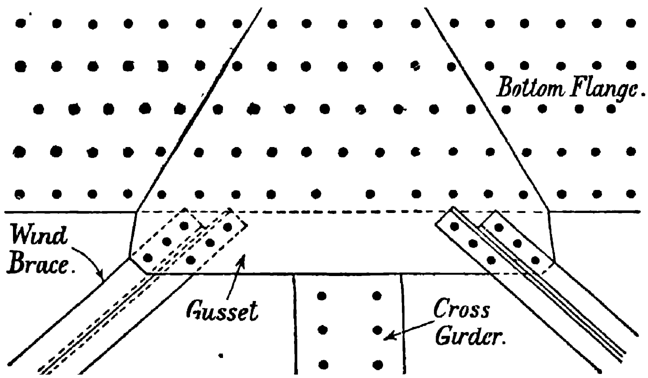


FIG. 377.

upon as forming the web of the wind girder. Frequently, however, a

separate and distinct braced web is provided. Fig. 376 is a skeleton diagram of such a web, this being of lattice pattern, since the wind may blow in either direction. Fig. 377 shows in detail the connection of the wind braces to the bottom flange or boom at A, and Fig. 378 shows two methods of connecting the wind braces at their intersection B.

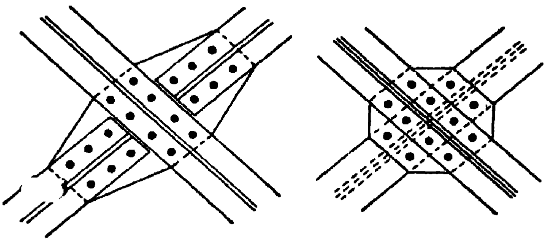


FIG. 378.

The stresses in the wind girder under both conditions (*a*) and (*b*) (Art. 224) are determined in the usual manner, and the web members designed to carry them. The stresses in the booms due to the wind are suitably combined with those due to other causes, and the booms are designed to carry the resultant stresses. All the wind pressure stresses should be treated as live load stresses.

226. Overhead and Sway Bracing.—In "through" bridges the top booms are often connected together by overhead bracing. If head room is limited, this overhead bracing takes the form of a curved girder, such

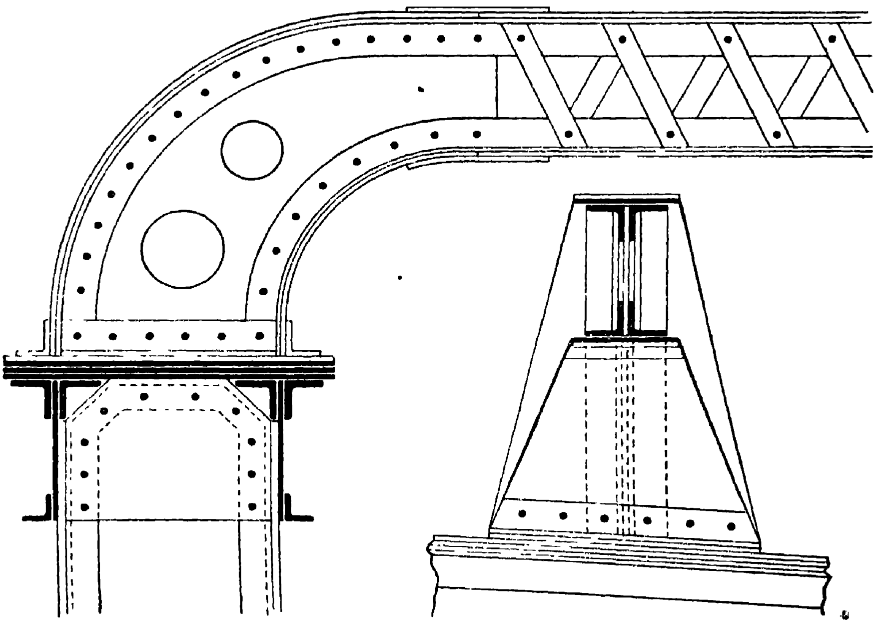
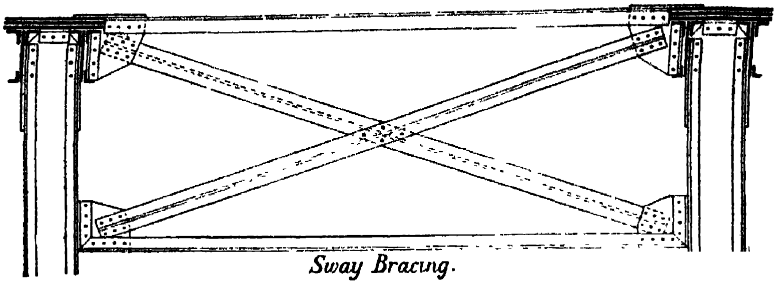


FIG. 379.

as is shown in Fig. 379. Should height permit, sway bracing of the form shown in Fig. 380 may be used.

This overhead bracing may be looked upon as tending to equalise

between the two main girders the wind pressure acting on the bridge. It



Sway Bracing.

FIG. 380.

also has the effect of resisting the distortion of the bridge due to the deflection of the cross girders as the travelling load passes.

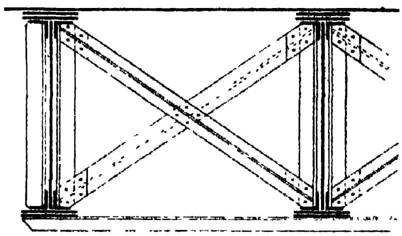


FIG. 381.

When the trusses are too low to admit of any overhead bracing, gussets may be introduced instead, connecting the vertical members with the cross girders, as shown in Fig. 370, p. 243.

To resist lateral distortion, deck bridges are invariably braced, as shown in Fig. 381. See also Fig. 372, p. 243.

227. Open Web Girder Bridge—Worked Example.—To indicate the method of procedure, the design of an open web girder bridge to fulfil the following conditions will be considered. *Type*—single track, through bridge. *Span*—150 feet. *Travelling load*—a train of "eight-wheeled" coaches, headed by three locomotives of the type shown in Fig. 382.

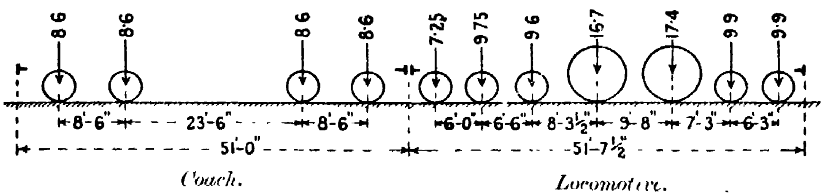


FIG. 382.

Type of Main Girders.—As typical of normal British practice for spans of the length given and carrying such a load, an **N** girder with curved top flange will be adopted.

Actual Span.—Fix as accurately as possible the actual span of the girder. Where rocking bearings are employed it is the distance from centre to centre of the pins, in this case 150 feet exactly. This span will be used for all calculations.

Depth of Girder and Number of Panels.—The depth at the centre of the span should be from one-twelfth to one-eighth of the span, the number of panels being chosen to correspond, due consideration having been given

to the spacing of the cross girders. Fig. 383 shows an outline of the truss as decided upon. The depth at the centre is 12 feet 6 inches, at the ends 7 feet 6 inches, and there are twenty panels, each 7 feet 6 inches in length.

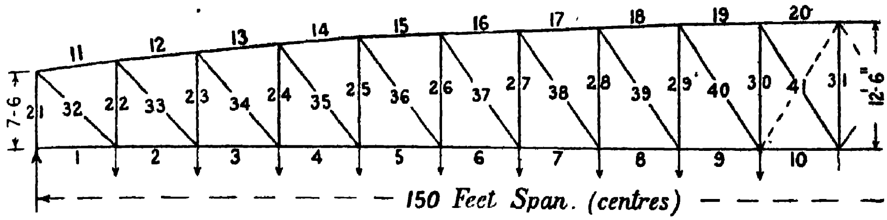


FIG. 383.

Outline of Main Girders and Cross Section.—Draw an outline of the bridge, showing the centre lines of the various members. Draw also an approximate cross section at the centre, showing the cross girders, rail bearers, flooring, etc. See Fig. 386, p. 256. Reference to an existing bridge is here desirable.

Lengths of Members.—Calculate the lengths of the centre lines of all the members of the main girders and tabulate them. See stress sheet, p. 257.

Travelling Load Effects.—Determine the maximum bending moment and shearing force diagrams, and also the equivalent uniform load due to the specified travelling load. The tracing paper method described in Art. 106, p. 98, may be used.

When the travelling load crosses the bridge the bending moment rises to a maximum of 4800 foot-tons, the equivalent uniform load being 256 tons. The maximum shear force occurs at the ends of the span, and is 133 tons.

Weight of Bridge.—The dead load due to the weight of the bridge and its floor must be estimated as closely as possible, basing the calculation if possible on an existing design. The following will indicate the method.

The area of bridge floor covered with ballast will be taken as 150 feet × 11½ feet (see cross section, p. 256). The weight of the bridge floor, exclusive of cross girders, is therefore—

	Tons.
150 feet of permanent way, including rails, chairs, and sleepers	11·0
3 inches of broken stone asphalt @ 140 lbs. per cubic foot	27·0
4 inches of ballast under sleepers @ 120 „ „	30·8
5 inches of ballast around sleepers @ 120 „ „	30·1
⅜ inch flooring plates, ballast guards, and fixings	17·1
Timber foot paths	3·8
Rail bearers (rolled steel joists @ 57 lbs. per foot-run)	8·4
Total	128·2

There are 20 panels and 21 cross girders. The dead load per cross girder is therefore 6·4 tons. Taking the maximum live load on a cross girder to be the heaviest axle load 17·4 tons, and the equivalent uniform load as 1½ times this, and doubling this latter figure to reduce it to a

dead load, the total equivalent uniform dead load upon a cross girder is $6{\cdot}4 + 52{\cdot}2 = 58{\cdot}6$ tons.

Using Unwin's formula (p. 223), assuming a ratio of depth to span of $\frac{1}{12}$, and a constant of 1500 for steel plate web girders; the span being about 16 feet, and the stress limited to 7 tons per square inch; the weight of each cross girder is 1·1 tons. Add 10 per cent. for gussets and fastenings at the ends. Hence the dead weight carried by the main girders, exclusive of their own weight, is—

	Tons.
Bridge floor	128·2
21 cross girders	25·4
Overhead wind bracing, etc., say	5·0
Total	158·6

The total equivalent uniform load on the two main girders is therefore $159 + 256 = 415$ tons, exclusive of their own weight. Applying Unwin's formula, the span being 150 feet, depth at centre $12\frac{1}{2}$ feet, and taking the safe working stress in compression at $5\frac{1}{2}$ tons per square inch, and the constant at 1900 for steel open web girders of the type under consideration, the weight of the two main girders is 87 tons.

Hence the total dead load on the main girders is $159 + 87 = 246$ tons.

Unit Load Stresses.—Find the stress in each member of the main girders with unit load at each panel point, preferably both graphically and analytically. Tabulate on the stress sheet.

Dead Load Stresses.—Tabulate also the stress in each member due to the actual dead loads at the panel points. This stress is 6·2 times the unit load stress, since the total dead load is 246 tons, and there are 2×20 panels.

Maximum Live Load Stresses in the Booms.—Find the maximum stresses in the boom members due to the live load. These will occur when the bridge is fully covered, and are obtained from the equivalent uniform load, which is 256 tons. The corresponding load at each panel point is 6·4 tons, and the stresses in the boom members are therefore 6·4 times the unit load stresses; they can therefore now be tabulated.

Maximum Live Load Stresses in the Web Members.—The maximum live load stresses in the web members must be obtained from the maximum shear force diagram (not 6·4 times the unit load stresses). Tabulate these stresses both for the front and back of the travelling load, giving to each its correct sign.

Wind Load Stresses.—Calculate and tabulate the wind load stresses under both conditions (*a*) and (*b*), Art. 224, p. 246.

Under condition (*a*) the exposed area is—

	Sq. Ft.
Twice the face area of the upper flange	450
The face area of the lower flange and floor system	420
Three times the face area of the verticals	480
Three times the face area of the diagonals	530
Total	1880

The distributed wind load at 56 lbs. per square foot is therefore 47 tons.

Under condition (*b*) the exposed area is—

	Sq. Ft.
The face area of the upper flange	225
The face area of the lower flange and floor system	420
One and a half times the area of the verticals	240
One and a half times the area of the diagonals	265
Total	1150

The distributed wind load at 30 lbs. per square foot is therefore 16 tons. The rolling wind load at 300 lbs. per foot-run is 20 tons. Hence the total wind load under these conditions is 36 tons.

In estimating the above areas reference may be made to an existing bridge, or since, in this case, the wind stresses will only affect the lower booms, the scantlings of the other members of the main girders may be calculated and their actual areas used.

The two bottom flanges of the main girders which constitute the booms of the wind girder are 18 feet centre to centre. Having found the total wind load, the stresses due to it under both conditions (*a*) and (*b*) can be calculated.

Maximum and Minimum Stresses.—All the stresses under each condition of loading are now tabulated. The maximum and minimum stress in each bar and the ratio $\frac{\text{minimum stress}}{\text{maximum stress}}$ is next determined. The maximum stress in a bar is the greatest stress whatsoever in one direction which can come on it. The minimum stress is the least stress in the same direction, or should the stress reverse, the greatest stress in the opposite direction. In the latter case the minimum stress is negative.

In finding the maximum and minimum stresses, however, care must be taken that they are the values between which the stresses actually alternate. Two examples taken from the stress sheet will be here considered.

Bar 10, tension boom. The maximum stress will occur when the bridge is fully covered by a train and is made up of dead load stress $= +184{\cdot}9$, live load stress $= +190{\cdot}9$, and wind load stress $= +37{\cdot}5$, total $413{\cdot}3$ tons. The minimum stress occurs when the bridge is quite empty and no wind blowing, and is $+184{\cdot}9$ tons. The stress will evidently alternate between these values, and their ratio is $+0{\cdot}44$.

Bar 40, web member. When the bridge is empty, the stress in this bar is $+6{\cdot}6$ tons. As a train rolls on the stress steadily decreases until the front of the train reaches the panel, when it has become $+6{\cdot}6-16{\cdot}1=-9{\cdot}5$ tons. It now begins to increase until, the rear of the train having just passed the panel, it reaches a positive maximum of $+6{\cdot}6+22{\cdot}7=+29{\cdot}3$ tons, decreasing again to $+6{\cdot}6$ tons as the train rolls off the bridge. The stress therefore alternates between a maximum of $+29{\cdot}3$ tons and a minimum of $-9{\cdot}5$ tons, and their ratio is $-0{\cdot}32$.

Working Stresses.—The safe working stress in a member depends not only on the maximum stress in it, but also on the range through which the stress alternates. Various methods and formulæ, based chiefly on

Wöhler's experiments, have been proposed to take this fact into account.*
Either of the two following may be used :—

f = safe working stress in the bar in tons per square inch.

r = the ratio $\dfrac{\text{minimum stress}}{\text{maximum stress}}$.

Claxton Fidler's Formula (slightly modified in form).—For the flanges of girders up to 100 feet span and for all web members, $f = \dfrac{9}{2-r}$ for tension members, and, $f = \dfrac{7}{2-r}$ for compression members.

For the flanges of girders over 100 feet span, $f = \dfrac{9}{1\frac{1}{2} - \frac{1}{2}r}$ for tension members, and, $f = \dfrac{7}{1\frac{1}{2} - \frac{1}{2}r}$ for compression members.

The Launhardt-Weyrauch Formula.—

$$f = 5\left(1 + \frac{r}{2}\right) \text{ for tension members.}$$

$$f = 4\tfrac{1}{2}\left(1 + \frac{r}{2}\right) \text{ for compression members.}$$

The gross area of compression members is to be taken, and suitable allowance made for the tendency of long struts to buckle.

The working stresses found by either, or both, of these formulæ are tabulated on the stress sheet.

Cross Sections of Members.—Using the safe working stresses as found above, design the members of the main truss in the following order :—

Tension Boom.—Find the necessary area at the centre, and determine the section there. Set out a diagram similar to the lower part of Fig. 331, p. 227, showing the variation in the force in the boom throughout its length, and show on this diagram the worth of each element in the boom section, using the safe working stress in each panel in turn. after deducting the area lost through rivet holes. This diagram determines the number and length of the boom plates.

Compression Boom.—Make a similar diagram for the compression boom, using, however, the gross area of the section. In the present design there is no need to make any allowance for buckling.

Joints in Booms.—Arrange for suitable grouped joints in the booms. The proposed lengths of plates should be shown on the diagrams. They must of course be convenient from a practical point of view. Design the riveted joints and find their efficiencies. They must equal in strength the plates which they connect. The safe working shear stress may be taken as 0·8 of the safe working stress in the bar. The safe working bearing stress may be double that of the shear stress.

Diagonal Ties.—Find a suitable section, distributing the load over one, two, or four bars as may appear necessary. Design the riveted joint in the end of the tie. If the stress reverses in a tie it becomes a strut, and it must also be considered as such.

* For a full discussion of this subject the student is referred to Professor Claxton Fidler's treatise on "Bridge Construction."

Example. Bar 32. Maximum load, 151·7 tons. Safe working stress, 5·9 tons (Claxton Fidler), or 6·2 tons (Launhardt-Weyrauch).

Try four bars 11 inches wide by $\frac{5}{8}$ inch thick.

Net area (less one rivet hole) = 25·3 square inches.

Actual stress in tension = 151·7 ÷ 25·3 = 6 tons per square inch.

Number of rivets required, 15 of $\frac{7}{8}$ inch diameter.

Minimum efficiency, 90·4 per cent. (tearing at second row, and shearing at first). Equivalent stress, 6·1 tons per square inch.

Compression Members.—Allowance may be made for the lengths of the struts by the following modification of the Rankine-Gordon formula. Using the notation of Art. 160, p. 166, $f = \frac{P}{A}\left\{1 + a\left(\frac{L}{k}\right)^2\right\}$

Secondary flexure in a strut braced as that shown in Fig. 384, p. 254, need not be considered.

To find the number of rivets in the end of the strut, the safe working stress for shear may be taken as 0·8 of the safe working stress in the bar, and the safe bearing pressure as double the safe shear stress.

Example. Bar 22. Maximum load, 95·3 tons. Calculated length, 8·45 feet. Equivalent length, 8·45 × 1·6 = 13·5 feet. Assume a section consisting of two B.S. channels, No. 19, 23·55 lbs. per foot-run, to each of which is riveted a plate 10 inches × $\frac{1}{2}$ inch (see Fig. 384, p. 254). Area of section = 23·8 square inches. Minimum I = 286, and $k^2 = 12$, both in inch units.

Safe stress, 4·5 tons per square inch (Claxton Fidler).

„ „ 5·5 „ „ „ (Launhardt-Weyrauch).

Actual stress $= \frac{95·3}{23·8}\left\{1 + \frac{(13·5 \times 12)^2}{36000 \times 12}\right\} = 4·3$ tons per square inch.

Safe stress in single shear on rivets = 0·8 × 4·5 = 3·6 tons per square inch. Number of rivets required = 43.

Gussets.—These should be thicker than the members which they unite, and of suitable shape to allow of good connections. The number of rivets through the ties and struts has already been determined. The number of rivets through the gussets into the boom is found as follows. The dead load which a gusset adds to the boom is determined from the dead load stress diagram, and the live load which it adds is found from the maximum shear force diagram. From these the minimum and maximum stresses and their ratio can be obtained. The safe working shear stress may be taken as eight-tenths of the safe working compressive stress, as found by the formulæ on p. 252.

Where a cross girder is attached to a gusset the additional load which it adds, including dead, live, and wind loads, must be compounded with that from the struts and ties, allowing for the fact that part of it will be in a direction perpendicular to the flange. The safe working stress can thus be determined, and then the number of rivets required. There will be local bending moments on the gussets and the rivets in them in almost every case, and it is well to err on the side of liberality when designing them.

Outline Drawing of Main Girders.—This can now be made. The centre lines are first set out, and the sections of the members shown upon them. Next the riveting is arranged, care being taken to get uniform

pitches, suitable joints, and provision for the attachment of the cross girders, etc. Some details may need revision. From this drawing the working drawings may be made.

To complete the bridge the following calculations are necessary.

Cross Girders and Rail Bearers.—These carry a definite portion of

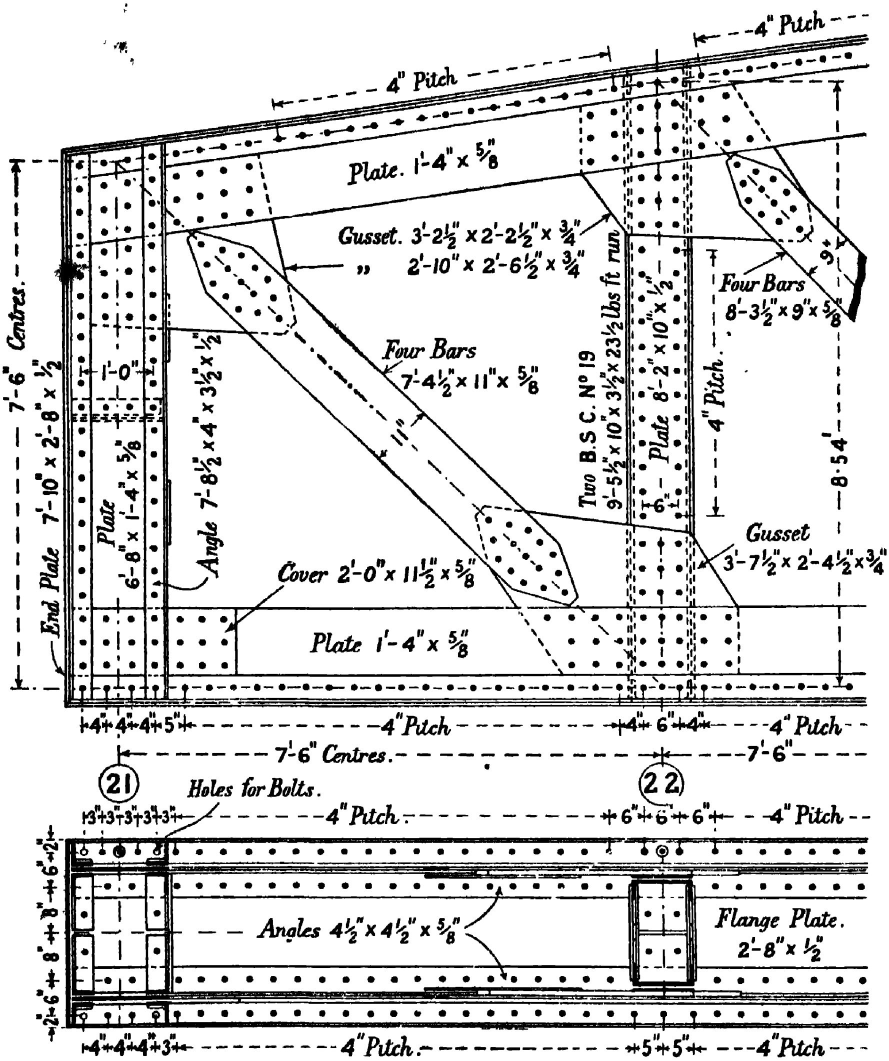


FIG. 384.

the bridge floor and a live load, consisting of one or more of the heaviest wheel loads. Design them as plate web girders (p. 216), doubling the live load to reduce it to an equivalent dead load. The working stress may be taken at $7\frac{1}{2}$ tons per square inch in tension.

Floor Plating.—This may be considered as carrying a certain dead

load per unit area in rectangular panels. The shear stress due to wind

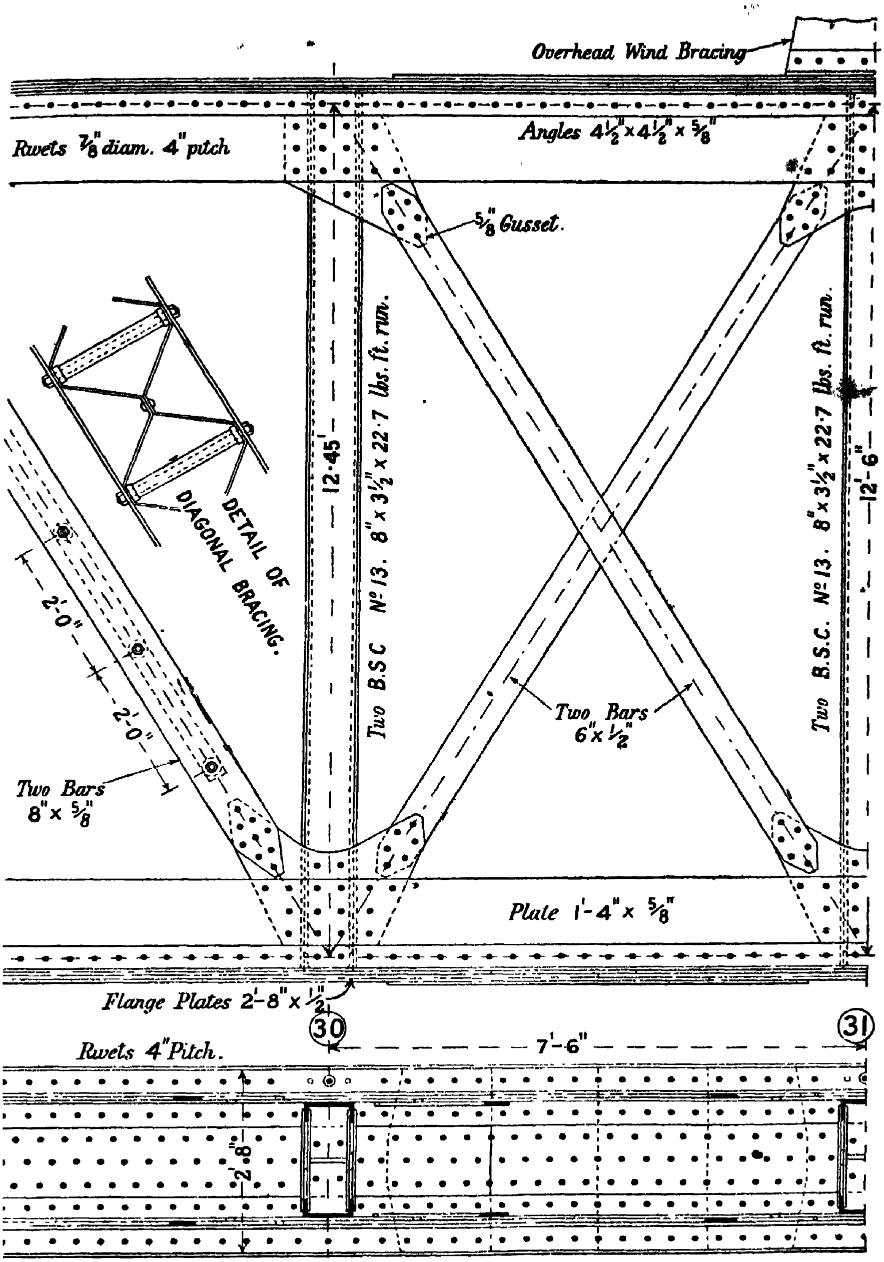


FIG. 385.

loads must also be examined. Practically, the thickness would be about ⅜ inch. As few sizes of plates as possible should be used (two only in

the example). Suitable connections to the cross girders and rail bearers should be arranged.

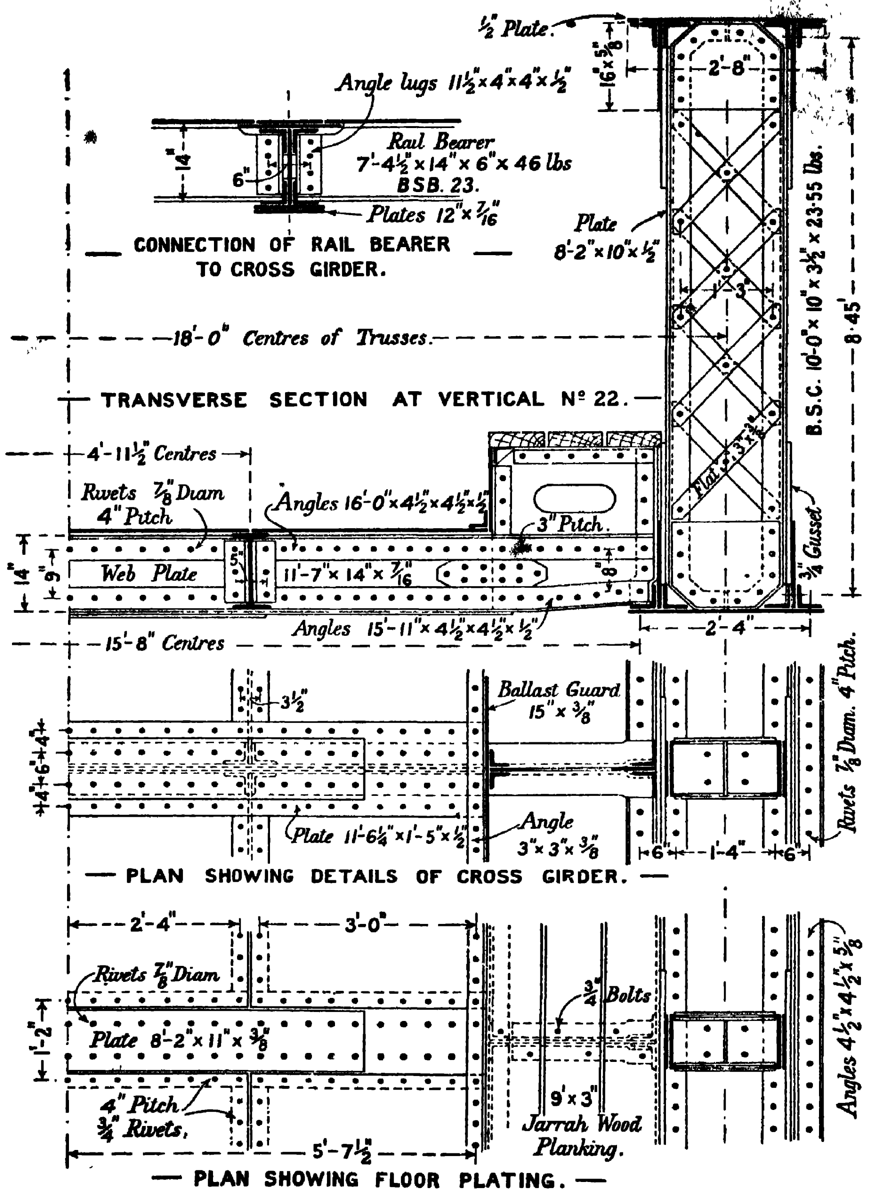


FIG. 386.

The Overhead Wind Bracing may be designed to carry one-half of

Single Track Railway Bridge, 150 feet Span. Extracts from Stress Sheet.

Bar number (see Fig. 383, p. 219)		5	10	15	20	22	29	32	40
Lettered (on stress diagram, not given)		I_1F	S_1K	J_1A	T_1A	B_1C_1	P_1Q_1	A_1B_1	Q_1R_1
Calculated length, in feet		7·50	7·50	7·52	7·50	8·45	12·30	10·61	14·41
Stress with unit load at each panel point		+ 22·43	+ 29·82	− 25·07	− 30·00	− 7·43	− 0·52	+ 11·93	+ 1·06
Actual stress, in tons, due to the dead load		+ 139·1	+ 184·9	− 155·4	− 186·0	− 46·1	− 3·2	+ 74·0	+ 6 6
Actual stress, in tons, due to the rolling live load.	In the flanges	+ 143·6	+ 190·9	− 160·4	− 192·0	—	—	—	—
	In the web in front of the load	—	—	—	—	+ 1·6	+ 15·1	—	− 16·1
	In the web at the rear of the load	—	—	—	—	− 49·2	− 18·2	+ 77·7	+ 22·7
Actual stress, in tons, due to wind pressure.	Empty bridge	± 36·8	± 49·0	—	—	—	—	—	—
	Loaded bridge	± 28·1	± 37·5	—	—	—	—	—	—
Maximum stress, in tons		+ 310·8	+ 413 3	− 315·8	− 378·0	− 95·3	− 21·4	+ 151·7	+ 29·3
Minimum stress, in tons		+ 139 1	+ 184·9	− 155·4	− 186·0	− 44·5	+ 11·9	+ 74 0	− 9·5
Ratio, minimum stress ÷ maximum stress		+ 0·44	+ 0·44	+ 0·49	+ 0·49	+ 0·47	− 0·56	+ 0·49	− 0·32
Safe stress, tons per square inch (Clayton Fidler)		7·0	7 0	5·5	5·5	4·5	2·7	5 9	3·8
Safe stress, tons per square inch (Launhart-Weyrauch)		6·1	6·1	5·6	5·6	5·5	3·2	6·2	4·2

the wind load on the windward girder to the leeward girder. The form shown in Fig. 379, p. 247, would be suitable in this example.

The Roller and Fixed Bearings and any other details will complete the design for the superstructure.

The probable deflection, necessary camber, quantities and weights will complete the calculations.

Should the finished weight come out much in excess of that estimated, it will be necessary to re-design the structure to allow for this.

Exercises XV.

1. A road bridge is 80 feet long and 15 feet clear width between the main girders. Each main girder is of the Warren type, and is divided into eight equal bays of 10 feet each. The weight per foot-run of each main girder may be taken as 4 cwt., and the total weight of cross girders, flooring, etc., per foot-run as 1 ton. The girder has to be designed to support a crowd of people weighing 1 cwt. per square foot of roadway, and also to be strong enough to sustain a traction engine. The wheel base of the traction engine may be taken as 14 feet, and the loads on the axles are 7 and 15 tons respectively. Estimate the greatest force to which each member is subjected. Also sketch a section of the booms and, starting from a point of support, proceed to determine the scantling of the members and to design the joints. Choose your own material, working stresses, and scales. Your calculations must be handed in with your drawings. [U.L.]

2. Design for a single track railway bridge. Span, 120 feet. Ratio of depth to span, $\frac{1}{10}$. The girders to be of uniform depth, divided into ten equal panels. Web bracing to be of **N** type. The bridge is to carry a uniform travelling load of 2 tons per foot-run, the maximum axle load being 18 tons.

3. Fig. 387 shows a hinged lifting bridge. The span is 40 feet, and it is divided into five equal bays, each of 8 feet length. The bridge load is equivalent to a uniformly distributed dead load of $\frac{5}{8}$ of a ton per foot-run, and to a uniformly distributed live load of $\frac{1}{4}$ ton per foot-run. Determine, (*a*) the stresses in the various bars of the bridge when it is closed and fully loaded; (*b*) when it is being lifted and is just clear of the free support, carrying then, of course, only the dead load.

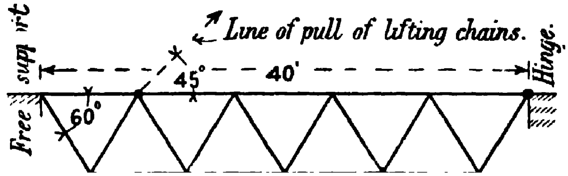


FIG. 387.

Choose your own working stresses, and design the top and bottom booms. All drawings to be neatly finished in pencil and fully dimensioned. All calculations must be handed in with the drawings. [U.L.]

4. Fig. 388 shows a bowstring girder for a proposed road bridge, which has also to carry a tram line; the span of the bridge is 140 feet, the depth at the centre 26 feet 6 inches; width of bridge from centre to centre of main girders, 18 feet. The dead load is to be 1300 lbs. per lineal foot, the live load 3100 lbs per lineal foot. Determine in any way you please the stresses in each member of the girder due to dead and live loads. Design the top and bottom booms. You are not required to draw the section of the booms, but to determine the necessary cross sectional area, and to sketch the sections. [U.L.]

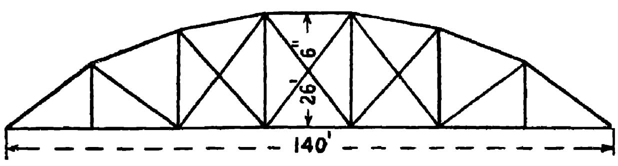


FIG. 388.

5. Design for a single track railway bridge of the American type (see p. 245).

To be of 160 feet span divided into eight equal panels. Ratio of depth to span, $\frac{1}{8}$. To carry a train consisting of locomotives of the type shown in Fig. 389.

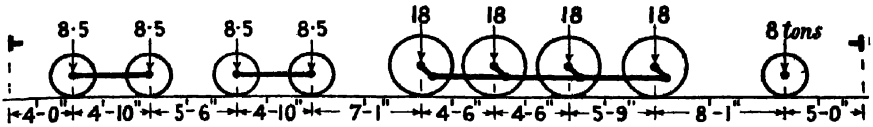


FIG. 389.

6. Design for a single track railway bridge of the type shown in the worked example (pp. 248-258). To be of 160 feet span. Ratio of depth to span about $\frac{1}{10}$. To carry a train of locomotives of the type shown in Fig. 390.

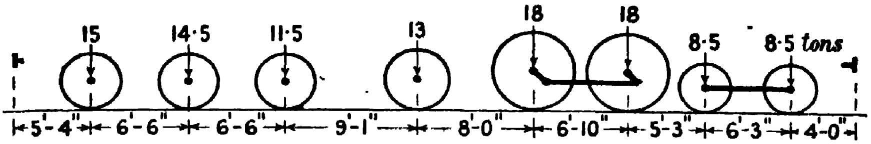


FIG. 390.

7. Design for a double track railway bridge of the Whipple-Murphy type. Span, 200 feet. It is required to carry trains of locomotives of the type shown in Fig. 390.

CHAPTER XVI

FRICTION AND LUBRICATION

228. Sliding Friction—Coefficient of Friction.—Friction is the resistance which comes into action when one body is made to slide over another. The *force of friction* (F) is the least force, acting parallel to the sliding surfaces of the bodies in contact, which will cause the one body to slide over the other. If Q is the mutual normal pressure between the bodies in contact, the ratio F/Q is called the *coefficient of friction*, and is denoted by μ. The following table gives some values of μ for moderate pressures and low speeds:—

Wood on wood,	dry . .	0·25 to 0·5	Leather on wood,	dry .	0·3 to 0·5
,, ,,	soaped .	0·1 ,, 0·2	Leather on metal,	dry .	0·3 ,, 0·6
,, ,,	greased	0·02 ,, 0·1	,, ,,	wet .	0·36
Metal on wood,	dry . .	0·2 ,, 0·6	,, ,,	greased	0·23
Metal on metal,	dry . .	0·15 ,, 0·3	,, ,,	oiled .	0·15
,, ,,	oiled intermittently	0·07 ,, 0·08	Hemp ropes on metal,	dry	0·2 to 0·34
Metal on metal,	oiled continuously	0·04 ,, 0·06	,, ,,	,, greased	0·15

The foregoing values of μ must be taken as approximate only. The results of experiments on friction are very discordant. It has been found that the coefficient of friction depends on the material of the sliding bodies, the state of their surfaces as regards smoothness, the intensity of the pressure between the surfaces, the velocity of sliding, the nature and quantity of the lubricant and the manner in which it is applied, and also on the temperature.

The friction at starting from rest or *statical friction* is greater than the friction of motion, and depends on the hardness of the bodies and the length of time during which they have been in contact.

The so-called *laws of friction* are—(1) The force of friction is directly proportional to the pressure between the surfaces in contact. (2) The force of friction is independent of the extent of the surfaces in contact. (3) The force of friction is independent of the velocity of sliding. These "laws" are approximately true when the intensity of the pressure between the surfaces is moderate, and when the speed of sliding is low.

229. Relations between the Forces on a Sliding Body.—Consider first the case of a body A of weight W resting on a fixed horizontal plane (Fig. 391). A is at rest under the action of two forces: (1) W, the pressure of A on the plane; (2) R, the pressure of the plane on A. In this case R is obviously equal and opposite to W. Suppose next that a

horizontal force P is applied to A, as shown in Fig. 392, and suppose that, on account of the friction between A and the plane, A remains at rest. P and W have a resultant S which makes an angle β with the normal to the plane and $\tan \beta = P/W$, also $S = \sqrt{P^2 + W^2} = W/\cos \beta$. To balance the force S there must be an equal and opposite force R exerted by the plane on A. If the force P be increased and A still remains at rest, R will increase, and so will the angle β. When P is increased until A begins to move, then $P/W = \mu$, by the definition of μ, and the angle β will have its maximum value ϕ, where $\tan \phi = \mu$. The angle ϕ is the angle which R makes with the normal to the plane when sliding begins, and is called the *friction angle*, the *limiting angle of resistance* or the *limiting angle of reaction.*

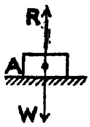


FIG. 391.

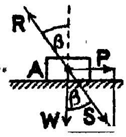


FIG. 392.

If the plane be tilted up through an angle β and A remains at rest on the plane (Fig. 393), R, the reaction of the plane on A, must balance W, and must therefore make an angle with the normal to the plane equal to β. The normal pressure of A on the plane is $W \cos \beta$, and P, the component of W parallel to the plane, is $W \sin \beta$. If the angle β be increased until A begins to slide down the plane, P will then be equal to $\mu W \cos \beta = W \sin \beta$, hence $\mu = \tan \beta = \tan \phi$, and ϕ, which has been called the friction angle, is also the maximum inclination which the plane can have consistent with the body A remaining at rest, or it is the minimum inclination which the plane can have consistent with the body sliding down the plane by the force of gravity. This inclination of the plane is called the *angle of repose*, and it is the same as the friction angle.

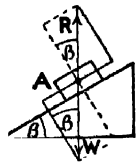


FIG. 393.

Next let A be beginning to slide on a horizontal plane, the force P being inclined at an angle θ to the horizontal (Fig. 394). The forces P, W, and R are in equilibrium, and R must be inclined to the normal to the plane at an angle ϕ. From the triangle of forces,

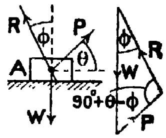


FIG. 394.

$$\frac{P}{W} = \frac{\sin \phi}{\sin (90 + \theta - \phi)} = \frac{\sin \phi}{\cos (\theta - \phi)}.$$

Hence for given values of W and ϕ, P will be least when $\cos (\theta - \phi)$ is greatest, that is, when $\theta = \phi$; the direction of P will then be perpendicular to that of R.

Consider next the case where a body of weight W is pulled up a plane which is inclined at an angle α to the horizontal by a force P acting parallel to the plane (Fig. 395), the motion being uniform. The forces which balance one another are P, W, and R, the latter force making an angle ϕ with the normal to the plane. From the triangle of forces

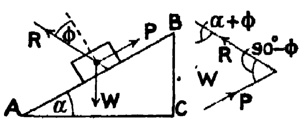


FIG. 395.

$$\frac{P}{W} = \frac{\sin (\alpha + \phi)}{(\sin 90 - \phi)} = \frac{\sin (\alpha + \phi)}{\cos \phi} = \frac{\sin \alpha \cos \phi + \cos \alpha \sin \phi}{\cos \phi} = \sin \alpha + \mu \cos \alpha,$$

If b, h, and l be the base AC, height BC, and length AB of the plane respectively, then $\sin \alpha = h/l$ and $\cos \alpha = b/l$, therefore $\frac{P}{W} = \frac{h}{l} + \frac{\mu b}{l}$ and $Pl = Wh + \mu Wb$, which shows that *the work done in drawing a body up an inclined plane is equal to the work done in lifting it against gravity through a height equal to the height of the plane, plus the work done in drawing it along the base of the plane against friction.* This is a useful rule to remember.

Here it may be pointed out that when α is comparatively small, as it generally is for most roads and railways, it is sufficiently accurate to assume that the base and length of the plane are equal.

A case of the inclined plane which is important in connection with the theory of the screw, is that in which the force P is parallel to the base of the plane (Fig. 396). The triangle of forces shows that $P = W \tan (\alpha + \phi)$.

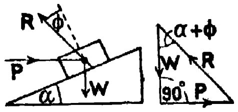


FIG. 396.

230. Efficiency of the Inclined Plane.—The efficiency of any machine being the ratio of the useful work done to the total work, this must be the same as the ratio of the effort when friction is neglected to the effort when friction is considered. Taking the case of the inclined plane shown in Fig. 395, where the effort P acts parallel to the plane, it has been shown that $P = \frac{W \sin (\alpha + \phi)}{\cos \phi}$ when friction is considered. If $\phi = 0$, $P = W \sin \alpha$, which is the value of the effort when friction is neglected. Hence the efficiency in this case is $\frac{\sin \alpha \cos \phi}{\sin (\alpha + \phi)}$.

For the case shown in Fig. 396, where the effort is horizontal, the efficiency is $\frac{\tan \alpha}{\tan (\alpha + \phi)}$.

231. Friction of Screws.—The connection between the inclined plane and the screw is shown clearly by Figs. 397 to 401. In Fig. 397 is shown a cylinder with one turn of a helix traced on its surface; the dotted right angled triangle is the development of the portion of the surface of the cylinder which is below the helix. p being the pitch of the helix, α its inclination, and d the diameter of the cylinder, $\tan \alpha = p/\pi d$.

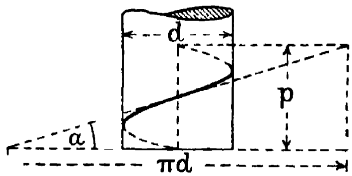


FIG. 397.

In Fig. 398 the inclined plane and the body sliding on it are two similar wedges which, when bent round a cylinder, as shown in Fig. 399, produce a form of screw and nut.

The connection between the inclined plane and a square double threaded screw and nut is shown in Figs. 400 and 401.

The force P in Figs. 398 to 401 is shown acting parallel to the base of the inclined plane or perpendicular to the axis of the screw, and in the case of the screw, P acts at a distance from the axis equal to the mean

radius of the screw. In each case, W being the load carried by the nut,

$$\frac{P}{W}=\tan\,(\alpha+\phi)=\frac{\tan\alpha+\tan\phi}{1-\tan\alpha\,\tan\phi}=\frac{p+\mu\pi d}{\pi d-\mu p}.$$

In practice the screw is usually rotated in the nut or the nut on the screw by a force Q acting on a wheel or lever of radius r attached to the screw or nut, and $Pd=2Qr$.

To reverse the motion of the screw or nut and lower the load W the effort P is reversed, and its value is then $P=W\tan\,(\phi-\alpha)$. When ϕ is

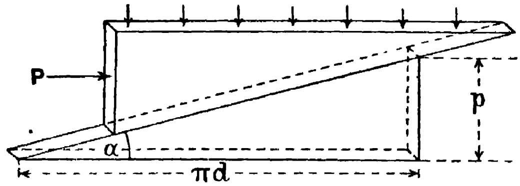


FIG. 398.

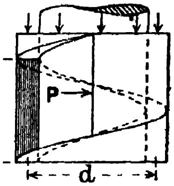


FIG. 399.

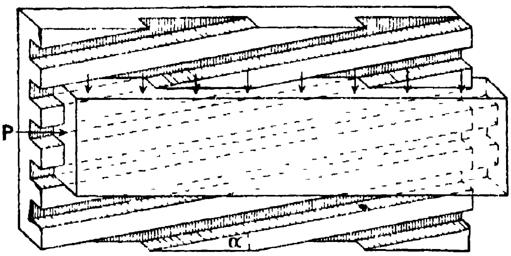


FIG. 400.

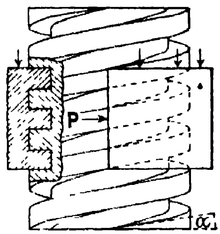


FIG. 401.

greater than α, P has a positive value, but when ϕ is less than α, P is negative, that is, it must act in the same direction for lowering as for raising, and if left to itself the load W will reverse the motion.

In the case of a screw thread of triangular section (Fig. 402), if R, the normal pressure on the thread, be resolved into two components, W parallel to the axis and S perpendicular to the axis of the screw, then $R=W/\cos\beta$, where β is the complement of the inclination of the side of the section of the thread to the axis. Now the friction is proportional to R; hence for a triangular thread $\tan\phi$ must be increased to $n\tan\phi$, where $n=1/\cos\beta$, and

$$\frac{P}{W}=\frac{\tan\alpha+n\tan\phi}{1-n\tan\alpha\,\tan\phi}=\frac{p+n\mu\pi d}{\pi d-n\mu p}.$$

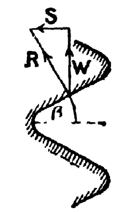


FIG. 402.

Also, since ϕ is generally a small angle, and n is less than $1\frac{1}{2}$ in ordinary cases, $n\tan\phi=\tan n\phi$ nearly, then $\frac{P}{W}=\tan\,(\alpha+n\phi)$ approximately. In the Whitworth thread $2\beta=55°$, and $n=1{\cdot}13$. In the Sellers thread $2\beta=60°$, and $n=1{\cdot}15$.

232. Efficiency of Screws.—*Square Thread.*—Since the efficiency is the ratio of the effort without friction to the effort with friction, efficiency $= \dfrac{\tan \alpha}{\tan (\alpha + \phi)}$; this is a maximum when $\alpha = 45° - \dfrac{\phi}{2}$, and the maximum efficiency is

$$\frac{\tan\left(45 - \dfrac{\phi}{2}\right)}{\tan\left(45 + \dfrac{\phi}{2}\right)} = \left(\frac{1 - \tan\dfrac{\phi}{2}}{1 + \tan\dfrac{\phi}{2}}\right)^2 = \frac{1 - \sin\phi}{1 + \sin\phi}.$$

Putting $\tan \phi = \mu$ and $\tan \alpha = p/\pi d$, where p is the pitch and d the mean diameter of the screw, then efficiency $= \dfrac{p(\pi d - \mu p)}{\pi d(p + \mu \pi d)}$.

The reversed efficiency, that is, the efficiency when W becomes the effort and P the resistance, is $\dfrac{\tan(\alpha - \phi)}{\tan \alpha}$; this is a maximum when $\alpha = 45° + \dfrac{\phi}{2}$, and the maximum efficiency is $\dfrac{\tan\left(45 - \dfrac{\phi}{2}\right)}{\tan\left(45 + \dfrac{\phi}{2}\right)}$, which is the same as the maximum direct efficiency.

Triangular Thread.—Without friction $P = W \tan \alpha = \dfrac{Wp}{\pi d}$.

With friction $P = \dfrac{W(\tan \alpha + n \tan \phi)}{1 - n \tan \alpha \tan \phi} = \dfrac{W(p + n\mu\pi d)}{\pi d - n\mu p}$.

Hence, efficiency $= \dfrac{\tan \alpha(1 - n \tan \alpha \tan \phi)}{\tan \alpha + n \tan \phi} = \dfrac{p(\pi d - n\mu p)}{\pi d(p + n\mu\pi d)}$.

The reversed efficiency is $\dfrac{\tan \alpha - n \tan \phi}{\tan \alpha(1 + n \tan \alpha \tan \phi)} = \dfrac{\pi d(p - n\mu\pi d)}{p(\pi d + n\mu p)}$.

The efficiencies of square and triangular threaded screws have been calculated for various values of α and three different values of μ, and the results have been plotted in Fig. 403. The full curves relate to the square threads, and the dotted curves to the triangular threads. It will be seen that for the same values of α and μ the efficiency of the triangular thread is not much less than that of the square thread; the difference is greater the greater the value of μ, but where $\mu = 0{\cdot}3$, the greatest difference is only about 4 per cent.

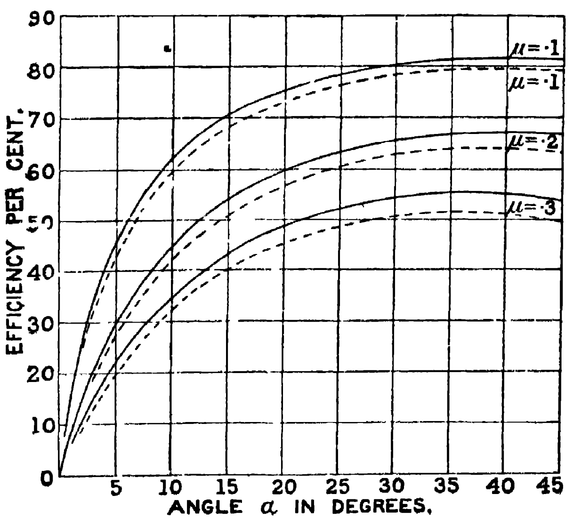


FIG. 403.

233. Friction of Pivots and Collars.—A thrust along the axis of a shaft is taken up by a pivot or collar bearing. A pivot must be on the end of a shaft, but a collar may be at any part of the length of the shaft. The rubbing surface of a pivot or collar may be any surface of revolution, the axis being the axis of the shaft. In the case of a pivot, the rubbing surface is generally either flat or conical. In a collar, the rubbing surface is generally flat.

In the present state of knowledge on the subject of friction, it is impossible to determine a correct expression for the friction of a pivot or collar. There is first of all the question of the distribution of the pressure on the rubbing surface to consider. When the bearing is new and there is perfect contact over the whole of the bearing surface, it is probable that the pressure is uniformly distributed, but since parts of the surface are at different distances from the axis, they must be moving with different velocities, and there is therefore, very probably, unequal wear, which will at once cause a redistribution of the pressure, and unequal distribution of pressure accompanied by different velocities will almost certainly result in variation in the coefficient of friction at different distances from the axis.

In what follows expressions will be found for the friction of pivots and collars on the assumption that the coefficient of friction is constant, and that either the pressure is uniformly distributed, or that the wear is uniform over the rubbing surfaces, and is directly proportional to the pressure and to the velocity. To say that the wear is uniform and directly proportional to the pressure and to the velocity is equivalent to stating that the product of the pressure and velocity is constant, or that the product of the pressure and radius is constant, because the velocity is proportional to the distance from the axis.

P = total axial load carried by pivot or collar.

p = intensity of normal pressure on rubbing surfaces when uniform, or at radius r when variable.

r = radius of an indefinitely narrow ring of the surface, and dr its width.

M = moment of friction on pivot or collar.

CASE I. *Flat Pivot* (Fig. 404).—(*a*) Uniform pressure $p = \dfrac{P}{\pi r_1^2}$. Load on ring of radius r and width $dr = 2p\pi r dr$. Moment of friction on ring $= 2\mu p\pi r^2 dr$,

$$M = 2\mu p\pi\int_0^{r_1} r^2 dr = 2\mu p\pi\frac{r_1^3}{3} = \frac{2}{3}\mu p\pi r_1^3 = \frac{2}{3}\mu P r_1 \cdot$$

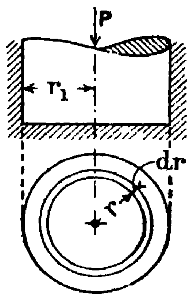


FIG. 404.

(*b*) Uniform wear. Let $pr = c$. Load on ring $= 2\pi p r dr = 2\pi c dr$.

$$\text{Total load} = P = 2\pi c\int_0^{r_1} dr = 2\pi c r_1, \text{ therefore } c = \frac{P}{2\pi r_1} \cdot$$

$$\text{Moment of friction on ring} = 2\pi c\mu r dr = \frac{2\mu\pi P r dr}{2\pi r_1} = \frac{\mu P r dr}{r_1} \cdot$$

$$M = \frac{\mu P}{r_1}\int_0^{r_1} r dr = \frac{\mu P}{r_1}\cdot\frac{r_1^2}{2} = \tfrac{1}{2}\mu P r_1 \cdot$$

CASE II. *Flat Collar or Recessed Footstep* (Fig. 405).—(*a*) Uniform pressure. Proceeding as for a flat pivot,

Moment of friction on ring

$$= 2\mu p\pi r^2 dr,$$

$$M = 2\mu p\pi\int_{r_2}^{r_1} r^2 dr = 2\mu p\pi\frac{r_1^3 - r_2^3}{3},$$

$$\text{but} \quad p = \frac{P}{\pi(r_1^2 - r_2^2)},$$

$$\text{therefore } M = \frac{2\mu P(r_1^3 - r_2^3)}{3(r_1^2 - r_2^2)}.$$

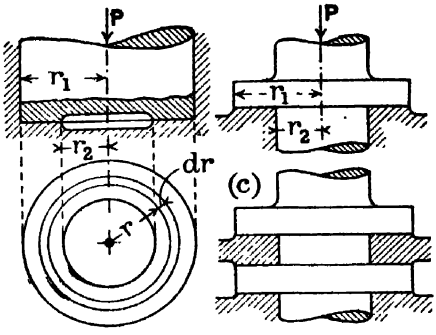


FIG. 405.

(*b*) Uniform wear. Let $pr = c$. Load on ring $= 2\pi prdr = 2\pi cdr$.

$$\text{Total load} = P = 2\pi c\int_{r_2}^{r_1} dr = 2\pi c(r_1 - r_2), \text{ therefore } c = \frac{P}{2\pi(r_1 - r_2)}.$$

$$\text{Moment of friction on ring} = 2\pi c\mu rdr = \frac{\mu P rdr}{r_1 - r_2},$$

$$M = \frac{\mu P}{r_1 - r_2}\int_{r_2}^{r_1} rdr = \frac{\mu P}{r_1 - r_2}\cdot\frac{r_1^2 - r_2^2}{2} = \tfrac{1}{2}\mu P(r_1 + r_2).$$

To increase the amount of rubbing surface, and so diminish the intensity of the pressure, it is better to use two or more collars, as shown at (*c*), Fig. 405, rather than have one large collar.

CASE III. *Conical Pivot* (Fig. 406).—(*a*) Uniform pressure. Area of surface of cone $= \dfrac{\pi r_1^2}{\sin\theta}$.

$$P = p\sin\theta\frac{\pi r_1^2}{\sin\theta} = p\pi r_1^2, \text{ therefore } p = \frac{P}{\pi r_1^2}$$ as for a flat pivot. The normal pressure on the surface of the cone is therefore independent of the angle at the vertex of the cone.

$$\text{Load on ring} = \frac{2p\pi rdr}{\sin\theta}.$$

$$\text{Moment of friction on ring} = \frac{2\mu p\pi r^2 dr}{\sin\theta},$$

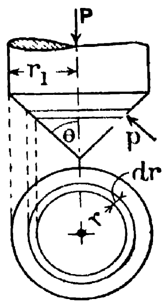


FIG. 406.

$$M = \frac{2\mu p\pi}{\sin\theta}\int_0^{r_1} r^2 dr = \frac{2\mu p\pi}{\sin\theta}\cdot\frac{r_1^3}{3} = \frac{2\mu P r_1}{3\sin\theta}.$$

(*b*) Uniform wear. From the above and Case I. it follows that

$$M = \frac{\mu P r_1}{2\sin\theta}.$$

CASE IV. *Conical Collar* (Fig. 407).—It is obvious from the preceding cases that in this case

$$M = \frac{2\mu P(r_1^3 - r_2^3)}{3\sin\theta(r_1^2 - r_2^2)}$$ for uniform pressure, and

$$M = \frac{\mu P(r_1 + r_2)}{2\sin\theta}$$ for uniform wear.

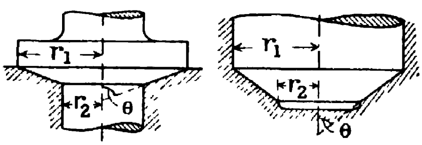


FIG. 407.

Comparing the different cases for uniform pressure and uniform wear, it will be seen that the moment of friction for uniform pressure is $\frac{4}{3}\left\{1 - \frac{r_1 r_2}{(r_1+r_2)^2}\right\}$ of the moment for uniform wear, and when $r_2 = 0$, this ratio becomes $\frac{4}{3}$.

Footstep or Pivot bearings are frequently fitted with loose discs, as shown in Fig. 408. Under normal conditions these discs will all rotate in the same direction, but with different velocities, consequently the relative velocity between the pivot and the disc next it, or between two discs, will be less than between the pivot and a fixed bearing. The total moment of friction is, however, probably not altered by the presence of the discs. If, however, one of the discs should heat up and seize, the next will act and give the first a better chance of cooling. A similar arrangement may be applied to collar bearings.

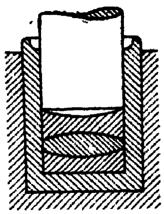


FIG. 408.

234. Schiele's Pivot.—The form of pivot known as *Schiele's pivot* was designed to give uniform wear in the direction of the axis with uniform pressure, the coefficient of friction being assumed to be constant. Let A (Fig. 409) be a point on the surface of the pivot, r the radius, and AB the tangent at A, YY being the axis. Let AC be the amount of vertical wear taking place at A; then if CD be drawn parallel to and AD perpendicular to AB, AD will be the amount of wear normal to the surface of the pivot at A. Let p = the intensity of the pressure normal to the surface of the pivot, p being assumed to be constant. The wear AD is assumed to be proportional to p and to the velocity of rubbing at A, and therefore AD is proportional to pr. Let AD $= kpr$, where k is a constant. By similar triangles

$$\frac{AC}{AD} = \frac{AB}{r}, \text{ therefore } AC = \frac{AD \cdot AB}{r} = kp \cdot AB.$$

Hence if AC is to be the same for every point on the pivot surface, AB must be constant. The curve which has the property that its tangent AB is of constant length is known as the *tractrix*, and also as the *anti-friction* curve.

It is evident that if a pivot wears equally *in the direction of its axis* it will preserve its shape, and there is a better chance of p, the intensity of the pressure, remaining uniform; also if p is uniform, the lubricant is more likely to remain between the rubbing surfaces.

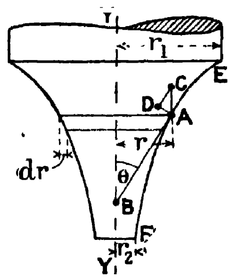


FIG. 409.

The curve EAF will never meet the axis YY, consequently this form of pivot cannot be brought to a point and have its proper shape to the end.

To find the moment of the friction of a Schiele pivot, consider a ring of the surface of radius r and width dr measured at right angles to the axis. The area of this ring is $2\pi r\frac{dr}{\sin\theta} = 2\pi l dr$, where $l = AB$. Moment of friction on ring $= 2\pi l\mu p r dr$.

Hence $$M = 2\pi l\mu p\int_{r_2}^{r_1} r\,dr = 2\pi l\mu p\frac{r_1^2}{\quad} = \pi l\mu p(r_1^2 - r_2^2).$$

The portion of the load P carried by the ring of radius r, already referred to, is

$$2\pi r\frac{dr}{\sin\theta}p\sin\theta = 2\pi prdr, \text{ therefore } P = 2\pi p\int_{r_2}^{r_1} rdr = \pi p(r_1^2 - r_2^2).$$

Hence $M = \mu Pl$, and this will be smaller, the smaller l is. Taking $l = r_1$, $M = \mu Pr_1$.

For a flat pivot it was shown that for uniform pressure $M = \frac{2}{3}\mu Pr_1$, and for uniform wear $M = \frac{1}{2}\mu Pr_1$. It would therefore appear that the friction of the Schiele pivot is greater than that of the flat pivot, but it is claimed for the Schiele pivot that the wear and pressure being uniform at every point, the surfaces always fit one another, and the lubricant is not forced out.

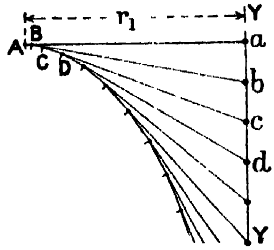


Fig. 410.

An approximate method of drawing the tractrix is shown in Fig. 410. Take points a, b, c, etc., on the axis YY. The distances ab, bc, etc., may be made equal to about one-tenth of r_1. The constant length of the tangent is taken $= a\text{A} = l = r_1$. With centre b and radius $= l$ describe an arc to cut aA at B; join bB. With centre c and radius $= l$ describe an arc to cut bB at C; join cC, and so on. A, B, C, etc., may be taken as points on the tractrix.

It may be mentioned that a tractrix is the involute of a catenary, a fact which suggests a method of drawing the curve.

A simple mechanical method of drawing the tractrix accurately is shown in Fig. 411. CD is an arm carrying a steel pin E with rounded ends, and a pencil F, the lead of which is very hard and sharpened to a knife edge, the edge being slightly convex in the direction of its length. The distance between the axes of the pin E and pencil F is equal to l, the length of the tangent of the tractrix to be drawn. The edge of the pencil point lies in the plane of the axes of the pin and pencil. The arm CD carries a weight W as shown; this weight may weigh about one pound A straight edge HK is placed with one edge parallel to the axis YY, and at a distance from it equal to the radius of the pin E. Holding the straight edge HK firmly with one hand, the pin E, held loosely between the thumb and fore-finger of the other hand, is drawn along the edge of the straight-edge, the pin being kept vertical. The pencil F traces out the tractrix AB. It is essential that the drawing edge of the pencil shall be so sharp that it

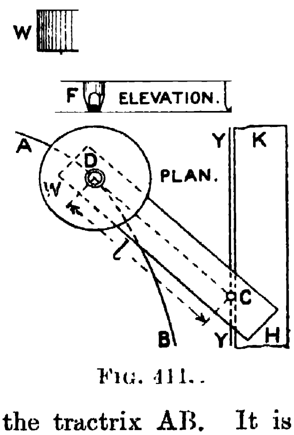


Fig. 411.

will only move easily over the paper in the direction of the edge. Instead of a drawing pencil, a piece of steel having a razor edge may be used.

235. Tower's Experiments on the Friction of Pivot and Collar Bearings. — The table below gives the results of the experiments on the friction of a pivot bearing carried out by Mr. Beauchamp Tower, and described in the fourth report of the Research Committee of the Institution of Mechanical Engineers on friction.* The pivot experimented with was of steel 3 inches in diameter, and flat ended. The bearing, which was of manganese bronze, is shown in Fig. 412. The oil was introduced, as shown, through a single central hole, and distributed over the bearing by a single diametrical groove, terminating at each end within $\frac{1}{16}$th of an inch of the circumference of the bearing. It was found that the oil circulated automatically, the pivot and bearing acting like a centrifugal pump.

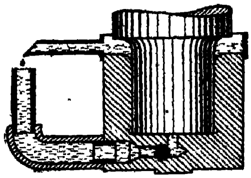


FIG. 412.

The coefficients of friction in the table below were calculated from the observed frictional moments, on the assumption that the mean leverage of the friction was two-thirds of the radius of the pivot, which would be correct if the pressure on the bearing was uniformly distributed, and the friction was independent of the velocity. The circulation of the oil varied from 20 to 56 drops per minute at the lowest speed to a continuous stream at the higher speeds.

Revs. per Min.	Load in Lbs. per Square Inch.							
	20	40	60	80	100	120	140	160
	Coefficients of Friction.							
50	0·0196	0·0147	0·0167	0·0181	0·0219	0·0221	..	..
128	0·0080	0·0054	0·0053	0·0063	0·0077	0·0083	0·0093	0·0113
194	0·0102	0·0061	0·0051	0·0045	0·0044	0·0052	0·0062	0·0068
290	0·0178	0·0107	0·0078	0·0064	0·0056	0·0048	0·0046	0·0044
353	0·0167	0·0096	0·0073	0·0063	0·0057	0·0053	0·0053	0·0054

The results of Mr. Tower's experiments on the friction of a collar bearing † showed that the friction in this type of bearing is practically independent of the speed. The adjoining table gives the mean values of the coefficient of friction (μ) obtained with different intensities of pressure (p) on the bearing ring, in lbs. per square inch. The mean leverage of the friction was taken as the mean radius of the ring. It was found in these experiments that the greatest load which the bearing would carry was 75 lbs. per square

p	15	30	45	60	67·5	75	82·5
μ	0·054	0·046	0·037	0·036	0·035	0·035	0·034

* *Proceedings of the Institution of Mechanical Engineers*, 1891.
† *Ibid.*, 1888.

inch at the highest speed, and 90 lbs. per square inch at the lowest speed.

236. Friction of an Axle.—AB (Fig. 413) is a body mounted on an axle whose axis is O and radius r. The body AB is either fixed to the axle and the axle rotates in a bearing, or the axle is fixed and AB rotates on the axle. The resultant of the forces which resist the rotation of AB is a force Q, acting at a perpendicular distance $OB = b$ from O. An effort P acts at a perpendicular distance $OA = a$ from O, and is just able to cause AB to rotate in the direction of the arrow e. The lines of action of P and Q meet at C, and make an angle θ with one another.

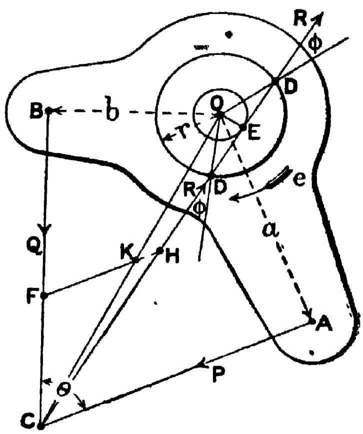


FIG. 413.

R, the resultant of the pressure of the bearing on the axle when the axle rotates (lower part of figure), or the resultant of the pressure of the axle on the lever when the axle is fixed (upper part of figure), must be equal and opposite to the resultant of P and Q, and therefore its line of action must pass through C, and must make with the normal OD to the sliding surfaces an angle equal to ϕ; also, the line of action of R will evidently lie between O and A.

Draw OE at right angles to the line of action of R. Then OE is evidently equal to $r \sin\phi$. In most cases ϕ is so small an angle that $\sin\phi$ may be taken equal to $\tan\phi$ or μ without sensible error. Let $OE = r\sin\phi = s$. If a circle be described with centre O and radius s, the line of action of R will obviously be tangential to this circle. Hence the construction for finding the line of action of R is to draw through C a tangent to the circle whose centre is O and radius s. This circle is called the *friction circle*.

If CF be made equal to Q, and FH be drawn parallel to CA to meet CE at H, the triangle CFH will be the triangle of forces for Q, P, and R, and FH will be equal to P and CH equal to R.

If there were no friction the line of action of R would be CO, and then FK would be equal to P and CK equal to R. Hence the efficiency of the mechanism is equal to FK/FH. The moment of the friction is $R \times s$ or $CH \times OE$.

Proceeding analytically, $R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta}$. Taking moments about O, $Pa = Qb + Rs = Qb + s\sqrt{P^2 + Q^2 + 2PQ\cos\theta}$, a quadratic equation which gives

$$P = \frac{Q}{a^2 - s^2}\{ab + s^2\cos\theta \pm s\sqrt{a^2 + b^2 + 2ab\cos\theta - s^2\sin^2\theta}\}.$$

The + sign in front of the surd is to be taken when P overcomes Q, and the − sign when Q overcomes P.

When $\theta = 0°$, OA and OB are in the same straight line, and P and Q are on *opposite sides* of O, then $P = Q\frac{b \pm s}{a \mp s}$, the upper sign to be taken

when P overcomes Q, and the lower sign when Q overcomes P. In this case $R = P + Q$.

When $\theta = 180°$, OA and OB are in the same straight line, and P and Q are on the *same side* of O, then $P = Q\frac{b \pm s}{a \pm s}$, the upper sign to be taken when P overcomes Q, and the lower sign when Q overcomes P. In this case $R = P - Q$.

237. Friction Axis of a Link.—The friction of a pin joint, such as is common in links or rods in mechanisms, is of the same character as the friction of an axle discussed in the preceding Article. It has been seen that in the case of an axle when the axle rotates in its bearing, or when the piece carried by the axle rotates on it, the resultant force on the axle does not intersect the axis, but is a tangent to its friction circle. So in a link, like the connecting-rod of a steam-engine, with pin joints at its ends, the line of thrust or pull on the rod will not coincide with the axis of the rod,* but will be tangential to the friction circles of the pins of its joints. This actual line of thrust or pull on the rod is called the *friction axis* of the rod or link, the change of the line of action of the thrust or pull from the geometrical axis of the rod to the friction axis being due to the friction of its pin joints.

Since four tangents may in general be drawn to two circles, it follows that a rod with pin joints has four different possible friction axes, and the one which is to be taken in any particular case will depend on the directions of the external forces on the link, and on the directions of its motions relative to the pins of the joints or to the bearings of the pins. This point is made clear by Fig. 414, which shows the connecting-rod AB and the crank BC of a steam-engine in four different positions (a), (b), (c), and (d) during a revolution of the crank.

The friction circles of the pins at A and B are shown greatly enlarged for the sake of clearness. In each position A′B′ is the friction axis of the connecting-rod. There is rotary motion of the pins in their bearings at A and B, and the point to be remembered is, that since friction always opposes motion, the force acting along the friction axis at a joint must have a moment about the axis of the pin to overcome the friction which tends to prevent the rotation at that joint.

In the position (a), Fig. 414, where the connecting-rod is exerting a thrust on the pins at A and B, the angle BAC is increasing, and will go

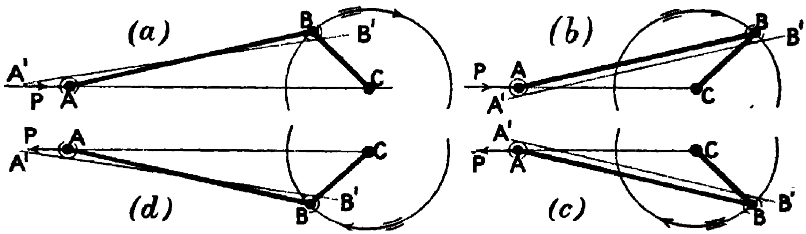


FIG. 414.

on increasing until the crank has turned through 90° from its inner

* The axis of the rod is here the line joining the centres of the pin joints at its ends.

dead centre, and the connecting-rod has therefore anti-clockwise motion about the pin at A. Hence the friction axis must touch the friction circle of the pin at A above the axis of that pin. Still referring to th position (*a*), the angle ABC is diminishing, and will go on diminishin until the crank is at its outer dead centre, and the connecting-rod ha therefore anti-clockwise motion about the pin at B. Hence the friction axis must touch the friction circle of the pin at B below the axis of that pin.

The student should now have no difficulty in reasoning out the positions of the friction axis for each of the remaining cases (*b*), (*c*), and (*d*).

238. Tower's Experiments on the Friction of Journal Bearings. —The results of Mr. Beauchamp Tower's experiments * showed that the coefficient of friction is approximately proportional to the square root of the velocity, and inversely proportional to the intensity of the pressure when the journal runs in an oil bath. Thus $\mu = \frac{c\sqrt{v}}{p}$, where μ is the coefficient of friction, v the velocity of the surface of the journal in feet per second, p the intensity of the pressure in lbs. per square inch of projected area of the bearing, and c a coefficient which has the following values for the lubricants mentioned:—Olive oil, 0·289; lard oil, 0·281; mineral grease, 0·431; sperm oil, 0·194; rape oil, 0·212; mineral oil, 0·276.

For syphon lubrication $\mu = c'/p$, where $c' = 2{\cdot}02$ for rape oil.

For pad lubrication, μ is approximately constant, and equal to 0·01 for rape oil.

The following results were obtained by Mr. Tower with a steel journal 4 inches in diameter and 6 inches long, at a speed of 150 revolutions per minute, or 157 feet per minute. The "brass" was of gun-metal, and embraced nearly one-half of the circumference of the journal, and was placed on the top. The lubricant used was rape oil.

Method of Lubrication.	p	μ
Oil bath	263	0·0014
Syphon lubricator	252	0·0098
Pad under journal	272	0·0090

With the same journal Mr. Tower obtained the results shown in the annexed table at a speed of 20 revolutions per minute, or 21 feet per minute in a bath of mineral oil.

p	454	342	216	91
μ	0·00132	0·00168	0·00247	0 0044

Mr. Tower's experiments on friction at different temperatures indicate a very great diminution in the friction as the temperature rises. Thus, in the case of lard oil, taking a speed of 450 revolutions per

* *Proceedings of the Institution of Mechanical Engineers*, 1883 and 1885.

minute, the coefficient of friction at a temperature of 120° Fahr. was only one-third of what it was at a temperature of 60° Fahr.

The following figures show the comparative friction with various lubricants tried by Mr. Tower under as nearly as possible the same conditions:—Temperature, 90° Fahr. Lubrication by oil bath—sperm oil, 0·484; rape oil, 0·512; mineral oil, 0·623; lard oil, 0·652; olive oil, 0·654; mineral grease, 1·048. These figures are the means of the actual frictional resistances at the surface of the journal (4 inches diameter) in lbs. per square inch of bearing at a speed of 300 revolutions per minute (314 feet per minute), with all nominal loads from 100 to 310 lbs. per square inch. They also represent the relative thickness or body of the various oils, and also in their order, though perhaps not exactly in their numerical proportions, their relative weight-carrying power. Thus sperm oil, which has the highest lubricating power, has the least weight-carrying power, and though the best oil for light loads, would be inferior to the thicker oils if heavy pressures or high temperatures were to be encountered.

239. Work Lost in Friction in Journal Bearings.—Let R = resultant load on journal in lbs., d = diameter of journal in inches, V = surface velocity of journal in feet per minute, N = revolutions of journal per minute, ϕ = friction angle, and μ = coefficient of friction.

The moment of R is $\frac{1}{2}Rd\sin\phi$, which may be written $\frac{1}{2}Rd\mu$, since ϕ is a very small angle. The work done per minute on friction is therefore $\frac{R\pi dN\mu}{12} = RV\mu$ ft.-lbs. The horse-power lost in friction is $\frac{RV\mu}{33000}$.

240. Methods of Lubricating Bearings.—There are two principal methods of lubricating bearings. In one method the oil is allowed to flow in at ordinary atmospheric pressure, while in the other the oil is forced in under sufficient pressure, generally by a pump employed for that purpose. When the oil enters at atmospheric pressure it should be delivered to the bearing at the place where the pressure on the bearing is least, but with forced lubrication the oil should be delivered to the bearing at the place where the pressure is greatest.

The well-known *needle lubricator* is shown in Fig. 415. B is an inverted glass bottle or reservoir containing oil. S is a wooden stopper, one end of which fits into the neck of the bottle, while the other end fits into a hole over the bearing of the journal J to be lubricated. N is the needle, which fits loosely into a hole in the stopper S. The lower end of the needle rests on the journal. When the shaft is at rest capillary action prevents the oil leaving the bottle, but when the shaft is rotating the vibration set up causes the oil to flow slowly on to the journal. The needle N is simply a straight piece of wire flattened at its upper end to prevent it falling out when the lubricator is removed from the bearing.

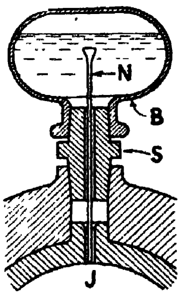


FIG. 415.

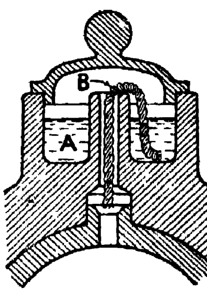


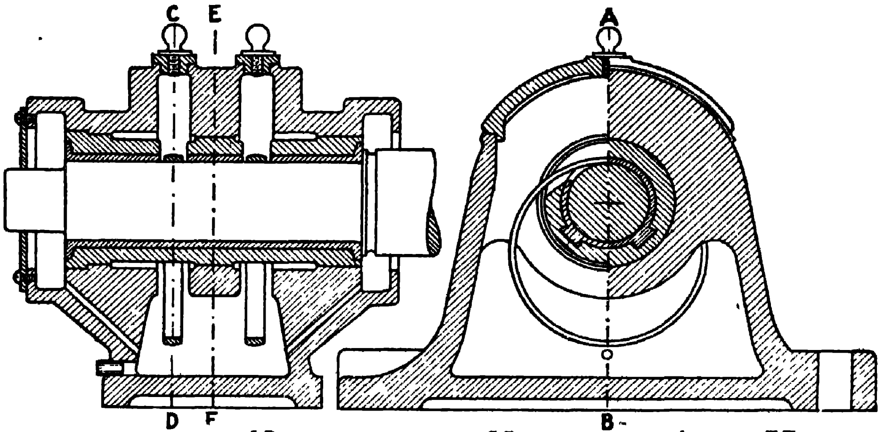
FIG. 416.

A *syphon lubricator* is shown in Fig. 416. The oil is stored in the

cup or box A, and is delivered slowly to the bearing through the wick B, which acts as a sypnon. It is important that the end of the wick which delivers the oil should be below the free surface of the oil in the cup, otherwise the oil will not flow through the wick.

In *pad lubrication* a part of the bearing surface upon which there is no pressure is dispensed with, and its place is taken by a soft pad, which is kept saturated with lubricant. In *bath lubrication* the bearing contains a space filled with oil, which is in contact with a portion of the journal.

Ring lubrication is illustrated by Fig. 417. In this bearing the journal carries two loose rings which rotate, being driven by frictional contact with the journal. These rings dip into an oil bath and carry oil to the top of the journal. The oil flows over the surface of the journal through oil grooves in the bearing, and finally returns to the bath below.



Section at AB. *Section at CD.* *Section at EF.*

FIG. 417.

An example of *forced lubrication* is shown in Fig. 418. This illus-

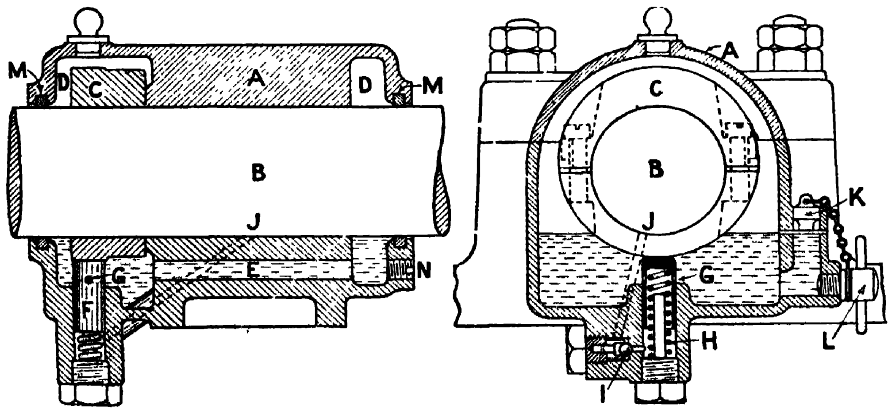


FIG. 418.

trates Tilston's system as applied to a journal bearing. A is the bearing,

and B the shaft. C is an eccentric clamped to the shaft. DD are end chambers connected by the passage E. F is a pump plunger, made from steel tubing forged on to a solid end. G are inlet holes in the plunger, which allow oil to pass to the inside of the pump when the plunger is at and near the top of its stroke. The eccentric drives the pump plunger, the latter being kept up to the former by the spring H. I is a non-return ball valve, and J an outlet from the pump to the shaft. K is a sight feed plug supported by a cross pin beneath it. L is a screwed plug to drain off spent oil and dirt. MM are leather washers to prevent oil travelling along the shaft. N is a screwed plug giving access to the passage E for cleaning purposes. As the plunger descends, the inlet holes G are cut off by the casing, and oil is forced past the non-return valve and through the outlet J to the shaft, and thence to the end chambers DD.

Forced lubrication has been used with great success on high-speed steam-engines. The various bearings are connected by pipes and passages to an oil pump driven by the engine. The oil after being used passes through a filter back to the reservoir which supplies the pump.

Splash lubrication is common and simple, but crude, and is used on high-speed vertical engines, especially on petrol engines. The engine is enclosed, and the crank case contains oil, into which the cranks splash as they rotate, throwing the oil over the various bearings.

241. Friction of Sliding Keys.—In machines it is frequently necessary to move a piece longitudinally on a shaft, while there is a torque between the piece and the shaft. In such cases a sliding key may be fixed to the sliding piece and fit easily into a keyway in the shaft, or the key may be fixed to the shaft and fit easily into a keyway in the sliding piece, as shown in Fig. 419, where the looseness of fit between the piece A and the shaft B, and between the key C and the keyway in A, is exaggerated. If the piece A is driven in the direction of the arrow D by a torque T, the forces which transmit this torque to the shaft are the equal forces P and Q at a distance r from one another, so that $Pr = T$. If two keys be used, as shown in Fig. 420, the equal forces P and Q will now be at a distance $2r$ from one another, and $2Pr = T$. Hence the force causing the sliding friction in the second arrangement is only half what it is in the first arrangement. To get the full advantage of the two keys it is necessary that they be very accurately fitted, so that they transmit the whole of the torque without any pressure between the sliding piece and the shaft itself.

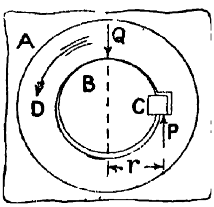


FIG. 419.

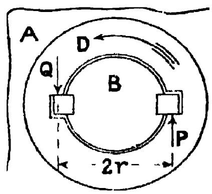


FIG. 420.

242. Rotating Guides for a Sliding Piece.—It is well known that a piece mounted loosely on a shaft may be made to slide along the shaft by the application of a smaller force when the shaft is rotating than when the shaft is at rest, and the greater the speed of the shaft, the smaller is

the force required to produce the sliding of the piece mounted on it. A convenient way of applying this principle to the guiding of a sliding piece so as to reduce the force required to slide it is shown in Fig. 421,

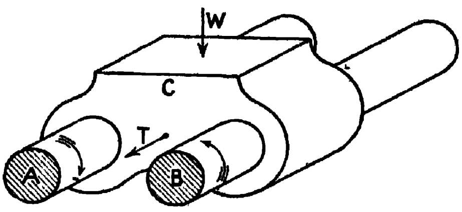


FIG. 421.

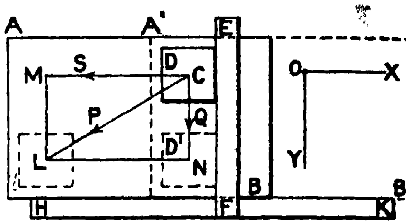


FIG. 422.

where A and B are two parallel shafts or spindles, which are rotated, preferably in opposite directions, and which support the piece C, which is made to travel along the shafts by a force T.

The theory of the action of the rotating guides is as follows. Let AB (Fig. 422) represent a horizontal flat plate, upon which rests a body D. In a given time, let AB travel a distance AA′ in the direction OX into the position A′B′. In the same time, let the body D be made to slide on AB a distance CN in the direction OY at right angles to OX into the position D′. The motion of D relative to AB will be the same as if while it slides the distance CN in the direction OY it be made to slide the distance CM equal to A′A in the direction XO. These simultaneous motions given to D will result in a motion of D relative to AB in the direction CL, and equal to CL where CL is the diagonal of the rectangle MN. Now the force P, acting along CL, necessary to slide D along CL, is equal to μR where R is the force, normal to AB, and pressing D on AB. But the force P, represented to scale by CL, may be replaced by the forces Q and S represented to the same scale by CN and CM respectively, and the ratio of the force Q to the force S is evidently the same as the ratio of the velocity of D in the direction OY to the velocity of AB in the direction OX. Applying this to a rotating guide, a force equal and opposite to S is the tangential force at the surface of the guide in the direction of its motion necessary to drive it, and Q is the force on the sliding piece in the direction of its motion necessary to make it slide.

To prevent D being carried in the direction OX when AB moves under it in that direction a fixed guide EF is necessary, and the force pressing D against this guide is evidently equal to S, which will cause a resistance equal to μS in the direction YO. Hence the resultant force necessary on D in the direction OY is $Q+\mu S$. By using two rotating guides rotating in opposite directions, the tractive force on the sliding piece is reduced from $Q+\mu S$ to Q for each guide.

To prevent AB moving in the direction OY when D moves over it in that direction a fixed guide HK is necessary, and the force pressing AB against this guide is evidently equal to Q, which will cause a resistance equal to μQ in the direction XO. Hence the resultant force necessary on AB in the direction OX is equal to $S+\mu Q$. In a rotating guide the resistance which would correspond to the resistance μQ would be the

resistance to rotation at the thrust bearing of the shaft, but in that case the resistance, reduced to the surface of the shaft, may be either greater or less than μQ, depending on the effective radius of the collar or pivot of the thrust bearing used.

Consider now the work done in the given time when two guides are used, as in Fig. 421, each guide carrying half the load W. Neglecting the work done at the thrust bearings of the guides, the work done is $2P \cdot CL = 2 \times \frac{1}{2}\mu W \cdot CL = \mu W \cdot CL = U$. If V is the surface velocity of the rotating guides, and v the velocity of the sliding piece,

$$\frac{CL}{CN} = \frac{\sqrt{V^2 + v^2}}{v}, \text{ and } U = \mu W \cdot CN \frac{\sqrt{V^2 + v^2}}{v}.$$

The force T is equal to $\mu W \cdot \frac{CN}{CL} = \mu W \frac{v}{\sqrt{V^2 + v^2}}$. If the guides are at rest, $V = 0$, and $T = \mu W$.

From the foregoing, it is seen that the work done with rotating guides is greater than the work done with ordinary sliding guides in the ratio of $\sqrt{V^2 + v^2}$ to v, and therefore the rotating guides would not be introduced to economise power. It would be absurd, for example, to use rotating guides in a planing machine. Rotating guides are useful in certain recording instruments, where a pen or pencil has to be guided in a straight line and moved by a small force.

The same principle is also applied when it is required to reduce the sliding friction of a piston or plunger in the direction of the axis, by giving the piston or plunger a simultaneous rotary motion. Kinematically, the mechanism in this case is the same as that discussed above.

243. Friction of a Band on a Pulley.—Let a band ABCD (Fig. 423) passing over a pulley have a tension T_1 in the part AB and a tension T_2 in the part CD, and let the band be just on the point of slipping on the pulley in the direction from C to B. T_1 will be greater than T_2 on account of the friction between the band and the pulley. Let θ be the angle subtended by the arc of contact BC at the centre of the pulley. Consider an indefinitely small portion bc of BC subtending an angle $d\theta$ at the centre of the pulley. Let T be the tension in the band at c, and $T + dT$ the tension at b. Let S be the resultant of the pressure of the pulley on the portion bc of the band, and let μ be the coefficient of friction between the band and the pulley. Then

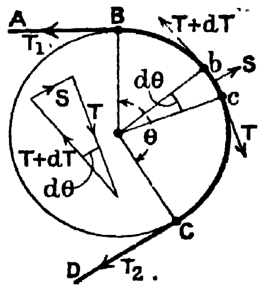


FIG. 423

$$T + dT - T = dT = \mu S, \text{ but } \underline{S = T d\theta}, \text{ therefore } dT = \mu T d\theta, \text{ and } \frac{dT}{T} = \mu d\theta.$$

$$\text{Integrating } \int_{T_2}^{T_1} \frac{dT}{T} = \mu \int_0^\theta d\theta, \text{ therefore } \log_e \frac{T_1}{T_2} = \mu\theta, \text{ or } \frac{T_1}{T_2} = e^{\mu\theta}.$$

In the above equations, θ is in circular measure, and the logarithm

is the Napierian or hyperbolic logarithm. Using common logarithms, $\log \frac{T_1}{T_2} = 0{\cdot}4343\mu\theta$, and if n is the measure of the angle θ in degrees, then $\log \frac{T_1}{T_2} = 0{\cdot}00758\mu n$.

If the band lies in a **V** groove on the pulley, as shown in Fig. 424, this has the effect of increasing the resistance to slipping, because slipping must now take place on two surfaces (the sides of the groove), upon each of which the normal pressure is greater than half the normal pressure on a flat pulley. Considering the element bc (Fig. 423), the resistance to slipping in the **V** groove is $2\mu R = \mu S$ cosec α, where 2α is the angle of the groove, but for a flat band the resistance to slipping of this element is μS. Hence the equations given above for a flat band will apply to a band in a **V** groove if μ be altered to μ_1, where $\mu_1 = \mu$ cosec α.

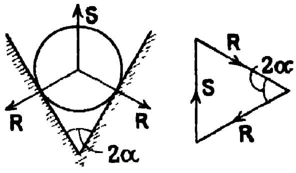


FIG. 424.

Exercises XVI.

1. Prove the formulæ given under Figs. 425, 426, 427, and 428, the motion of the body of weight W being uniform and up the plane.

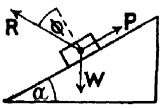


FIG. 425.

$$\frac{P}{W} = \frac{\sin(\alpha + \phi)}{\cos\phi}.$$

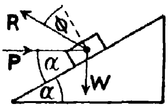


FIG. 426.

$$\frac{P}{W} = \tan(\alpha + \phi)$$

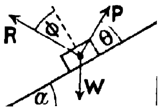


FIG. 427.

$$\frac{P}{W} = \frac{\sin(\alpha + \phi)}{\cos(\theta - \phi)}.$$

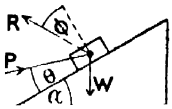


FIG. 428.

$$\frac{P}{W} = \frac{\sin(\alpha + \phi)}{\cos(\theta + \phi)}.$$

2. Referring to Fig. 427 for given values of W, α, and ϕ, what is the value of θ when P is least?

3. Prove the formulæ given under Figs. 429, 430, 431, and 432, the motion of the body of weight W being uniform and down the plane.

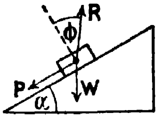


FIG. 429.

$$\frac{P}{W} = \frac{\sin(\phi - \alpha)}{\cos\phi}.$$

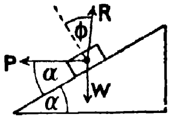


FIG. 430.

$$\frac{P}{W} = \tan(\phi - \alpha).$$

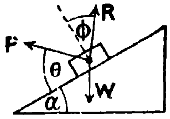


FIG. 431.

$$\frac{P}{W} = \frac{\sin(\phi - \alpha)}{\cos(\phi - \theta)}.$$

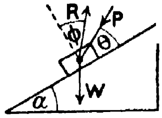


FIG. 432.

$$\frac{P}{W} = \frac{\sin(\phi - \alpha)}{\cos(\phi + \theta)}.$$

4. For the key or cotter shown in Fig. 433, prove that the force P required to drive the key in is Q{tan $(\alpha + \phi)$ + tan ϕ}, and that the force required to drive the key out is Q{tan $(\phi - \alpha)$ + tan ϕ}.

5. What is the greatest taper which a key may have consistent with the friction holding the key in position? Take $\mu = 0{\cdot}07$, and express the taper in the form 1 in x, where x is a length.

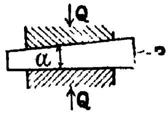


FIG. 433.

6. A piece slides on a bar of square section by the action of a force P, as shown in Fig. 434. If Q is the force pressing the sliding piece on the bar, show that $P=\mu Q\sqrt{2}$.

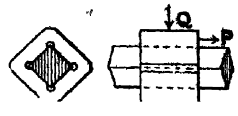


FIG. 434.

7. An inclined plane has a base 90 feet long, and is 20 feet high; the coefficient of friction between it and a body weighing 800 lbs. placed on it is 0·3: how many foot-pounds of work are done in drawing the body up the whole length of the plane, and how many in drawing it down the plane, the pulling force being parallel to the plane?

8. What must be the effective horse-power of a locomotive which moves at the steady speed of 45 miles per hour on level rails, the resistances being 15 lbs. per ton, and the weight of the engine and train 220 tons? If the rails were laid at a gradient of 1 in 130, what additional horse-power would be required?

9. If the engine of the preceding exercise exerts the same power on the incline as on the horizontal, at what speed, in miles per hour, would it ascend an incline of 1 in 180 with the same train, assuming the frictional resistances to be unaltered?

10. Calculate the horse-power required to drive a motor car weighing 1 ton up an incline of 1 in 14 at 24 miles an hour, supposing it to reach the same velocity when running freely down the incline. [U.L.]

11. A window sash (Fig. 435), of height h, is counterbalanced by weights; show that it can be raised by a vertical force, if its point of application is not further than $\frac{1}{2}h\cot\phi$ from the centre, where ϕ is the angle of friction. [B.E.]

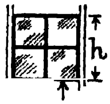


FIG. 435.

12 A square threaded screw, whose mean diameter is $1\frac{1}{2}$ inches, and pitch $\frac{1}{2}$ inch, has its axis vertical, and carries at its upper end a weight W, which is raised by the application of a torque T to the screw. It was found by experiment that the relation between T and W was given by the equation $T=\frac{W}{4}+3$, where T is in inch-lbs. and W in lbs. Determine the values of μ for the screw and nut when, (1) W = 50 lbs., (2) W = 100 lbs., (3) W = 200 lbs.

13. Particulars are given in the following table of certain standard Whitworth

Outside diameter (d), inches	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$
Diameter at bottom of thread, inches	0·393	0·840	1·287	1·715	2·180	2·631	3·106
Number of threads per inch	12	8	6	$4\frac{1}{2}$	4	$3\frac{1}{2}$	$3\frac{1}{4}$

screws in which the angle of the V thread is 55°. Calculate the efficiencies of these screws, taking $\mu=0·15$, and plot the results, taking efficiencies for ordinates, and d for abscissæ

14. A simple screw-jack (Fig. 436) has a square threaded screw whose mean diameter is 1·8 inches and pitch 0·4 inch. If the coefficient of friction between the screw and nut is 0·12, what force at the end of a lever 24 inches long, measured from the axis of the screw, will raise a load of 2 tons? Assume that the load rotates with the screw, thus eliminating collar friction. Calculate also the efficiency. What force at the end of the lever will be necessary to lower the load of 2 tons?

15. A weight W is carried by a square threaded screw and nut, as shown in Fig. 437 Outside diameter of screw, 1·5 inches; pitch, 0·4 inch; thickness and depth of thread, 0·2 inch. Outside diameter of collar on nut, 3 inches; inside ditto, 1·5 inches. The nut is rotated by a force of 80 lbs at the end of a spanner 18 inches long. Find the load W, in lbs., (1) when friction is neglected, (2) when μ for the collar and for the screw is 0·2.

FIG. 436.

16. A flat pivot has to carry a load of 5000 lbs., and the intensity of the pressure (assumed to be uniform) is to be

120 lbs. per square inch. What horse-power will be absorbed by the friction of this pivot when running at 200 revolutions per minute with a coefficient of friction equal to 0·005?

17. The thrust shaft of a marine engine indicating 4500 horse-power has 8 collars $26\frac{1}{4}$ inches diameter, the diameter of the shaft between the collars being $16\frac{3}{4}$ inches. Taking the coefficient of friction at 0·04, the intensity of the thrust pressure at 50 lbs. per square inch, and the speed of the shaft 80 revolutions per minute, what horse-power is absorbed by friction in the thrust bearing, and what percentage is it of the horse-power of the engine?

18. A straight lever mounted on an axle 2 inches in diameter has arms 5 inches and 10 inches long, measured from the centre of the axle. There is a load W of 100 lbs. at the end of the short arm, and a vertical force P at the end of the long arm. Taking the coefficient of friction $\mu=0·1$, find P, (1) to just raise W, (2) to just lower W.

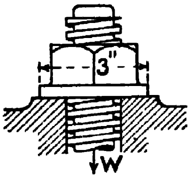


FIG. 437.

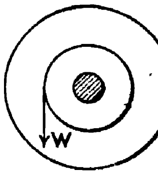


FIG. 438.

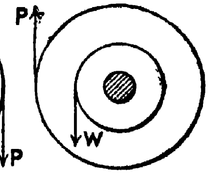


FIG. 439.

19. A weight W of 500 lbs. hangs by a rope which is coiled round a barrel whose effective diameter is 12 inches. The barrel is fixed on an axle whose diameter is 3 inches. W may be raised or lowered by a vertical force P acting at the circumference of a wheel 30 inches in diameter, also fixed to the axle. Find P to raise W with a uniform velocity, (*a*) when P acts as shown in Fig. 438, (*b*) when P acts as shown in Fig. 439. Also find P to lower W with a uniform velocity in each case. $\mu=0·1$.

20. Referring to Fig. 413, p. 270. $a=30$ inches, $b=10$ inches, $r=1·5$ inches, $\theta=60°$, angle CBO = angle CAO = 90°, $\sin\phi=0·1$, and Q = 600 lbs. Find P (*a*) to just raise Q, (*b*) to just lower Q.

21. Fig. 440 shows a bent lever AOB. The fulcrum at O is in a loose cylindric bearing 4 inches diameter. AO is 12 inches, BO is 24 inches; the force Q of 1000 lbs. acts at A. What force P acting at B will just overcome Q, (1) when friction is neglected, (2) when the coefficient of friction is 0·3. Find also the line of action and magnitude of the force R acting on the lever at O. [B.E.]

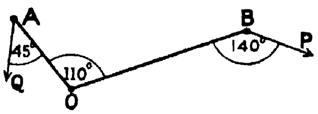


FIG. 440.

22. A crank disc (Fig. 441) receives an oscillating motion through an angle AOB by a "gab" ended rod CD driven by an eccentric. At (*a*) the crank pin is shown below, and at (*b*) above the centre of the disc. Show that in one of these arrangements the effect of the friction between the pin and the "gab," during both the forward and return strokes, will be to throw the "gab" off the pin, while in the other the effect will be to keep the "gab" on the pin.

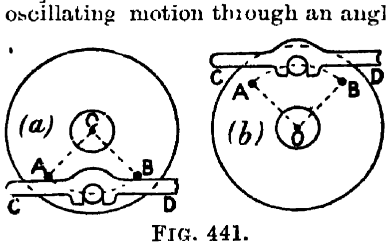


FIG. 441.

23. An ordinary horizontal direct acting steam-engine mechanism is shown in a particular position in Fig. 442, AB being the connecting-rod, and BC the crank. The diameters of the journals at A, B, and C are 5, 8, and $7\frac{1}{2}$ inches respectively. The force P transmitted through the piston-rod to the cross-head is 6000 lbs. Q, the useful resistance to the motion of the crank shaft, acts in a vertical direction at a perpendicular distance of 12 inches from the axis of the shaft, as shown.

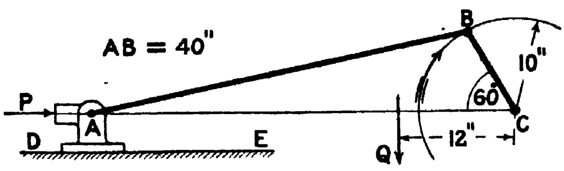


FIG. 442.

Find the magnitude of the force Q, (*a*) neglecting friction, (*b*) allowing for friction at the journals A, B, and C and also at the guide DE, neglecting the weights of the moving parts. Take $\mu = 0{\cdot}05$.

24. A horizontal pump, stroke 4 inches, is driven by means of an eccentric, $11\frac{1}{2}$ inches diameter, keyed to a shaft 4 inches diameter. The shaft is driven by a vertical belt on a 14 inch diameter pulley. The belt embraces an arc of 180°, and the coefficient of friction between belt and pulley is 0·25. If the tension on the tight side of the belt is 350 lbs., find the maximum horizontal force that can be delivered to the pump when the radius of the eccentric is at 60° to the inner dead centre. Assume the eccentric rod to be very long. Coefficient of friction between eccentric sheave and strap and between shaft and bearing is 0·1. [U.L.]

25. A pulley weighing 1000 lbs. is supported on a 5 inch shaft midway between two bearings. The mass centre of the pulley is $\frac{3}{4}$ inch from the axis of the shaft. Neglecting the effect of the deflection of the axis of the shaft from the axis of revolution, calculate the horse-power required to overcome the friction of the bearings in consequence of the error of balance in the pulley when the shaft makes 200 revolutions per minute. Assume the coefficient of friction between the bearing surfaces to be 0·05. [U.L.]

26. The journals of a shaft are 6 inches in diameter. The shaft carries a load of 8 tons, and makes 75 revolutions per minute. If the coefficient of friction between the journals and bearings is 0·05, at what rate, in B.Th.U. per minute, is heat being generated at the bearings?

27. The radius of gyration of a fly-wheel and crank shaft is 10 feet. The shaft journals are 12 inches in diameter. The turning effort on the shaft is withdrawn when the speed is 65 revolutions per minute. There being no resistance except the friction at the journals, find how many revolutions the wheel and shaft will make before coming to rest after the effort is withdrawn. Take $\mu = 0{\cdot}065$.

28. A shaft 8 inches in diameter carries a vertical load of 6 tons and a horizontal load of 8 tons. Taking $\mu = 0{\cdot}05$, find the horse-power lost in friction at the journals when the shaft is driven by a pure torque at 100 revolutions per minute.

29. A wheel under a torque of 2000 inch-lbs. is mounted on a shaft along which it has to slide. The rotary motion of the wheel is transmitted to the shaft through two accurately fitting sliding keys which are opposite to one another, as shown in Fig. 420, p. 275. The resultant force on each key is at a distance of $1\frac{1}{4}$ inches from the axis of the shaft. If the coefficient of friction is 0·08, what force acting on the boss of the wheel, parallel to the axis of the shaft, will be necessary to slide the wheel along the shaft?

30. Referring to Fig. 421, p. 276, the rotating guides are horizontal, and are each 0·3 inch in diameter. The weight of the sliding piece is 0·5 lb. Taking the coefficient of friction as 0·05, find the tractive force T when the guides rotate at 600 revolutions per minute, and the sliding piece travels 5 inches in 20 seconds, and express it as a fraction of the tractive force when the guides are at rest. Find also the ratio of the work done when the guides are rotating at the above speed to the work done when the guides are at rest. Neglect the friction of the thrust bearings of the guides.

31. A solid cast-iron disc, 40 inches in diameter, and 8 inches thick, is rotating at a uniform speed of 240 revolutions per minute. If the air frictional resistance is assumed to be equal to KV^2 lbs. per square foot, where V is the linear velocity of any point, obtain an expression for the horse-power required to keep the disc in rotation. [B.E.]

32. A flat band laps half round a fixed pulley. From one end of the band there hangs a weight W of 100 lbs., while the other is pulled by a force P. If $\mu = 0{\cdot}3$, what is the smallest value of P which will raise W, and what is the value of P which will lower W with a uniform velocity?

33. Find the answers to the preceding question if the band is round and lies in a **V**-groove on the pulley, the angle of the **V** being 45°.

34. If a cord hanging in a vertical plane over a fixed horizontal cylinder with 20 lbs. at one end and 10 lbs. at the other be on the point of slipping, what is the coefficient of friction between the cord and the cylinder?

35. How many times must a hemp rope $1\frac{1}{2}$ inches in diameter be passed round a post if a force of 5 lbs. at the slack end is just to hold it when it is

about to break on the tight side? The breaking strength of a $1\frac{1}{8}$ inch hemp rope may be taken as 18,000 lbs., and $\mu=0{\cdot}4$. Prove the formula you employ. [U.L.]

36. A brake strap (Fig. 443) $\frac{1}{8}$ inch thick, embracing two-thirds of the circumference of a pulley 16 inches in diameter, has one end attached to the end B of a lever whose fulcrum is at C. The other end of the strap is attached to the lever at C. AC = 15 inches. BC = 3 inches. A weight W hangs by a rope $\frac{5}{8}$ inch in diameter, which is coiled round a barrel 10 inches in diameter. The pulley and barrel are fixed to the same axle. BD is perpendicular to AB. The weight W is held up by a force P of 50 lbs. acting at A at right angles to AB. Taking $\mu=0{\cdot}2$, find the greatest value of W (a) when the weight hangs as shown, (b) when the weight hangs from the other side of the barrel.

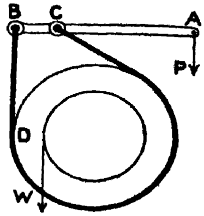


Fig. 443.

CHAPTER XVII

EFFORT, ACCELERATION, AND VELOCITY DIAGRAMS

244. Effort.—If a force P, acting on a body A (Fig. 444) which moves or may move in a definite direction BC, be resolved into two components, one Q parallel to BC, and the other R at right angles to BC, the component Q is called the *effort* of P on the body A.

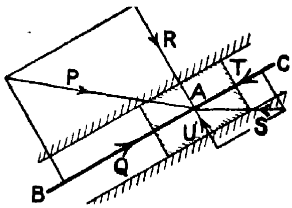


FIG. 444.

245. Unbalanced Effort.—If the motion of the body A (Fig. 444) is opposed by a force S, whose components parallel and perpendicular to BC are T and U respectively, then Q − T is the *unbalanced effort* on A, and this unbalanced effort will accelerate the speed of A, the work done by it appearing as an increase in the kinetic energy of A. If Q − T is negative, then the acceleration of the speed of A will also be negative, and the kinetic energy of A will decrease by an amount equal to that required to overcome the resistance T − Q.

246. Effort-Space Diagram.—In the well-known diagram representing the work done by a force or an effort acting through a given distance, the base represents the distance or space, and the ordinates represent the effort. Such a diagram is shown in Fig. 445, where lengths on the base OX represent distances or spaces through which a body A is moved by an effort P, whose magnitude for any position N of the body is represented by the ordinate Np of the curve BpDE. The same figure also shows that the motion of A is opposed by a resistance R, whose magnitude for any position N of the body is represented by the ordinate Nr of the curve FrDH. From O to N the work done by P is represented by the area of the figure OBpN, and the work done on R is represented by the area of the figure OFrN. The difference between these two areas, namely, the area of the figure FBpr, represents the excess work which goes to increase the kinetic energy, and therefore also to increase the speed of A. Let W = weight of A, v_0 = speed of A when at O, v = speed of A when at N, and K = work represented by the area FBpr, then $\frac{W}{2g}(v^2 - v_0^2) = K$, and $v = \sqrt{\frac{2gK}{W} + v_0^2}$.

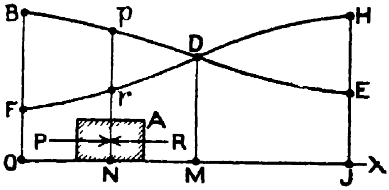


FIG. 445.

The speed will increase so long as P is greater than R, and the speed

will be a maximum when A is at M, where P is equal to R. Beyond M the speed will diminish, and if the body comes to rest at J, then $\frac{Wv_0^2}{2g}$ + work represented by area OBDEJ = work represented by area OFDHJ, or $\frac{Wv_0^2}{2g}$ + work represented by area BDF = work represented by area EDH, and if $v_0 = 0$, area BDF = area EDH.

It is sometimes convenient to plot the unbalanced effort pr on a straight space base, as shown in Fig. 446. The area of this diagram between the ordinates through O and L then represents the increase in the kinetic energy of the body when it has moved through the distance OL, areas above OL being reckoned as positive, and areas below OL as negative. The same result is represented by the portion FBDEHDF of the original diagram (Fig. 445), areas above FDH being reckoned as positive, and areas below FDH as negative.

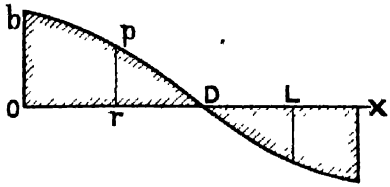


FIG. 446.

247. Effort - Time Diagram.—Referring to Fig. 445, if OX is a time base, that is, if ON represents the time during which the body A has been moving while the effort changes from OB to Np, then the area of the figure OBpN represents the momentum added to A by the effort P during the time ON. If the resistance R be also plotted on the time base OX, the result being the curve FrDH, then the area of the figure OFrN represents the decrease in the momentum of A during the time ON, and the area of the figure FBpr represents the net increase in the momentum of A, due to the simultaneous action of P and R during the time ON.

If the unbalanced effort be plotted on a straight time base OX (Fig. 446), then the area of the diagram between ordinates through O and L represents the net increase in the momentum of A during the time OL.

248. Space Average and Time Average of a Force.—When the magnitude of a force, acting on a body in the direction of its motion, is plotted on a straight base which represents the *space* or *distance* through which the force acts, the average value of the magnitude of the force, or the mean height of the diagram, is the *space average of the force.* Again, when the magnitude of the force is plotted on a straight base which represents the *time* of the motion, the average value of the magnitude of the force, or the mean height of the diagram, is the *time average of the force.*

When the space average of a force is used, it is a question of work; and when the time average of the force is used, it is a question of momentum.

The space and time averages of a force are obviously equal when the magnitude of the force is constant; they are also equal when the body upon which the force acts moves with a uniform velocity, however the force may vary in magnitude; but if the velocity of the body is not uniform, and the magnitude of the force varies, the space and time averages of the force may be very different.

An example of some interest is the relation between the space and time averages of the pressure on the piston of a steam-engine. The ordinary indicator diagram is a force-space diagram, and the mean pressure found from it is a space average, and this space average is used in calculating the work done in the cylinder. In an indicator invented by Professor Ripper,* the mean pressure is shown directly by a pointer on a dial; but this mean pressure is a time average, unless the proper correction has been made by adjusting the instrument to convert the time average into the space average. The necessary correction to convert the time average into a space average will depend on the way in which the pressure varies in the cylinder.

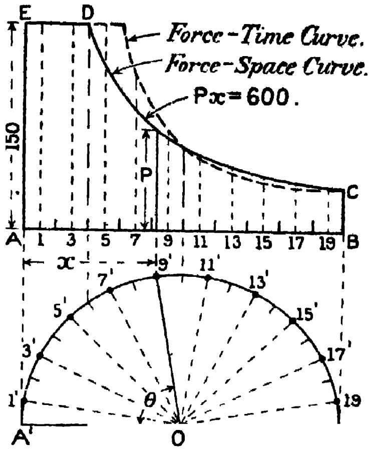


FIG. 447.

The following example will show the relation between the space and time averages of the pressure in a particular case. ABCD (Fig. 447) is the force-space diagram on a base AB, which represents the stroke of the piston (20 inches). The steam pressure is 150 lbs. per square inch for the first 4 inches of the stroke, after which the pressure follows the law $Px = 600$. Dividing the stroke into 10 equal parts of 2 inches each, and calculating the pressures at the middle points of these parts, the following table is constructed:—

x(space)	1	3	5	7	9	11	13	15	17	19
P . .	150	150	120	85·71	66·67	54·55	46·15	40	35·29	31·58

from which the mean value of P is 78·0, and this is the *space average* of P.

To find the time average of P, construct the semicircle A′9′B′, which represents the path of the crank pin for one stroke of the piston, and divide this into 10 equal arcs; then assuming that the crank pin is moving with a uniform velocity, each of these arcs will be described by the crank pin in equal intervals of time. Bisect these equal arcs at the points 1′, 3′, 5′, etc. Then, neglecting the effect of the obliquity of the connecting-rod, the position of the piston, measured by its distance from the beginning of its stroke when the crank makes an angle θ with A′O, is $x = 10(1 - \cos\theta)$, and $P = \dfrac{600}{10(1 - \cos\theta)}$, but is not greater than 150.

Let the base AB now represent the *time* taken by the piston to make one stroke, and let it be divided into 10 equal parts, representing equal intervals of time, and let these be bisected at the points 1, 3, 5, etc. The points 1, 3, 5, etc., will be the positions of the pressure ordinates on

* See the *Proceedings of the Institution of Mechanical Engineers*, 1899.

a *time base* corresponding to the positions 1′, 3′, 5′, etc., respectively of the crank pin. The following table may now be constructed :—

Position of crank pin	1′	3′	5′	7′	9′	11′	13′	15′	17′	19′
x (time) . .	1	3	5	7	9	11	13	15	17	19
θ	9°	27°	45°	63°	81°	99°	117°	135°	153°	171°
P	150	150	150	109·9	71·11	51·88	41·27	35·15	31·73	30·19

from which the mean value of P is 82·12, and this is the *time average* of P.

The time average of the pressure on the piston is therefore, in this case, 4·12 lbs. per square inch, or 5·3 per cent. greater than the space average. These results are not quite accurate, but they are sufficiently approximate for practical purposes. More exact results would of course be obtained by making the space and time intervals shorter, and correspondingly more numerous.

If the effect of the obliquity of the connecting-rod be considered, and the length of the connecting-rod is five times the length of the crank, it will be found that in the foregoing example the time average of the pressure is only 1·56 per cent. greater than the space average.

249. Acceleration-Time Diagram.—In Fig. 448, OD is a time base, and the curve ABC is such that any ordinate FN represents the acceleration f of the motion of a body after the lapse of time t, represented by the abscissa ON. If f_m is mean acceleration during the interval of time t, or the mean height of the curve AF above ON, then the area of the diagram OAFN represents $f_m t$, or v the added velocity. If v_1 and v_2 are the velocities at the beginning and end of the interval of time ON $= t$, then $v = v_2 - v_1 = f_m t$.

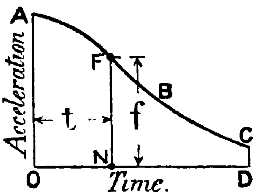


FIG. 448.

Since acceleration is proportional to the force producing it, it is evident that a curve of unbalanced effort will also be a curve of acceleration, but to a different scale.

250. Velocity-Time Diagram.—OABCD (Fig. 449) is a velocity-time diagram for the motion of a body. An ordinate BN of the velocity curve ABC represents the velocity v after the lapse of time t, represented by ON.

The area of the diagram between the ordinates AO and BN represents the distance travelled by the body in the time t, for if v_m is the mean velocity between O and N, the distance travelled in the time t is $v_m t$; but v_m is the mean height of the curve AB above ON, and the area of OABN is therefore $v_m \times \text{ON} = v_m t$.

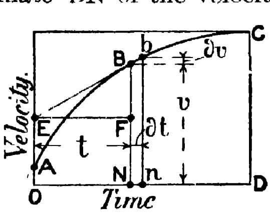


FIG. 449.

The slope of the curve ABC at any point B is equal to the acceleration at the time ON, for if a point b be taken on the curve ABC near to B, and if the ordinate $bn = v + \delta v$, and the abscissa $\text{O}n = t + \delta t$, the slope of Bb is $\frac{\delta v}{\delta t}$, and in the limit when b coincides with B, the slope of Bb

becomes $\frac{dv}{dt}$, the slope of the tangent to the curve at B, namely, $\frac{BF}{EF}$. But $\frac{dv}{dt}$ is the rate of increase of the velocity, and is therefore equal to the acceleration f. In measuring the slope of BE, the height BF must be measured with the velocity scale, and the base EF with the time scale.

251. Space-Time Diagram.—OBCD (Fig. 450) is a space-time diagram for the motion of a body. An ordinate BN of the space or distance curve OBC represents the distance s travelled after the lapse of time t, represented by the abscissa ON.

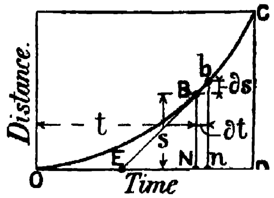


FIG. 450.

The slope of the curve at any point B is equal to the velocity at the time ON, for if a point b be taken on the curve near to B, and if the ordinate $bn = s + \delta s$, and the abscissa $On = t + \delta t$, the slope of Bb is $\frac{\delta s}{\delta t}$, and in the limit when b coincides with B, the slope of Bb becomes $\frac{ds}{dt}$, the slope of the tangent to the curve at B, namely, $\frac{BN}{EN}$. But $\frac{ds}{dt}$ is the rate of change of position, and is therefore equal to the velocity v. In measuring the slope of BE, the height BN must be measured with the distance scale, and the base EN with the time scale.

252. Acceleration-Space Diagram.—Fig. 451 shows an acceleration-space diagram, any ordinate BN of the curve ABC representing the acceleration when the distance moved by the body is represented by the abscissa ON. Consider an indefinitely narrow vertical strip of the diagram. Let ds be the width of this strip and f its height, then its area is fds. But $f = \frac{dv}{dt} = \frac{ds}{dt} \cdot \frac{dv}{ds} = v\frac{dv}{ds}$, therefore $fds = vdv$.

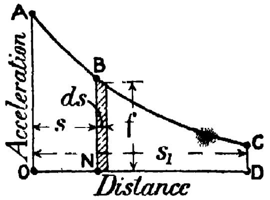


FIG. 451.

Let v_0 be the velocity of the body when at O, and v_1 its velocity when at D, then the area OABCD $= \int fds = \int_{v_0}^{v_1} vdv = \frac{v_1^2 - v_0^2}{2}$. If f_m is the mean acceleration between O and D, and OD $= s_1$, then $f_m s_1 = \frac{v_1^2 - v_0^2}{2}$, or twice the area of the diagram represents the difference between the squares of the velocities of the body at the ends of the space base. If $v_0 = 0$, or the body is at rest when at O, then $f_m s_1 = \frac{v_1^2}{2}$, or twice the area of the diagram represents the square of the velocity of the body at the other end of the space base.

253. Velocity-Space Diagram.—ABC (Fig. 452) is a curve such that any ordinate BN represents the velocity v of a body when it has moved a distance s, represented by the abscissa ON.

It was shown in the preceding Article that $f = v\frac{dv}{ds}$, therefore $\frac{f}{v} = \frac{dv}{ds}$. But $\frac{dv}{ds}$ is the slope $\frac{BF}{EF}$ of the tangent BE to the curve ABC at B. If BL is the normal to the curve ABC at B, then $\frac{LN}{BN} = \frac{BF}{EF} = \frac{f}{v}$, but $BN = v$, therefore $LN = f$; or the sub-normal of a velocity-space curve represents the acceleration.

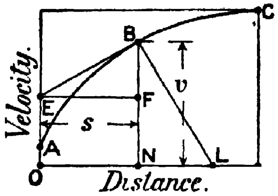


FIG. 452.

The scale with which to measure LN must now be determined. Let the velocity scale be 1 inch to m feet per second, the distance scale 1 inch to n feet, and the acceleration scale 1 inch to x feet per second per second. Let BF, EF, BN, and LN denote the lengths of these lines *in inches*. Then, $f = LN \times x$ feet per second per second, and $v = BN \times m$ feet per second; BF represents a velocity $BF \times m$ feet per second, and EF represents a distance $EF \times n$ feet. Hence $\frac{f}{v} = \frac{LN \times x}{BN \times m} = \frac{BF \times m}{EF \times n}$, therefore $\frac{x}{m} = \frac{m}{n}$, and $x = \frac{m^2}{n}$.

254. Conversion of Space-, Velocity-, and Acceleration-Time Diagrams.—It was shown in Art. 250 that the slope of the velocity-time curve represented the acceleration, and in Art. 251 that the slope of the space-time curve represented the velocity. These properties may be made use of in constructing any two of the three curves, space-time, velocity-time, or acceleration-time, from the third.

The curve OABC (Fig 453) is a space-time curve plotted from the data in the following table:—

t . . .	0	2	4	6	8	10	12	14	16	18
s . . .	0	7	22	41	64	90	122	160	197	228

where t is the time in seconds, and s the distance moved from rest in feet. Let A and B be two points on the curve OABC, the points being sufficiently near to one another to warrant the assumption that the part AB of the curve is straight. Drawing AD perpendicular to the ordinate through B, BD is the space covered during the interval of time AD, and the mean velocity during that interval is $BD \div AD$. In Fig. 453 AD is 2 seconds and BD is 26 feet, therefore the velocity at the time 9 seconds, the middle of the interval AD, is 13 feet per second, and if the ordinate NP be made equal to 13 on the velocity scale, a point P on the velocity curve is determined. If equal intervals of time be taken, it is only necessary to take the distance BD in the dividers and step it out a fixed number of times on the mid ordinate to obtain a point on the velocity curve.

The velocity curve in Fig. 453 has been found by taking intervals of one second, and making the mid ordinate ten times the increase in space for each second.

The acceleration curve is determined in like manner from the velocity curve, NQ being made a fixed number of times FH, or, as has been done in Fig. 453, the mid ordinate has been made four times the increase in velocity for each second.

Conversely, the velocity curve may be determined from the acceleration curve, and the space curve from the velocity curve, beginning in each case at the zero point.

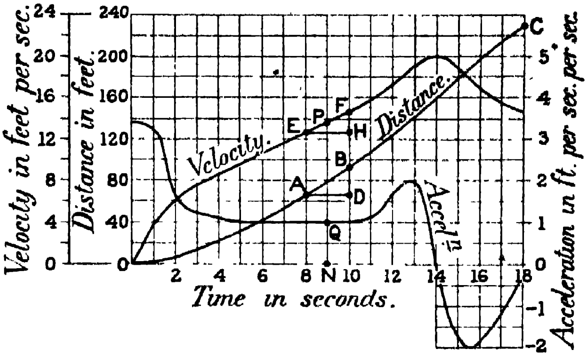


FIG. 453.

The student should work out the foregoing example carefully to, say, the following scales:—Time, 1 inch to 2 seconds; space, 1 inch to 50 feet; velocity, 1 inch to 5 feet per second; and acceleration, 1 inch to 1 foot per second per second.

The properties of the curves on a time base which have been made use of in the foregoing example are applied in a slightly different manner in Fig. 454. Suppose the velocity curve to be given. Divide the diagram into vertical strips A, B, C, etc., and draw the mid ordinates shown by dotted lines. Project the mid ordinate points of the velocity curve on to the vertical through O, thus obtaining the points a, b, c, etc. Choose a pole P on the time base, and join P to a, b, c, etc. Starting at O, draw across the strips A, B, C, etc., continuous lines parallel to Pa, Pb, Pc, etc., respectively; a fair curve through the junctions of these lines will be the space-time curve.

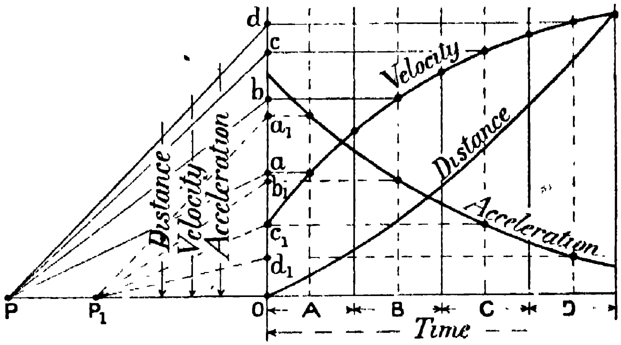


FIG. 454.

Choose a pole P_1 on the time base, and draw P_1a_1, P_1b_1, P_1c_1, etc., parallel to the portions of the velocity curve across the strips A, B, C, etc., respectively, to meet the vertical line through O at a_1, b_1, c_1, etc. Horizontal lines from a_1, b_1, c_1, etc., to cut the mid ordinates of the strips A, B, C, etc., respectively, determines points on the acceleration curve.

The relations between the different scales are found as follows. Let the interval of time A be δt seconds, and at the end of that interval let the increase in space be δs feet, and the increase in velocity δv feet per second. At the middle of the interval A let the velocity be v feet per

second, and the acceleration f feet per second per second. Let the scales be—space, 1 inch to l feet; velocity, 1 inch to m feet per second; acceleration, 1 inch to n feet per second per second; and time, 1 inch to q seconds. Also let $OP = p$ inches, and $OP_1 = p_1$ inches.

Then from the simple geometry of the figure

$$\frac{\delta s}{l} \div \frac{\delta t}{q} = \frac{v}{m} \div p, \text{ or } \frac{\delta s}{\delta t} \times \frac{q}{l} = \frac{v}{mp} \quad . \quad . \quad . \quad (1)$$

also
$$\frac{\delta v}{m} \div \frac{\delta t}{q} = \frac{f}{n} \div p_1, \text{ or } \frac{\delta v}{\delta t} \times \frac{q}{m} = \frac{f}{np_1} \quad . \quad . \quad . \quad (2)$$

but $\frac{\delta s}{\delta t} = v$ (approximately) . . . (3), and $\frac{\delta v}{\delta t} = f$ (approximately) . . . (4).

Therefore from (1) and (3) $\frac{q}{l} = \frac{1}{mp}$, and from (2) and (4) $\frac{q}{m} = \frac{1}{np_1}$. Hence if q, p, p_1, and, say, m are given, l and n can be found, or if l, m, n, and q are given, p and p_1 can be found.

255. Angular Motion Diagrams.—In the preceding Articles of this chapter only linear distance or displacement, linear velocity, and linear acceleration have been referred to, but all that has been said about the relations between these also applies to the relations between angular displacement, angular velocity, and angular acceleration. Angular displacement, measured in radians, angular velocity in radians per second, and angular acceleration in radians per second per second, may be plotted on a straight time base, and angular velocity, angular acceleration, and time may be plotted on a straight base representing angular displacements, exactly as for linear motion.

256. Examples.—(1) The tractive force on a car weighing 10 tons is P lbs., and there is a uniform resistance R lbs., so that the unbalanced effort F is P − R lbs. Values of F at intervals of 2 seconds are given in the second column of the table below. The car is at rest when the time t is 0. It is required to determine the acceleration f of the speed of the car in feet per second per second, the velocity v of the car in feet per second, and the distance s travelled from the starting point in feet at each of the given times t.

Time (t) Sec.	Unbalanced Effort (F) Lbs.	Acceleration (f) Feet per Sec. per Sec.	Mean Acceleration during Interval.	Increase in Velocity during Interval.	Velocity (v). Feet per Sec.	Mean Velocity during Interval.	Distance moved during Interval	Distance moved from Start (s). Feet.
0	610	0·877	0·848	1·696	0	0·848	1·70	0
2	570	0·819	0 775	1·550	1 696	2·471	4·94	1·70
4	508	0·730	0·664	1·328	3·246	3·910	7 82	6·64
6	416	0·598	0·522	1·044	4·574	5·096	10 19	14·46
8	310	0·446	0·433	0·866	5·618	6·051	12·10	24·65
10	292	0·420	0 393	0·786	6·484	6·877	13·75	36·75
12	255	0 367	0 370	0·740	7·270	7·640	15 28	50·50
14	260	0·374	0·402	0·804	8·010	8·412	16·82	65·78
16	300	0·431	0·438	0·876	8·814	9·252	18·50	82·60
18	309	0·444	0 459	0·918	9·690	10·149	20·30	101·10
20	330	0·474			10·608			121·40

The acceleration f is calculated from the formula $f = \frac{32 \cdot 2F}{22400}$.

The mean acceleration during any interval is half the sum of the accelerations at the beginning and end of the interval.

The increase in the velocity during any interval is the mean acceleration during the interval multiplied by 2, the length of the interval in seconds.

The velocity v at the end of any interval is the sum of all the interval increases of velocity up to and including that interval.

The mean velocity during any interval is half the sum of the velocities at the beginning and end of the interval.

The distance moved during any interval is the mean velocity during the interval multiplied by 2, the length of the interval in seconds.

The distance moved from the start at the end of any interval is the sum of all the distances moved during the intervals up to and including that interval.

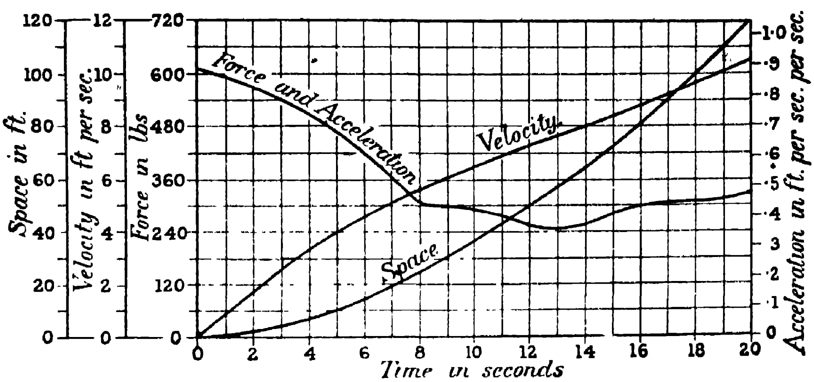


FIG. 455.

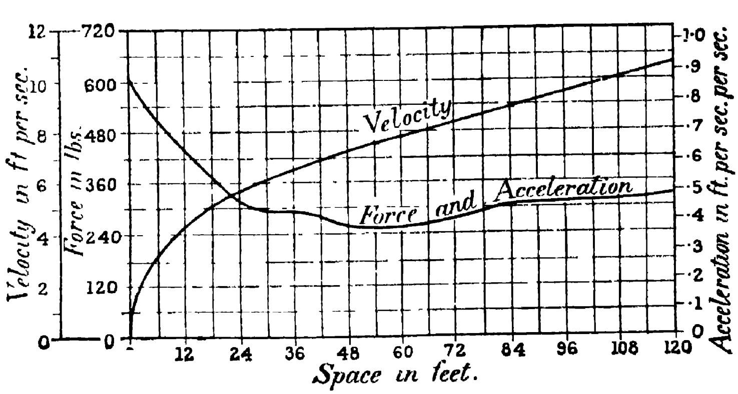


FIG. 456.

The results are shown plotted on a time base in Fig. 455, and F, f, and v are shown plotted on a space base in Fig. 456.

(2) A body weighing 1 ton is lifted vertically by a rope, there being a damped spring balance to indicate the pulling force F of the rope. There is a constant frictional resistance of 1000 lbs. to the motion of the body. When the body has been lifted x feet from its position of rest, the pulling force in lbs. is automatically recorded, and is given in the second column of the table below. It is required to find the velocity v of the body in feet per second for the given values of x, also the time t in seconds to rise the distance x.

Distance (x). Feet.	Effort (F). Lbs.	Unbalanced Effort (P). Lbs.	Mean Value of P during Interval.	Increase in Kinetic Energy during Interval. Ft.-lbs.	Total Kinetic Energy, K.	Velocity (v). Feet per Sec.	Time over Interval. Sec.	Time from Start (t). Sec.
0	5580	2340			0	0		0
10	5450	2210	2275	22,750	22,750	25·57	0·782	0·78
20	5260	2020	2115	21,150	43,900	35·53	0·327	1·11
30	5020	1780	1900	19,000	62,900	42·52	0 256	1 37
40	4810	1570	1675	16,750	79,650	47·85	0·221	1·59
50	4600	1360	1465	14,650	94,300	52·07	0·200	1 79
60	4430	1190	1275	12,750	107,050	55 48	0·186	1·97
70	4270	1030	1110	11,100	118,150	58·28	0·176	2·15

Unbalanced effort, $P = F - (2240 + 1000) = F - 3240$.

The mean value of P during any interval is half the sum of the values of P at the beginning and end of that interval.

The increase in the kinetic energy of the body during any interval is the work done by P during that interval, namely, the mean value of P during the interval multiplied by 10.

K, the total kinetic energy in the body at the end of any interval, is the sum of all the interval increases of kinetic energy up to and including that interval.

$$v = \sqrt{\frac{2 \times 32{\cdot}2K}{2240}}.$$

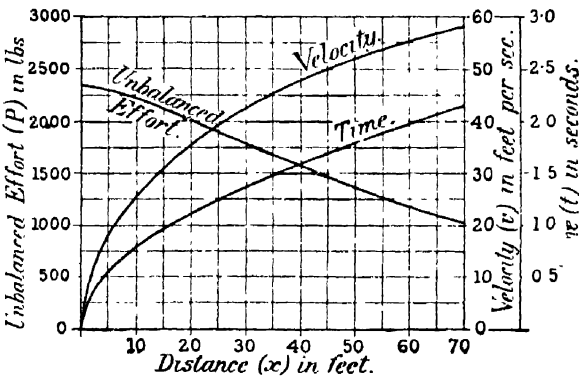


FIG. 457.

The time taken over any interval is 10, the distance moved, divided by the mean velocity during the interval. The mean velocity during an interval is taken as half the sum of the velocities at the beginning and end of that interval.

The time taken from the start to the end of any interval is the sum of all the interval times up to and including that interval.

The results are shown plotted on a distance base in Fig. 457.

257. The Bull Engine.—Towards the end of the eighteenth century William Bull invented a simple form of pumping-engine, which was developed and improved by his son Edward Bull and Richard Trevithick. This engine is now antiquated, but in its working it presents an exceedingly interesting problem in mechanics, which will now be considered.

The engine has an inverted cylinder *ab* (Fig. 458) placed directly over the pump well, and this piston-rod is attached directly to the pump rods or "pitwork." The up stroke is performed by the steam acting on the under side of the piston, but during this stroke no water is pumped, the work done by the steam, over and above that required to overcome the friction appearing in the energy of the raised heavy pitwork. During the down stroke the steam is led from the lower to the upper end of the cylinder, thus producing equilibrium on the piston, and the descent of the heavy pitwork raises the water.

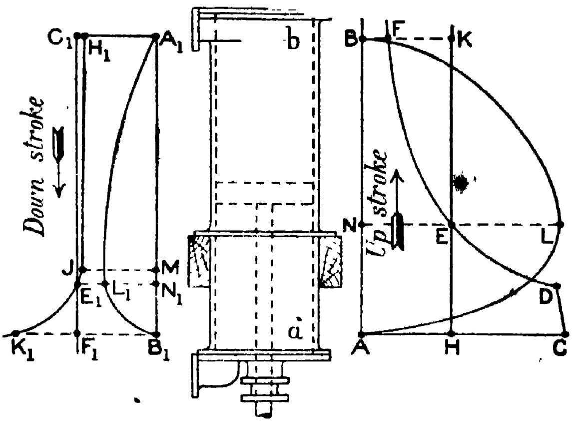


FIG. 458.

During the up stroke the steam is used expansively, and the end of the stroke is reached when the work done by the steam is equal to the work done on the resistance. The diagram ACDEFB to the right in Fig. 458 shows the effective pressure per square inch on the piston during the up stroke, AB being the length of the stroke. AH represents the total resistance, per square inch of piston, due to the weight of the piston, piston-rod, and pitwork, and the resistance of friction. HEK, the resistance line, is parallel to AB, and cuts the expansion curve at E. When the piston has moved to N, the effective pressure on the piston exactly balances the resistance, and from A to N the work done by the steam is represented by the area ACDEN, while the work done on the resistance is represented by the area AHEN. The excess work, represented by the area HCDE, is stored in the rising masses as kinetic energy, and the speed of the piston increases as it moves from A to N. Above N the effective pressure on the piston will continue to diminish as the piston rises, until the position B is reached, when it comes to rest.

The velocity curve ALB may be constructed as in the second example of the preceding Article, and the point B where this curve cuts the line of stroke AB determines the end of the up stroke.

A similar problem is presented during the down stroke of the piston. Referring to the diagram to the left of Fig. 458, the effort A_1C_1 is the weight of the pitwork, etc. (per square inch of piston), and the effort line $C_1E_1F_1$ is parallel to A_1B_1, the line of stroke. The resistance A_1H_1, at the beginning of the stroke, is due to the head of water, the size of the pump, and the resistance of the valves. As the speed increases the

friction of the water in the pipes will increase the resistance, and the line H_1J will not be parallel to A_1B_1. To stop the downward motion, the steam under the piston is shut in when the piston reaches the point M, and the resistance then increases, as shown by the line JE_1K_1, due to the compression of the steam on the lower side of the piston. In constructing the curve JE_1K_1, it must be remembered that while the steam below the piston is being compressed the steam above is expanding, and this must be allowed for, so that JE_1K_1 may show the effective increase in the resistance. The velocity curve $A_1L_1B_1$ is determined as before.

During the up stroke the steam above the piston is exhausted into the condenser. To prevent damage to the cylinder in the event of the prearranged stroke being exceeded, there are buffer beams against which the cross-head strikes.

Exercises XVIIa.

1. An effort-space diagram is drawn to the scales, 1 inch to 60 lbs. and 1½ inches to 1 foot. The length of the diagram is 2·3 inches, and its area is 2·45 square inches. How many ft.-lbs. of work does the area of the diagram represent, and what is the space average of the effort?

2. A body weighing 80 lbs. is moved from rest in a horizontal direction by an effort which varies uniformly from 112 lbs. at the beginning to 40 lbs. when the body has moved 8 feet. There is a uniform horizontal resistance of 60 lbs. Represent this by a diagram to the scales, 1 inch to 50 lbs., and 1 inch to 2 feet. Calculate the kinetic energy and the velocity of the body when it has moved 8 feet. Find also the maximum velocity.

3. In an effort-time diagram, the effort being the unbalanced effort, the length of the diagram is 5 inches, and represents 30 seconds. The scale for the effort is 1 inch to 100 lbs., and the area of the diagram is 12 square inches. If the weight of the body upon which the effort acts is 5 tons, what is the increase in its velocity, in miles per hour, in the time represented by the length of the diagram?

4. The pressure on the piston of a direct acting steam-engine is 150 lbs. per square inch for the first three-tenths of the stroke, and for the remainder of the stroke the pressure varies inversely as the distance of the piston from the beginning of the stroke. Draw on the same base, (*a*) the pressure-space diagram, (*b*) the pressure-time diagram, assuming an infinite connecting-rod, (*c*) the pressure-time diagram, taking the length of the connecting-rod twice the stroke of the piston. Find the mean pressure in lbs. per square inch of piston from each diagram. Assume that the crank is rotating at a uniform speed.

5. A certain acceleration diagram on a time base has an area of 2·1 square inches. The base is 3·5 inches long, and represents 17·5 seconds. The acceleration scale is 1 inch to 5 feet per second per second. If the velocity is 8 feet per second at the beginning, what is the velocity at the end of the 17·5 seconds?

6. In a diagram representing the unbalanced effort on a body weighing 800 lbs., the effort scale is 1 inch to 100 lbs. If the effort curve is also the acceleration curve, and the acceleration scale is x inches to 10 feet per second per second, find x.

7. The tangent at a certain point of a certain velocity curve on a time base is inclined at 35° to the base (tan. 35° = 0·7). If the time scale is 1 inch to 5 seconds, and the velocity scale is 1 inch to 20 feet per second, what is the acceleration in feet per second per second at the point considered.

8. The area of an acceleration diagram on a space base was measured with a planimeter and found to be 7·85 square inches. The base ON of the diagram was 4 inches long, and represented 20 feet. The acceleration scale was 1 inch to 10 feet per second per second, and the velocity at O was 2 feet per second. Calculate the velocity at N.

9. At a particular point on the curve of a velocity-space diagram the inclination of the tangent is 30°. The ordinate of the point represents a velocity of 7 feet per second, and the sub-normal measures 1·35 inches. If the distance scale

is 1 inch to 10 feet, what is the acceleration, in feet per second per second, at the position considered?

10. An electric street car was found to have moved from rest 66, 245, 490, and 750 feet in 5, 10, 15, and 20 seconds respectively from the start. Construct on a time base the displacement, velocity, and acceleration curves, and state the velocities in miles per hour, and the accelerations in miles per hour per second at the times 5, 10, 15, and 20 seconds from the start. [*Engineering News*, Oct. 14, 1897, and Durley's "Kinematics of Machinery," p. 47.]

11. The angular position θ (in radians) of a rocking shaft at any time t (in seconds) is measured from a fixed position. Successive positions at intervals of $\frac{1}{50}$ second have been determined as follows:—

t	0·0	0·02	0·04	0·06	0·08	0·10	0·12	0·14	0·16	0·18
θ	0·106	0·208	0·337	0·487	0 651	0·819	0·978	1·111	1·201	1·222

Find the change of angular position during the first interval from $t=0·0$ to $t=0·02$. Calculate the mean angular velocity during this interval in radians per second, and, on a time base, set this up as an ordinate at the middle of the interval. Repeat this for the other intervals, tabulating the results, and drawing the curve showing approximately angular velocity and time. In the same way find a curve showing angular acceleration and time.

Read off angular velocity in radians per second, and angular acceleration in radians per second per second, when $t=0·075$ second.

A wheel keyed to the shaft weighs 720 lbs, and has a radius of gyration of 1·5 feet. What is the torque tending to fracture the shaft when $t=0·16$ second? [B.E.]

12. A weight W of 1000 lbs. is made to move along a horizontal plane. The frictional resistance R is uniform and equal to 100 lbs. The driving force P in lbs. varies uniformly, and is given by the formula $P=Q(10-x)$, where x is the distance moved in feet from the starting point. Determine in each of the following cases (a) the distance moved in feet by W before coming to rest, (b) the maximum velocity of W in feet per second, (c) the distance in feet of W from the starting point when its velocity is a maximum, (d) the acceleration in feet per second per second when W is 2 feet from the starting point. Case I. $Q=15$; Case II. $Q=20$; Case III. $Q=30$.

13. A body A weighing 1000 lbs. is moved horizontally by a force P lbs. which is equal to 200 lbs. for the first 2 feet, and afterwards varies according to the law $Px=400$, where x is the distance moved from the starting point. The frictional resistance R is constant, and equal to 100 lbs. Determine (a) the distance moved by A, in feet, before coming to rest; (b) the maximum velocity of A, in feet per second; (c) the distance of A from the starting point, in feet, when its velocity is a maximum; (d) the acceleration, in feet per second per second, when A is 3 feet from the starting point. Plot $P-R$, and the velocity, on a space base

14. A body weighing 1610 lbs. is lifted vertically by a rope, there being a damped spring balance to indicate the pulling force F lb. of the rope. There is a constant frictional resistance of 740 lbs. to the motion of the body. When the body has been lifted x feet from its position of rest, the pulling force is automatically recorded as follows:—

x	0	11	20	34	45	55	66	76
F	4010	3915	3763	3532	3366	3208	3100	3007

Using squared paper, find the velocity v feet per second for values of x of 10, 30, 50, 70, and draw a curve showing the probable values of v for all values of x up to 80. In what time does the body get from $x=45$ to $x=55$? In what time does it get from $x=0$ to $x=75$? [B.E.]

15. A body weighing 322 lb. is lifted by a force F lb. which alters. When the body has risen through the distance x feet, the force in lb. for the several values of x is as follows (or would be if the body rose as far):—

x	0	1	2	3	4	5·5	7	9	11	12·5	14	17	20
F	540	540	540	530	500	460	310	220	190	190	190	190	190

Using squared paper, find the velocity in each position and the time taken by the body to get to each position counting from $x=0$, the velocity then being 5 feet per second. [B.E.]

16. A tram-car, weighing 15 tons, suddenly has the electric current cut off. At that instant the speed of the car was 16 miles per hour. Reckoning time from that instant, the following velocities, V (miles per hour), and times, t (seconds), were noted. $V=16$, $t=0$. $V=14$, $t=9·3$. $V=12$, $t=21$. $V=10$, $t=35$. Calculate the average value of the retarding force, and find the average velocity from $t=0$ to $t=35$. Also find the distance travelled between these times.

If the law of resistance be F (lb.) $=a+bV+cV^2$, where V is in miles per hour as before, indicate the method by which values of a, b, and c could be found from the above observations. Also calculate the relation between V and T (the time taken to come to rest from velocity V) for such tests. What is T when V is very large? [B.E.]

17. During the up stroke of the piston of a Bull engine the effective pressure p of the steam on the piston, in lbs per square inch, varies, as shown in the following table, where x is the distance of the piston from the bottom of its stroke in feet; the piston will however not rise so high as 10 feet. The weight

x	0·0	0·5	1·0	1·5	2·0	2·5	3·0	3·5	4·0	4·5	5·0	6·0	7·0	8·0	9·0	10·0
p	55	54	53·5	53	41	34	28	24	21	18	16	12·5	10·5	8·5	7	6

of the piston, piston-rod, and "pitwork" amounts to 22 lbs. per square inch of piston, and the frictional resistances are equivalent to 2 lbs. per square inch of piston. Draw the velocity curve for the piston on a stroke base, and find the length of the stroke. Find the time taken to make one up stroke, and draw the velocity curve on a time base.

258. Simple Harmonic Motion.—A (Fig. 459) is a point which is moving with a uniform velocity V along the circumference of the circle BACD. a is another point which is moving backward and forward along the diameter BOC of the same circle in such a manner that Aa is always perpendicular to BC; in other words, a is the projection of A on BC. Under these circumstances, the point a has *simple harmonic motion*.

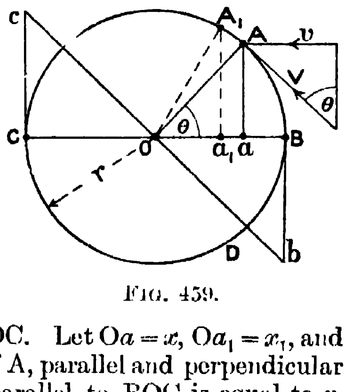


FIG. 459.

Let A and A_1 be two positions, near to one another, of the point which is moving round the circle, and a and a_1 corresponding positions of the point which has simple harmonic motion. Let v be the velocity of a at a, and v_1 the velocity of a at a_1 along BOC. Let $Oa=x$, $Oa_1=x_1$, and angle BOA $=\theta$. Resolving V, the velocity of A, parallel and perpendicular to BOC, it is evident that the component parallel to BOC is equal to v, the velocity of a, and $v=V\sin\theta=\frac{V}{r}\sqrt{r^2-x^2}$. Also $v_1=\frac{V}{r}\sqrt{r^2-x_1^2}$.

The *mean* velocity of a between a and a_1 is $\frac{1}{2}(v_1+v)$, and the time taken to travel from a to a_1 is $\frac{aa_1}{\frac{1}{2}(v_1+v)}=\frac{2(x-x_1)}{v_1+v}$. If f is the *mean* acceleration of a between a and a_1, $f=(v_1-v)\div\frac{2(x-x_1)}{v_1+v}=\frac{v_1^2-v^2}{2(x-x_1)}=\frac{\text{V}^2}{2r^2}(x+x_1)$. Now if AA_1 is made indefinitely small, x_1 becomes equal to x, and f becomes the acceleration of a at a. Hence $\underline{f=\frac{\text{V}^2x}{r^2}}$.

If f be plotted on the space base BOC, the straight line $b\text{O}c$ is the result, the maximum values of f being at B and C where $x=r$. At the centre O, where $x=0$, $f=0$. At B and C, $v=0$, and at O, v has its maximum value, and is there equal to V.

When A is at B or C, A and a coincide, and f for a becomes the radial acceleration of A, namely, $\frac{\text{V}^2}{r}$, a result which has been proved in another way in Art. 21, p. 17.

Since the acceleration of the point a is directly proportional to its displacement from its middle position, this property may be used as a test of simple harmonic motion. In fact, the definition of simple harmonic motion is better given as the motion which a point has when its acceleration is proportional to its displacement from its middle position, because this includes the case of a point oscillating in a curved path (Fig. 460), where the *arc* OB or the *arc* OC $=r$, and the *arc* O$a=x$.

FIG. 460.

A complete oscillation or vibration is a movement from one end of the path to the other and back again. The time of a complete oscillation is called the *periodic time*. If V is in feet per second, f in feet per second per second, r in feet, and t, the periodic time, in seconds, then

$$t=\frac{2\pi r}{\text{V}}\text{, and }f=\frac{\text{V}^2x}{r^2}=\frac{4\pi^2x}{t^2}.$$

Referring again to Fig. 459, $\text{A}a=r\sin\theta$, but $v=\text{V}\sin\theta$, therefore if the velocity scale be chosen so that r represents V, then Aa will represent

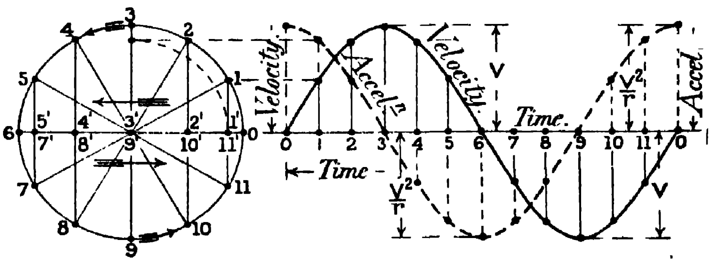


FIG. 461.

v on that scale, and the circle BACD will be the velocity diagram on the space base BOC for the point which has simple harmonic motion. Again, $f=\frac{\text{V}^2x}{r^2}$; therefore if the acceleration scale be so chosen that r

represents $\frac{V^2}{r}$, Oa or x will represent f on that scale. This also follows from Art. 253, p. 287, because x is the sub-normal of the velocity curve on a space base.

Fig. 461 shows the velocity v and acceleration f plotted on a time base. The constructions are obvious, and clearly shown in the figure.

259. Forces giving a Body Simple Harmonic Motion.—If a body weighing W lbs. has simple harmonic motion along the line BOC (Fig. 462) under the action of a force P, then since acceleration is proportional to the force producing it, $\frac{P}{W} = \frac{f}{g}$, and using the notation and results of the preceding Article, $\frac{P}{W} = \frac{V^2 x}{gr^2}$, and when $x = r$, $\frac{P}{W} = \frac{V^2}{gr} = \frac{4\pi^2 r}{gt^2}$.

The force P must always act towards the centre O, so that while the body is moving towards O, P is an effort, but when the body is moving away from O, P becomes a resistance.

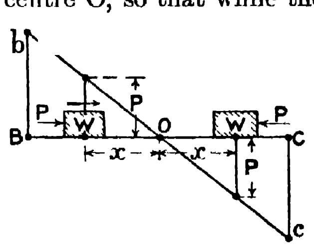


FIG. 462.

The force diagram on a space base is evidently a straight line one, like the acceleration diagram. In moving from B to O the work done by the effort is represented by the area of the triangle BbO, and is stored up in the body as kinetic energy, to be given out again in overcoming the resistance in moving from O to C, the work done on the resistance from O to C being represented by the area of the triangle OcC.

The foregoing results may be applied to the case of a body which has angular harmonic motion. Let O_1A (Fig. 463) be a bar upon which is mounted a mass M, the weight of the rod O_1A and the mass M being W, and let the whole body oscillate with harmonic motion about an axis O_1 perpendicular to the plane of the paper. Let O_1A be the central position, and O_1B, making an angle θ with O_1A, any other position. Let k be the radius of gyration of the whole body about the axis O_1, and let P be a force acting on the body at a distance from O_1 equal to k and in the direction of motion, which will give the harmonic motion.

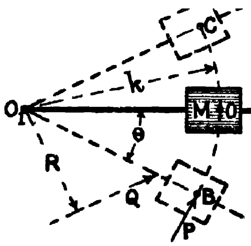


FIG. 463.

Applying the formula $\frac{P}{W} = \frac{4\pi^2 r}{gt^2}$ to this case, $r = \text{arc OB} = k\theta$. Hence $\frac{P}{W} = \frac{4\pi^2 k\theta}{gt^2}$, and $Pk = \frac{4\pi^2 W k^2 \theta}{gt^2} = \frac{4\pi^2 I\theta}{gt^2}$, where I is the moment of inertia of the body about the axis O_1. The product Pk is the turning moment of the force P about the axis O_1. If T denotes this turning moment, then $T = \frac{4\pi^2 I\theta}{gt^2}$. If the force which gives harmonic motion to the body be a force Q acting as shown, R being the perpendicular distance of its line of action from O_1, then $T = QR = \frac{4\pi^2 I\theta}{gt^2}$.

For example, take the case of the simple pendulum (Fig. 464), where a small body of weight w swings in a vertical plane in a small arc of a circle of radius l. $T = w \cdot \overline{O_1N} = wl \sin\theta$. Since θ is a small angle, $\sin\theta$ may be taken equal to θ, also $I = wl^2$, hence

$$T = wl\theta = \frac{4\pi^2 wl^2\theta}{gt^2}.$$

therefore

$$t^2 = \frac{4\pi^2 l}{g} \text{ or } t = 2\pi\sqrt{\frac{l}{g}},$$

a well-known result, which may be proved in other ways.

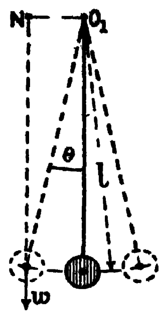


FIG. 464.

Exercises XVIIb.

1. A body of 60 lb. has a simple vibration, the total length of a swing being 3 feet; there are 200 *complete* vibrations (or double swings) per minute; calculate the forces which act on the body at the ends of a swing, and show on a diagram to scale what force acts upon the body in every position. [B.E.]

2. A weight of 5 lbs. is supported by a spring. The stiffness of the spring is such that putting on or taking off a weight of 1 lb. produces a downward or upward motion of 0·04 foot. What is the time of a complete oscillation, neglecting the mass of the spring? [B.E.]

3. A weight of 10 lbs. suspended by a spiral spring makes 107 complete vertical oscillations in 1 minute. What weight applied gradually will lengthen the spring 1 inch?

4. A U tube (Fig. 465) of uniform bore contains a liquid which fills a length of 2 feet of the tube. Find the time of a complete oscillation of the liquid in the tube.

5. A steel wire 0·16 inch diameter fixed at its upper end and guided at the lower end has a wheel weighing 12·3 lbs. fixed to it near its free end and 40 inches from its fixed end, as shown in Fig. 466. The wheel is turned through a small angle and then liberated, and it is then found to make 4 complete oscillations in 6 seconds. Taking the modulus of rigidity of the wire as 13,000,000 lbs. per square inch, calculate the radius of gyration of the wheel about its axis.

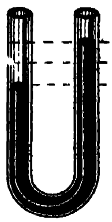


FIG. 465.

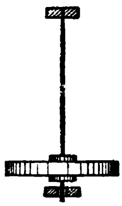


FIG. 466.

6. A fly-wheel weighs 5 tons, and its radius of gyration is 6·30 feet. It is at the end of a shaft 40 feet long, 5 inches diameter, modulus of rigidity of material 12,000,000 lbs. per square inch, what is the natural time of torsional vibration of the system, neglecting the inertia of the shaft itself? [B.E.]

7. A fly-wheel weighs 5 tons, and has a radius of gyration of 6 feet. It is at one end of a shaft, the other end of which is fixed. It is found that a torque of 200,000 lb.-feet is sufficient to turn the wheel through 1°. If the wheel is twisted slightly and then released, how many vibrations per minute will it make? [B.E.]

8. A uniform circular plate, 1 foot in diameter, and weighing 4 lbs., is hung in a horizontal plane by three fine parallel cords from the ceiling, and when set into small torsional vibrations about a vertical axis is found to have a period of 3 seconds. A body, whose moment of inertia is required, is laid horizontally across it, and the period is then found to be 5 seconds, the weight of the body being 6 lbs. Find the moment of inertia of the body about the axis of oscillation. [Inst. C.E.]

CHAPTER XVIII

PISTON OR SLIDER AND CONNECTING-ROD VELOCITY AND ACCELERATION DIAGRAMS

260. Piston or Slider Velocity Diagrams.—In the direct-acting engine the reciprocating motion of the piston is converted into the rotary motion of the crank shaft by means of the crank and connecting-rod. In what follows it is really the motion of the cross-head which is studied, but the piston and cross-head have exactly the same motion. Referring to Fig. 467, A is the axis of the cross-head pin, AB the axis of the connecting-rod, B the axis of the crank pin, BC the crank, and C the axis of the crank shaft. The line of stroke of the piston when produced is assumed to pass through C. Let O_1 be the instantaneous centre for the

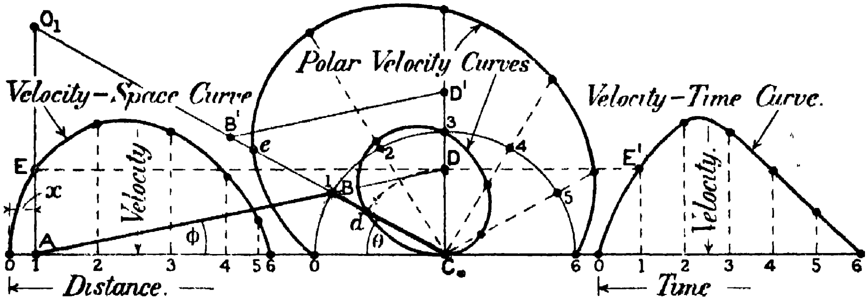


FIG. 467.

connecting-rod when in the position shown. Let V be the linear velocity of the crank pin, and v the velocity of the cross-head, then $\frac{v}{V}=\frac{O_1A}{O_1B}$.

Through C draw CD′ perpendicular to AC. Make CB′ = V to any convenient velocity scale, and draw B′D′ parallel to AB to meet CD′ at D′, then, since the triangles O_1AB and CD′B′ are similar, $\frac{CD'}{CB'}=\frac{O_1A}{O_1B}=\frac{v}{V}$, therefore CD′ = v. Since V, the velocity of the crank pin, is usually uniform, it is generally convenient to select the velocity scale such that CB = V, then if AB be produced to meet CD′ at D, CD = v. Drawing DE parallel to the line of stroke to meet AE perpendicular to the line of stroke at E determines a point on the piston velocity-space curve. If C*d* (on the crank) be made equal to CD, then *d* is a point on the polar velocity curve for the piston. A point on another form of the polar velocity curve is obtained by making B*e* (on the crank produced) = CD.

The diagram to the right in Fig. 467 shows the velocity of the piston, during one stroke, plotted on a crank angle, or time base.

If the connecting-rod is of infinite length, AB becomes parallel to the line of stroke, BD is perpendicular to CD, and v is equal to V sin θ. If V is constant the piston has simple harmonic motion, and the velocity-space and the velocity-time diagrams for the piston are the same as those shown in Fig. 461, p. 297. The polar velocity diagram becomes a circle described on the crank as diameter when the latter is perpendicular to the line of stroke.

The arrangement shown in Fig. 468 is the equivalent of an infinite connecting-rod. This arrangement is frequently found in steam pumps, one rod M being the steam piston-rod, and the other N the pump plunger. The crank in this case must be an overhung one, or the crank shaft must be divided to allow the slotted piece KL to pass. If the slotted piece have two slots at right angles, as shown in Fig. 469, the crank may be placed anywhere on the shaft without altering the shaft in any way.

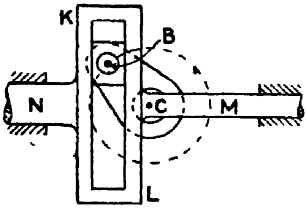


FIG. 468.

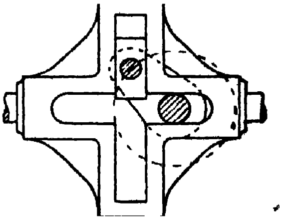


FIG. 469.

261. Piston or Slider Acceleration.—Since CD (Figs. 467 and 470) represents the piston velocity if CB represents the crank pin velocity, it follows that since acceleration is rate of increase of velocity, the velocity of the point D along CD will be the rate of increase of CD, and will therefore be the piston acceleration.

Consider the point D as a point in the connecting-rod produced, then D must be moving at the instant in a direction perpendicular to OD with a velocity equal to $V \cdot \frac{OD}{OB}$. This velocity may be found by construction as follows. On OD make OB′ = OB. Draw B′C′ perpendicular to OD and equal to BC. Join OC′ and produce it to meet DE, a perpendicular to OD, at E, then $DE = V \cdot \frac{OD}{OB}$.

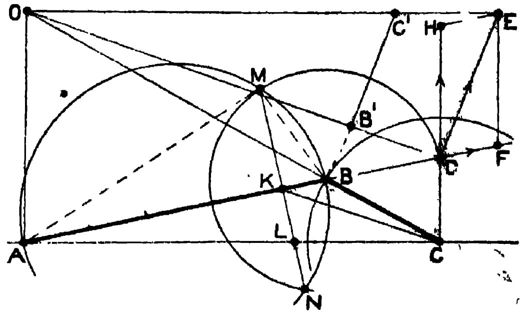


FIG. 470.

If DE be resolved into components DF along AB, and DH along CD, then DH is the velocity of D along CD, and therefore represents the piston acceleration. Draw CK parallel to OD to meet AB at K, and draw KL perpendicular to AB to meet AC at L, then it will be shown that CL = DH. The triangles CKB and OBD are similar, and

$$\frac{CK}{BC} = \frac{OD}{OB} = \frac{OD}{OB'} = \frac{DE}{B'C'} = \frac{DE}{BC},$$

therefore CK = DE. The triangles CKL and DEH are similar, because the sides of CKL are respectively perpendicular to the sides of DEH, and $\frac{CL}{CK} = \frac{DH}{DE}$, but CK = DE, therefore CL = DH. If, therefore, the point O is accessible, CL, the piston acceleration, is found by drawing CK parallel to OD, and KL perpendicular to AB. But for a considerable portion of the motion of the piston the point O is either at an inconvenient distance or is quite inaccessible, and some other construction for finding the point K is desirable.

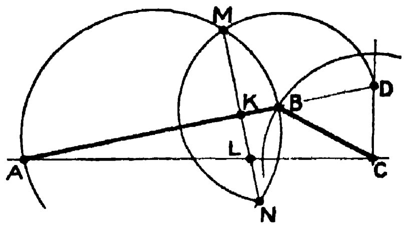


FIG. 471.

What is generally known as *Klein's construction* is the most convenient for finding KL. Klein's construction is as follows. On AB as a diameter describe a circle. With centre B and radius BD describe another circle, cutting the former at M and N. Join MN. The line MN coincides with the line KL of the former construction. For, referring to K as found by the first construction, $\frac{BK}{BD} = \frac{BC}{OB}$, because the triangles CBK and OBD are similar. Also, $\frac{BC}{OB} = \frac{BD}{AB}$, because the triangles CBD and OBA are similar, therefore $\frac{BK}{BD} = \frac{BD}{AB}$, or $BK \cdot AB = BD^2$. Referring now to K as found by Klein's construction,

$$\frac{BK}{BD} = \frac{BK}{BM} = \frac{BM}{AB} = \frac{BD}{AB}, \text{ or } BK \cdot AB = BD^2 \text{ as before.}$$

For the sake of clearness, the essential lines of Klein's construction are shown separately in Fig. 471.

262. Piston Acceleration at Ends of Stroke.—When the piston is at either end of its stroke the crank and connecting-rod are in a straight

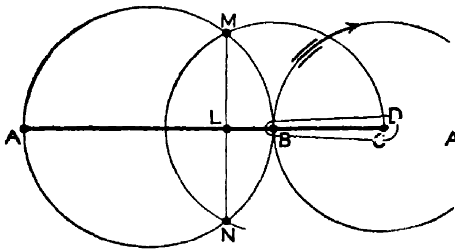


FIG. 472.

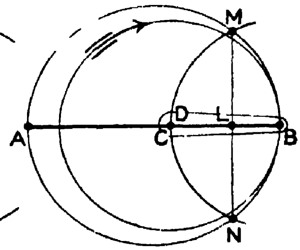


FIG. 473.

line, and Klein's construction gives the result shown in Fig. 472 for the outer end of the stroke, and the result shown in Fig. 473 for the inner end. Referring to Fig. 472, the angular velocity of the connecting-rod in this position is V/l, and A has an acceleration in the direction AC due to this and equal to V^2/l. Also the angular velocity of the crank is V/r, and in the position shown A has an acceleration in the direction AC due

to this and equal to V^2/r. Hence the total acceleration of A in the direction AC is equal to $V^2/r + V^2/l$. Referring to Fig. 473, it follows in the same way that the acceleration of A in the direction CA is equal to $V^2/r - V^2/l$. Hence $f = V^2\left(\frac{1}{r} \pm \frac{1}{l}\right) = \frac{V^2}{r}\left(1 \pm \frac{r}{l}\right) = \frac{V^2}{r}\left(1 \pm \frac{1}{n}\right)$, where n is the ratio of the length of the connecting-rod to the length of the crank: the plus sign applies to the outer and the minus sign to the inner end of the stroke. For example, if $V = 10$ feet per second, $r = 10$ inches, and $l = 50$ inches, $f = \frac{10^2 \times 12}{10}\left(1 \pm \frac{10}{50}\right) = 144$ feet per second per second at the outer end, and 96 feet per second per second at the inner end of the stroke.

263. Piston Acceleration Diagrams.—Having shown how the piston acceleration may be determined at any point of the stroke, the results for a number of points may be plotted, and acceleration curves obtained corresponding to the piston velocity curves described in Art. 260. Fig. 474 shows the various piston acceleration curves for the case where the length of the connecting-rod is $2\frac{1}{4}$ times the length of the crank.

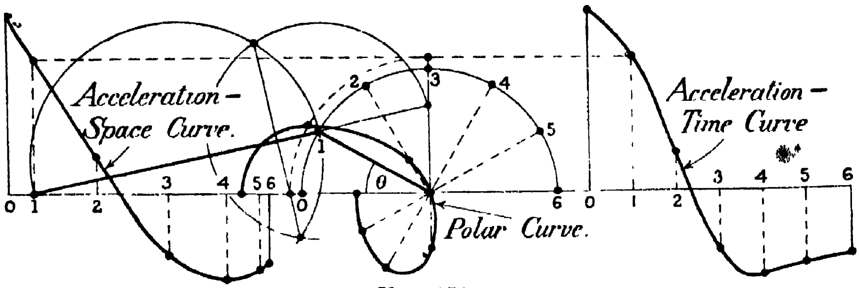


FIG. 474.

264. Piston Acceleration Scale.—It was shown in Art. 261 that $BK \cdot AB = BD^2$ (Fig. 471), and when the piston is at the outer end of its stroke (Fig. 472) this becomes $BL \cdot AB = BC^2$ or $BL = \frac{BC^2}{AB}$. Now

$$CL = BC + BL, \text{ therefore } CL = BC + \frac{BC^2}{AB} = BC\left(1 + \frac{BC}{AB}\right) = BC\left(1 + \frac{1}{n}\right).$$

But it was shown in Art. 262 that the piston acceleration at the outer end of the stroke is equal to

$$\frac{V^2}{r}\left(1 + \frac{1}{n}\right) = f. \quad \text{Hence if } CL = f, \; BC\left(1 + \frac{1}{n}\right) = \frac{V^2}{r}\left(1 + \frac{1}{n}\right) \text{ or } BC = \frac{V^2}{r}.$$

Therefore the scale on which CL will measure the acceleration of the piston is one on which a length equal to BC represents $\frac{V^2}{r}$, the radial acceleration of the crank pin. For example, if BC on the drawing measures $2\frac{1}{2}$ inches (on a full size scale), and if $V = 10{\cdot}5$ feet per second, and $r = 9$ inches $= 0{\cdot}75$ foot, then $\frac{V^2}{r} = \frac{10{\cdot}5^2}{0{\cdot}75} = 147$ feet per second per second, and the acceleration scale is such that 1 inch represents $147 \div 2{\cdot}5 = 58{\cdot}8$ feet per second per second, or 100 feet per second per second is represented by $100 \div 58{\cdot}8 = 1{\cdot}7$ inches.

265. Position of Piston for Maximum Velocity and Zero Acceleration.—Still assuming that the velocity of the crank pin is uniform, it is evident that when CD (Figs. 467 and 470) ceases to increase, the position for maximum velocity and zero acceleration of piston has been reached, and this will obviously happen when the angle ODA is a right angle (Fig. 475). No direct geometrical construction has yet been found for drawing the figure so that the angle ODA may be a right angle, but by analysis it may be shown that when ODA is a right angle, θ, the angle ACB, is given by the equation

$$\sin^6\theta - n^2\sin^4\theta - n^4\sin^2\theta + n^4 = 0,$$

which is a cubic equation in $\sin^2\theta$. Mr. G. A. Burls * has solved this equation for a considerable number of values of n, and placed the results in a table, of which the following is an abstract:—

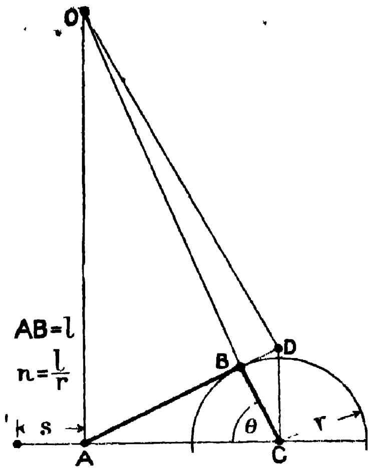


FIG. 475.

n	$s \div r$	θ	n	$s \div r$	θ	n	$s \div r$	θ
1·0	2·0000	90° 0′ 0″	2·0	0·8474	67° 42′ 0″	6·0	0·9218	80° 47′ 40″
1·1	1·0530	64 57 50	2·5	0·8550	70 43 46	7·0	0·9321	82 3 3
1·144	1·0000	64 5 11	3·0	0·8674	73 10 31	8·0	0·9389	82 56 30
1·2	0·9564	63 35 5	4·0	0·8906	76 43 24	9·0	0·9468	83 47 12
1·5	0·8681	64 20 38	5·0	0·9085	79 6 34	10·0	0·9524	84 24 59

s is the distance of the piston from the outer end of its stroke when its velocity is a maximum or its acceleration zero.

$$\frac{s}{r} = n + 1 - \sqrt{1 - \sin^2\theta} - \sqrt{n^2 - \sin^2\theta}.$$

Students are referred to Mr. Burls' paper for the complete discussion of this problem.

For practical purposes, when n has values common in direct-acting engines, it is usually sufficient to assume that the position of the piston for maximum velocity and zero acceleration is that for which the crank and connecting-rod are at right angles to one another.

Professor Unwin's formula,† $\sin^2\theta = \dfrac{n^2}{n^2+1} + \dfrac{2n^2+1}{(n^2+1)(n^4+4n^2)}$, gives, for values of n usual in practice, an extremely close approximation to the true value of θ.

266. Analytical Determination of Piston Velocity and Acceleration.—Piston velocity and acceleration diagrams are most readily drawn by the accurate constructions which have been given in preceding Articles,

* *Proceedings Inst. C.E.*, vol. cxxxi. p. 338.

† *Ibid.*, vol. cxxv. p. 366.

but the velocity or acceleration of the piston for any position of the crank may be calculated by means of the formulæ now to be proved.

Referring to Fig. 467, p. 300, it is readily seen that

$$\frac{v}{V}=\frac{\sin(\theta+\phi)}{\cos\phi}=\frac{\sin\theta\cos\phi+\cos\theta\sin\phi}{\cos\phi}=\sin\theta+\frac{\cos\theta\sin\phi}{\cos\phi}.$$

But $\sin\phi=\frac{r}{l}\sin\theta=\frac{\sin\theta}{n}$, and $\cos\phi=\sqrt{1-\sin^2\phi}=\frac{1}{n}\sqrt{n^2-\sin^2\theta}$.

Therefore $$\frac{v}{V}=\sin\theta+\frac{\sin\theta\cos\theta}{\sqrt{n^2-\sin^2\theta}}=\sin\theta+\frac{\sin 2\theta}{2\sqrt{n^2-\sin^2\theta}}.$$

For values of n usual in direct-acting engines it will be sufficiently accurate to take $\sqrt{n^2-\sin^2\theta}=n$, then $v=V\left(\sin\theta+\frac{\sin 2\theta}{2n}\right)$ approximately.

From this approximate expression for v the acceleration f is found as follows:—

$$f=\frac{dv}{dt}=V\left(\frac{d\theta}{dt}\cos\theta+\frac{d\theta}{dt}\cdot\frac{2\cos 2\theta}{2n}\right),\ V \text{ being constant,}$$

$$=V\left(\frac{V}{r}\cos\theta+\frac{V}{r}\cdot\frac{\cos 2\theta}{n}\right)=\frac{V^2}{r}\left(\cos\theta+\frac{\cos 2\theta}{n}\right).$$

267. Angular Velocity of Connecting-rod.—The connecting-rod has a motion of translation along with the piston, and also an angular motion, the angle ϕ which it makes with the line of stroke changing from zero to a maximum, and back again to zero during one stroke of the piston. ϕ is evidently a maximum when the crank is perpendicular to the line of stroke, and it is zero when the crank is on the line of stroke.

Referring to Fig. 476, O is the instantaneous centre of the connecting-rod when in the position shown, and if BC represents V, the velocity of the crank pin, CD represents v, the velocity of the piston. Imagine a velocity equal to v to be impressed on the connecting-rod in the direction CA. The point A will now be at rest, and the connecting-rod will only have angular motion. The point B has now a velocity which is the resultant of the velocities BC′ perpendicular to BC, and $=V=$ BC, and C′D′ parallel to CA, and $=v=$ CD. This resultant will evidently be perpendicular to AB and = BD. The angular velocity of AB in the given position is therefore equal to BD′/AB = BD/AB, and as AB is constant, the angular velocity is represented by BD.

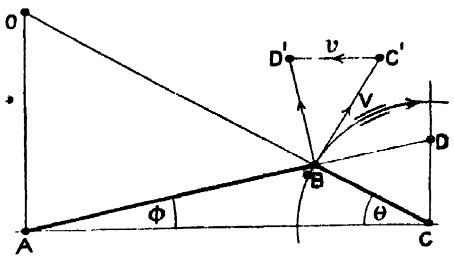


FIG. 476.

The foregoing result is also obvious when it is remembered that, at the instant considered, the connecting-rod is rotating about O, and its angular velocity about O is equal to BC′/OB = BC/OB = BD/AB, and this will also be the rate of change of the angle ϕ.

The angular velocity of AB may be plotted on the crank CB from the pole C, or on the piston or cross-head stroke as a base, but preferably

on the connecting-rod, as shown to the right in Fig. 477. The cross-head end of the connecting-rod is placed at C, and assumed to be at rest. The

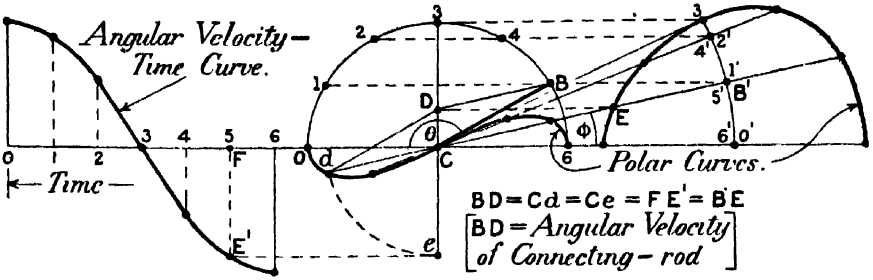


FIG. 477.

different values of ϕ corresponding to different values of θ are readily obtained by the construction shown, and which may be briefly described as follows. With C as centre, and radius equal to the length of the connecting-rod, measured on the linear scale, describe the arc $0'3'$. Let CB be one position of the crank. Draw BB′ parallel to C0′ to meet $0'3'$ at B′. Join CB′, then angle $B'C0' = \phi$. Draw BD parallel to B′C to meet the perpendicular to C0′ from C at D. Then BD represents the angular velocity of the connecting-rod when the crank is at CB. Make, on B′C, B′E = BD. Then E is a point on a polar curve of angular velocity of connecting-rod. If Cd = BD be marked off on B′C from C as a pole, then d is a point on another polar curve of angular velocity of connecting-rod. Observe that when BD is positive B′E is measured from the arc $0'3'$ on the side opposite to C, and when negative it is measured from the arc $0'3'$ towards C. Also for the other polar diagram, when BD is positive Cd is measured from C towards the arc $0'3'$, and when negative it is measured from C in the opposite direction.

The construction for the angular velocity curve on a time base is obvious, and clearly shown in the figure.

The scale on which BD will measure the angular velocity of the connecting-rod is found as follows. Let BC, BD, and AB denote the lengths of these lines on the drawing, measured in inches on the full size scale. Let the scale for angular velocity be 1 inch to x radians per second, and let the linear scale of the drawing be 1 inch to y feet. Also let r be the true radius of the crank in feet, and let V be in feet per second. Then

$$\frac{\text{Angular velocity of connecting-rod}}{\text{Angular velocity of crank}} = \frac{\mathrm{BD} \div \mathrm{AB}}{\mathrm{BC} \div \mathrm{BC}} = \frac{\mathrm{BD}}{\mathrm{AB}}.$$

But angular velocity of crank $= \dfrac{\mathrm{V}}{r}$, therefore angular velocity of connecting-rod $= \omega = \dfrac{\mathrm{BD}}{\mathrm{AB}} \cdot \dfrac{\mathrm{V}}{r}$.

But $\mathrm{AB} = n \cdot \mathrm{BC} = nr/y$, therefore $\omega = \dfrac{\mathrm{BD} \cdot y}{nr} \cdot \dfrac{\mathrm{V}}{r} = \mathrm{BD} \cdot x$, and therefore $x = \dfrac{y\mathrm{V}}{nr^2}$.

Referring to Fig. 467, $\frac{BD}{BC}=\frac{\sin(90-\theta)}{\sin(90-\phi)}=\frac{\cos\theta}{\cos\phi}$, but $\sin\phi=\frac{\sin\theta}{n}$ and $\cos\phi=\frac{\sqrt{n^2-\sin^2\theta}}{n}$. Therefore

$$\frac{BD}{BC}=\frac{n\cos\theta}{\sqrt{n^2-\sin^2\theta}}, \text{ and } \omega=\frac{BD}{AB}\cdot\frac{V}{r}=\frac{BD}{n\cdot BC}\cdot\frac{V}{r}=\frac{\cos\theta}{\sqrt{n^2-\sin^2\theta}}\cdot\frac{V}{r}$$

268. Angular Acceleration of Connecting-rod.—Referring to Fig. 470, p. 301, just as DH or CL represents the rate of increase of CD, so DF or KL represents the rate of increase of BD, or the rate of increase of BD′ (Fig. 476), and therefore KL/AB represents the angular acceleration of the connecting-rod. The figure CBKL (Fig. 470) is a linear acceleration diagram, the scale of which was shown (Art. 264) to be such that BC represents V^2/r, the linear acceleration of B in the direction BC. Hence if KL and BC be measured in inches, and r and nr are the true lengths in feet of the crank and connecting-rod respectively, angular acceleration of connecting-rod $=a=\frac{KL}{BC}\cdot\frac{V^2}{r}\div nr=\frac{KL}{BC}\cdot\frac{V^2}{nr^2}$, and since KL is the only variable in the expression for a, KL will represent a on a certain scale. Let this scale be 1 inch to z radians per second per second, and let the linear scale of the drawing be 1 inch to y feet, then

$$a=KL\cdot z=\frac{KL\cdot y}{r}\cdot\frac{V^2}{nr^2}=KL\cdot y\cdot\frac{V^2}{nr^3}, \text{ and } z=\frac{yV^2}{nr^3}.$$

The angular acceleration of the connecting-rod may be plotted in a manner similar to that described for the angular velocity in the preceding Article.

269. Case where Line of Stroke does not Intersect Axis of Crank Shaft.—The illustrations of the direct acting engine mechanism which have been given in preceding Articles have shown the line of stroke passing through the axis of the crank shaft, and this is the usual arrangement, but in a single-acting engine, that is, an engine in which all the work is done on one side of the piston, there are advantages in arranging the mechanism as shown in Fig. 478, where pq, the line of stroke, when produced, does not pass through C, the axis of the crank shaft.

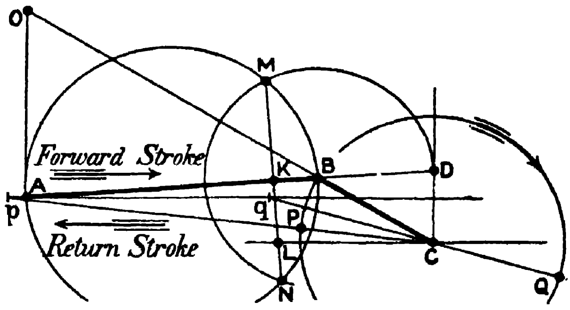


FIG. 478.

One result of altering the position of the line of stroke, as shown in Fig. 478, is that during the forward or working stroke the obliquity of the connecting-rod is diminished, and in consequence of this the pressure on the cross-head guide is diminished, and the turning moment on the crank is slightly more uniform. The diminished pressure on the guide means of course less work lost in friction at that part. During the return stroke the obliquity of the connecting-rod is increased by this

new arrangement, but as the engine is single acting, the forces to be dealt with during the return stroke are much smaller. It is easy to show that the stroke of the piston is now greater than twice the radius of the crank. The "dead centres" still occur when the piston is at the ends of its stroke, but the two dead centres are no longer on the same diameter of the crank pin circle, and, as will be seen by reference to Fig. 478, the time taken for the return stroke is less than that taken for the forward stroke if the crank is turning uniformly.

The construction for determining the piston velocity and piston acceleration are unaltered, except that CL, the acceleration, is now shown on a line through C parallel to the line of stroke, instead of on the line of stroke.

270. Inversions of the Direct-Acting Engine Mechanism.—What, in preceding Articles, has been called the direct-acting engine mechanism is also known as one form of the *slider-crank chain*, namely, the *turning slider-crank chain*. The slider-crank chain consists of four parts, three *links* and a *block* or *slider*. In the direct-acting engine mechanism (Fig. 479) the links are, the crank *a*, the connecting-rod *c*, and the

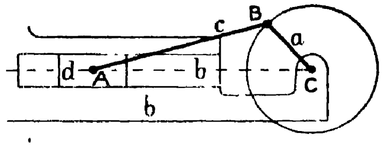


FIG. 479.

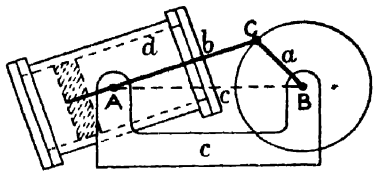


FIG. 480.

frame *b* of the engine. The cross-head *d* is the block or slider, but the slider may include the piston-rod and piston, and, as in most internal combustion engines, the slider may consist of the piston only.

Various mechanisms may be obtained from the slider-crank chain by the process of *inversion*, which involves the exchange of one fixed part or link for another.

The oscillating engine mechanism (Fig. 480) is obtained from the direct-acting engine mechanism by making *c* the fixed link instead of *b*. The crank is still *a*, but the crank shaft is now at B instead of at C. The block *d* is now the cylinder which oscillates on trunnions at A. The link *b* oscillates with the cylinder, but the relative motion between *d* and *b* is still that of sliding; in fact, the motion of any one link relative to that of any other link of the chain is unaltered by the process of inversion. The oscillating engine mechanism is known as the *swinging-block slider-crank chain*.

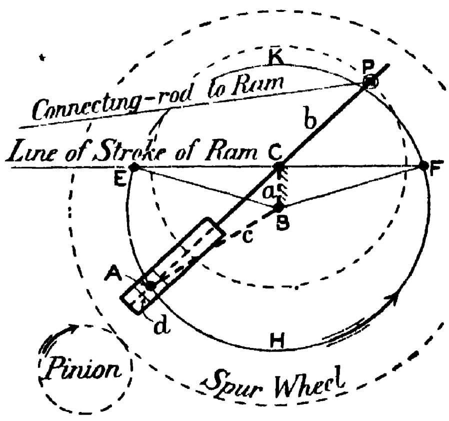


FIG. 481.

The mechanism shown in Fig. 481 is known as the *Whitworth quick*

return motion or *turning-block slider-crank chain*, and is obtained by making a the fixed link. c becomes a crank, the rotary motion of which is communicated to b, but the angular velocities of b and c are unequal, except at two points during a revolution. This mechanism is used for driving the ram of a shaping machine or the ram of a slotting machine. The link or crank c is really a spur-wheel rotating about an axis at B, and carrying a pin A projecting from one side, on which fits the block d, which in turn fits in a slot formed in b. The link b rotates about an axis at C, and carries a pin P, the position of which may be varied to suit the required stroke of the ram, which carries the cutting tool at one end. The reciprocating motion of the ram is obtained from the pin P through a connecting-rod. The line of stroke of the ram is shown passing through C, and cutting the circle described by the pin A at E and F. The pin A has uniform velocity, and the times of the forward or cutting stroke, and the return or idle stroke of the ram, are to one another as the arc EHF is to the arc FKE.

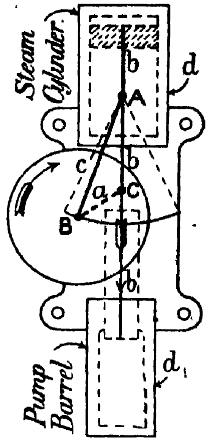


FIG. 482.

The only other possible inversion of the slider-crank chain is that obtained by fixing the block d. This inversion, called the *swinging slider-crank*, is not very important, but one interesting application of it is found in the pendulum pump, shown in Fig. 482. The block d has become the steam cylinder, pump barrel, and frame. The link b has become the piston, piston-rod, and plunger. The connecting-rod c swings about a pin A fixed on the side of the steam cylinder. The crank a has become a fly-wheel, which rotates about a pin B attached to the lower end of the swinging link c, and it also rotates about a pin C, which is attached to the sliding link b. The stroke of the piston and plunger is evidently twice the radius of the crank a.

Exercises XVIII.

1. Construct the piston velocity diagrams, as shown in Fig. 467, p. 300, for the following cases:—(1) $l/r=\infty$, (2) $l/r=4\cdot5$, (3) $l/r=2$, where l = length of connecting-rod, and r = radius of crank. The three sets of diagrams to be drawn on the same corresponding bases, or, in the case of the polar diagrams, from the same pole, in order to show the differences due to variations in the value of l/r. Take $r=10$ inches, velocity of crank pin 10 feet per second, and linear scale $\frac{1}{4}$. Construct on the diagrams the velocity scale, showing feet per second. Take from the diagrams the values of v in feet per second when $\theta=45°$, and state the results.

2. In a direct-acting engine mechanism the radius of the crank is 10 inches, and the velocity of the crank pin 10 feet per second, find, by calculation, the answers to the queries in the following table:—

l (inches)	∞	∞	45	45	45	45	20	20	20	20
θ (degrees)	30	...	30	150	...	..	30	150	...	...
x (inches)	?	18	?	?	2	18	?	?	2	18
v (feet per second)	?	?	?	?	?	?	?	?	?	?

where l is the length of the connecting-rod, θ the angle between the crank and

the inner dead centre radius, x the distance of the piston from the outer end of its stroke, and v the velocity of the piston.

3. When the length of the connecting-rod is equal to that of the crank, show that the stroke of the piston is four times the length of the crank. Also, if the crank has uniform velocity, show that the piston has simple harmonic motion, and that the maximum velocity of the piston is twice the velocity of the crank pin.

4. In a direct-acting engine the connecting-rod is 50 inches, and the crank 10 inches long. If the crank makes 120 revolutions per minute, calculate the mean velocity of the piston, in feet per minute, also the velocity of the piston, in feet per minute, when the crank and connecting-rod are at right angles to one another.

5. Estimate the greatest and least forward velocity of the piston of a locomotive engine, relative to the rails, when the train is running at 50 miles per hour, the diameter of the driving wheels being 66 inches, the length of stroke 27 inches, and the length of the engine connecting-rod 54 inches. [Inst.C.E.]

6. Construct the piston acceleration diagrams, as shown in Fig. 474, p. 303, for the following cases:—(1) $l/r=\infty$, (2) $l/r=4{\cdot}5$, (3) $l/r=2$, where l=length of connecting-rod, and r=radius of crank. The three sets of diagrams to be drawn on the same corresponding bases, or, in the case of the polar diagrams, from the same pole, in order to show the differences due to variations in the value of l/r. Take $r=10$ inches, $V=10$ feet per second, and linear scale $\frac{1}{4}$. Construct on the diagrams the acceleration scale, showing feet per second per second. Take from the diagrams the values of f, the piston acceleration, in feet per second per second, when $\theta=30^\circ$, and state the results.

7. Same as preceding exercise, but for the following cases:—(1) $l/r=1{\cdot}1$, (2) $l/r=1{\cdot}144$, (3) $l/r=1{\cdot}2$.

8. If the acceleration of a piston is 350 feet per second per second when it has moved 4 inches from one end of its stroke, which is 24 inches, at what speed is the crank shaft running, in revolutions per minute? Assume an infinite connecting-rod.

9. In a direct-acting engine, l=length of connecting-rod, r=radius of crank, $n=l/r$, x=distance of piston from outer end of stroke, V=velocity of crank pin, v=velocity of piston, and f=acceleration of piston. Show that—

(1) when the crank is perpendicular to the line of stroke,

$$\frac{x}{r}=n+1-\sqrt{n^2-1},\quad v=V,\quad \text{and } f=\mp\frac{V^2}{r}\cdot\frac{1}{\sqrt{n^2-1}};$$

(2) when the crank is at right angles to the connecting-rod,

$$\frac{x}{r}=n+1-\sqrt{n^2+1},\quad \frac{v}{V}=\frac{\sqrt{n^2+1}}{n},\quad \text{and } f=\pm\frac{V^2}{r}\cdot\frac{\sqrt{n^2+1}}{n^4};$$

(3) when $x=r$, $\dfrac{v}{V}=\dfrac{n\sqrt{4n^2-1}}{2n^2-1}$, and $f=\mp\dfrac{V^2}{r}\cdot\dfrac{n(4n^4-6n^2+1)}{(2n^2-1)^3}$.

The upper sign in the value of f in each case applying to the "in" stroke, and the lower sign to the "out" stroke.

10. If ω is the angular velocity, and α the angular acceleration of the connecting-rod, then, using the notation of Exercise 9, show that (1) when the crank and connecting-rod are in a straight line $\omega=V/nr$, and $\alpha=0$; (2) when the crank is perpendicular to the line of stroke, $\omega=0$, and $\alpha=-\dfrac{1}{\sqrt{n^2-1}}\cdot\dfrac{V^2}{r^2}$; and (3) when the crank is perpendicular to the connecting-rod,

$$\omega=\frac{V}{n^2r},\quad \text{and}\quad \alpha=\frac{n^4-1}{n^5}\cdot\frac{V^2}{r^2}.$$

11. Construct the connecting-rod angular velocity diagrams, as shown in Fig. 477, p. 306, for the following cases:—(1) $l/r=4{\cdot}5$, (2) $l/r=2$, (3) $l/r=1$, where l=length of connecting-rod, and r=radius of crank. Take $r=10$ inches, velocity of crank pin 10 feet per second, and linear scale $\frac{1}{4}$. Construct the angular velocity scale, showing radians per second. Take from the diagrams the values of ω in radians per second when $\theta=30^\circ$, and state the results.

12. In an ordinary steam-engine the stroke is 18 inches, the length of the connecting-rod is 36 inches, and the revolutions are 400 per minute. The

diameters of the crank shaft journal, the crank pin, and the cross-head pin are $7\frac{1}{4}$, $7\frac{3}{4}$, and $5\frac{1}{2}$ inches respectively. Find the velocity of the piston and the velocity of rubbing of each journal in feet per minute in the position of the mechanism, for which the crank arm has turned through an angle of 30° from the inner dead centre. [U.L.]

13. Taking the same cases and the same particulars as in Exercise 11, construct the connecting-rod acceleration diagrams, and construct the acceleration scale, showing radians per second per second. Take from the diagrams the values of a in radians per second per second when $\theta = 75°$, and state the results.

14. In a direct-acting engine the line of stroke is at a perpendicular distance of 4 inches from the axis of the crank shaft. If the radius of the crank is 8 inches, and the length of the connecting-rod is 30 inches, find the length of the piston stroke. On the stroke of the piston as base, construct the piston velocity and piston acceleration curves for both the forward and return strokes. The speed of the engine being 200 revolutions per minute, construct the velocity and acceleration scales.

15. The table of a small planing machine is driven from a crank through a connecting-rod, which is 9 inches long. The axis of the crank shaft is 3 inches below the line of stroke. If the stroke of the table is 6 inches, find the radius of the crank. Construct the velocity curves for both the cutting and return strokes of the table, on a stroke base, the crank rotating uniformly at 20 revolutions per minute. What are the velocities of the table, in feet per minute, at mid-stroke (a) when cutting, (b) when returning? Also, what is the time ratio of the cutting and return strokes.

16. In an oscillating engine the piston has a stroke of 6 feet, and the distance between the axis of the trunnions and the axis of the shaft is 10 feet 6 inches. The shaft makes 35 revolutions per minute. Find (a) the maximum angular velocity of the cylinder in radians per second. (b) the piston speed in feet per minute at mid-stroke, and (c) the mean piston speed in feet per minute.

17. Fig. 483 shows the swinging-block slider-crank chain as applied to a shaping machine. The pinion E drives the wheel F, which rotates on the fixed pin B, and carries the pin C. The pin C carries the block b, which works in the slot of the lever dd, which oscillates about the fixed pin A. The upper end of the lever dd carries a pin H, from which a connecting-rod transmits motion to the ram carrying the cutting tool. The stroke of the ram is varied by altering the distance of the pin C from B. For the given dimensions find the length of the stroke of the ram. Construct on a stroke base the velocity curves for the cutting and return strokes. The wheel F makes 20 revolutions per minute. What is the time ratio of the cutting and return strokes?

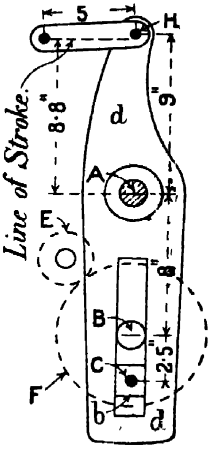


FIG. 483.

18. Referring to the Whitworth quick return motion, shown in Fig. 481, p. 308, $BC = 1\frac{1}{4}$ inches, $AB = 5$ inches, $CP = 5$ inches, and the connecting-rod to the ram is 16 inches long. The line of stroke passes through C, and is perpendicular to BC. The driving wheel makes 15 revolutions per minute. Construct on a stroke base the tool velocity curves for the cutting and return strokes. What is the time ratio of the cutting and return strokes?

19. A and B (Fig. 484) are fixed centres. The crank BC revolves uniformly with an angular velocity of 10 radians per second about the centre B. The end C is pivoted to a block, which can slide along AD. AD revolves about the centre A. The point E moves along EA. Determine the velocity of sliding at both C and E when BC is at right angles to AB, and find also the maximum velocity of E. Show how the mechanism can be applied as a quick return motion for a shaping machine, and determine the ratio between the times of cutting and return. [U.L.]

20. The crank AB (Fig. 485) rotates uniformly at 150 revolutions per minute. The end D of the rod BD is constrained to move in the straight line GH. The end E of the rod CE moves on the straight line EK. Determine the velocity of the point E for the given position of the mechanism. Indicate how you would determine the acceleration of the point E. [U.L.]

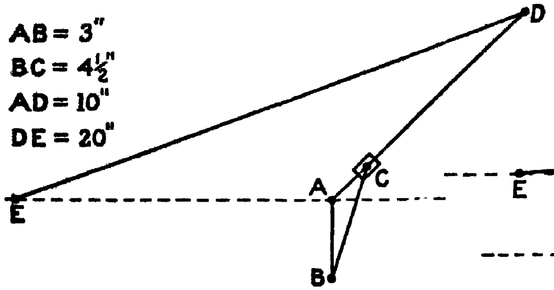


FIG. 484.

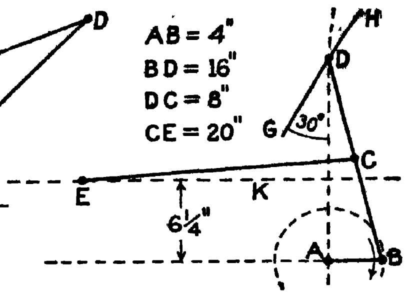


FIG. 485.

21. In a horizontal steam-engine the indicator reducing gear consists of a radial arm AB suspended from a fixed centre A above the line of stroke; the end B of the radial arm is connected to the cross-head by a link BC, and the cord passes off from a sector fixed to the arm AB, its centre being A and the radius AD. If, for any position of the gear, a vertical line drawn through A cuts the link BC in N, show that $\frac{AN}{AD}$ gives the ratio of the piston speed to the cord speed. [U.L.]

22. In the four-bar mechanism shown in the sketch (Fig. 486), the bar A is a fixed bar; the bars B and D rotate about the fixed centres OAB and OAD, and they are coupled together at their outer ends by the bar C; the bar B revolves in a clockwise direction, with uniform velocity round its fixed axis OAB at 50 revolutions per minute. Find in any way you please the positions of the bar D for twelve equidistant positions of the bar B during one complete revolution, and draw up a table showing (1) angle turned through by the bar B, in degrees;

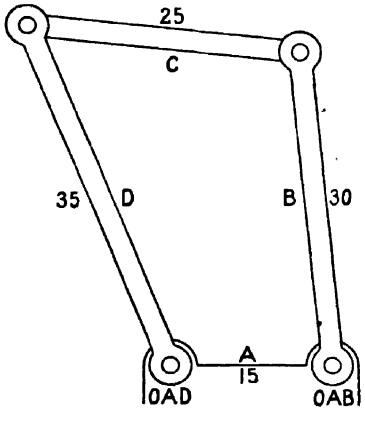


FIG. 486.

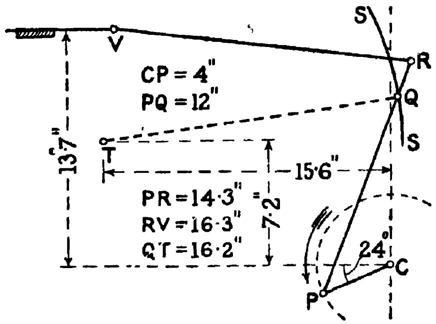


FIG. 487.

(2) angle turned through by the bar D, in degrees; (3) mean angular velocity of the bar D, in radians per second, during each interval; (4) mean angular acceleration of the bar D, in radians per second per second. Draw curves showing angular velocity and angular acceleration at any time. The lengths of the bars are 15, 30, 25, and 35 inches respectively. [B.E.]

23. The diagram (Fig. 487) shows a radial valve gear. The crank CP turns uniformly at 12 radians per second, and is pinned at P to the rod PR, the point Q in this rod being guided in the circular path SS, centre T. For the position of the mechanism shown in the diagram, determine and measure the velocities and accelerations of the points R and V. [B.E.]

CHAPTER XIX

PISTON AND CRANK EFFORT DIAGRAMS

271. Piston Effort Diagrams.—An engine or machine worked by fluid pressure has usually a piston or ram which receives reciprocating motion in a cylinder. When the piston or ram is single-acting, the pressure of the fluid introduced into the cylinder causes the piston or ram to move outwards, and the return stroke is usually performed either by the energy stored up in a fly-wheel, or by the pressure of the fluid in another cylinder, through intermediate mechanism. In this case the effort on the piston or ram at any instant is simply the force exerted on it by the fluid in the cylinder at that instant. When the piston is double-acting, the fluid is admitted into the cylinder on opposite sides of the piston alternately, and after doing its work it is allowed to escape. In this case the effort on the piston at any instant is the difference between the forces exerted by the fluid on the opposite sides of the piston at that instant.

When the fluid used is water, the pressure which it exerts is practically constant throughout the stroke, and the effort is therefore constant, and the effort diagram is a rectangle whose length represents the length of the stroke of the piston or ram.

In heat engines and in machines worked by compressed air the pressure of the fluid is generally variable throughout the stroke. In such cases the actual effort on the piston is obtained from *indicator diagrams*, which are simply the records of self-registering pressure-gauges, which show the pressure of the fluid at every point of the stroke of the piston. If the engine is double-acting two indicator diagrams are required, one for each side of the piston. The indicator diagram shows the intensity of the pressure of the fluid, generally in lbs. per square inch.

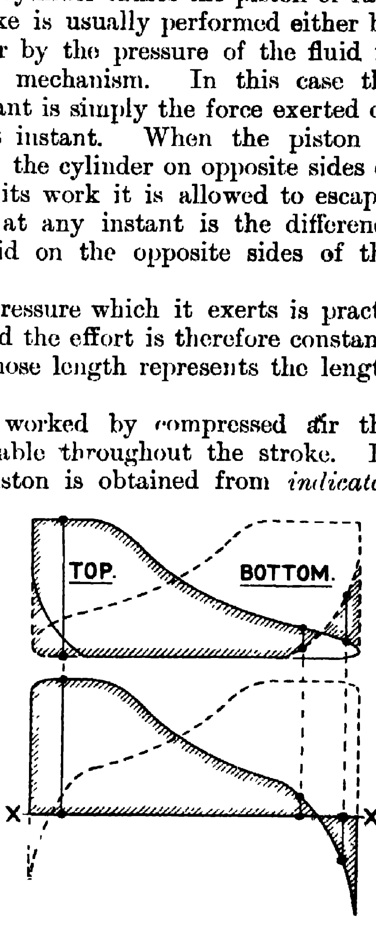


FIG. 488.

Let p_1 and p_2 be the intensities of the pressures on the front and back of the piston respectively at any instant, and let a_1 and a_2 be the effective areas of the front and back of the piston respectively, then the effort on the piston at the instant considered is $p_1a_1 - p_2a_2$. If $a_1 = a_2 = a$, then the effort is $a(p_1 - p_2)$. In double-acting engines a_1 is not generally equal to a_2 on account of the presence of the piston-rod on one side.

The upper part of Fig. 488 shows a pair of indicator diagrams from the cylinder of a vertical steam-engine. The full line diagram is from the top end, while the dotted line diagram is from the bottom end of the cylinder.

The effective pressure on the piston at any point of the stroke is shown by the vertical distance between the top of one diagram and the bottom of the other at that point. If this vertical distance be plotted on a straight base for a sufficient number of points in the stroke, a diagram is obtained which shows more clearly the effort on the piston during the stroke. In Fig. 488 the full line diagram on the base XX is the effort diagram for the down stroke, while the dotted line diagram is the effort diagram for the up stroke. Where the effort is negative, the diagram is below the base XX.

272. Reduction of Indicator Diagrams to same Effort Scale.—It was stated in the preceding Article, in referring to the indicator diagrams given in Fig. 488, that the effective pressure on the piston at any point of the stroke is shown by the vertical distance between the top of one diagram and the bottom of the other at that point. This, however, is only true when the effective areas of the top and bottom of the piston are equal, and when the pressure scales of the two diagrams are the same. If the pressure scales are the same, but the areas are unequal, then either the ordinates of the diagram for the smaller area of piston must be reduced in the ratio of the smaller to the larger area, or the ordinates of the diagram for the larger area of piston must be enlarged in the ratio of the larger to the smaller area.

The diagrams of Fig. 488 are repeated in Fig. 489, and the diagram for the bottom of the piston is shown corrected to the thicker dotted line diagram to allow for the area of the piston-rod on the under side. The effective force on the piston at any point of the down stroke is now represented by the vertical distance between the top of the full line diagram and the bottom of the thicker dotted line diagram at that point, and the effective force at any point of the up stroke is represented by the vertical distance between the top of the thicker dotted line diagram and the bottom of the full line diagram at that point. If the original indicator diagrams are not to the same pressure scale, it will of course be necessary to bring them to the same scale, in addition to correcting one of them for the difference between the areas of the top and bottom of the piston.

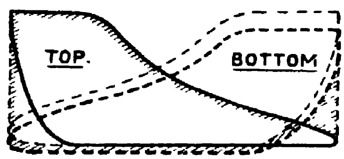


FIG. 489.

The indicator diagrams from the different cylinders of a compound or triple expansion engine are generally to different pressure scales; also when the strokes of the different pistons are the same, which is generally the case, their areas are different. Hence it is evident that in order that the effort diagram for one piston may be comparable with the effort diagram for another piston, the pressure scales must be the same, and their ordinates must be such as to give equivalent pressures on pistons of the same area.

Let A be an indicator diagram for one side of a piston, the effective area of that side being a_1, and let p_1 be the pressure scale of this diagram

in lbs. per square inch per inch. Also let B be an indicator diagram for the other side of the same piston, or for one side of another piston, the effective area of that side being a_2, and let p_2 be the pressure scale of this diagram in lbs. per square inch per inch. Then in constructing piston effort diagrams which shall be comparable, either the ordinates of B must be multiplied by $\frac{a_2p_2}{a_1p_1}$, or the ordinates of A must be multiplied by $\frac{a_1p_1}{a_2p_2}$.

273. Correction of Piston Effort Diagrams for Weight of Reciprocating Parts in Vertical Engines.—In a vertical engine the weight of the piston, piston-rod, cross-head, and a part of the connecting-rod increases the effort during the down stroke, and diminishes it during the up stroke by an amount equal to that weight. Let w equal the weight of the reciprocating parts in lbs. per square inch of piston, then the effort due to the fluid pressure per square inch of piston must be in-

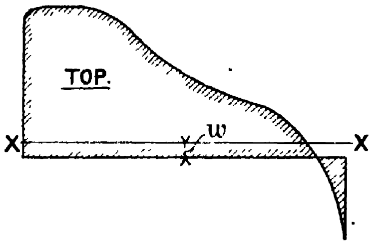


FIG. 490.

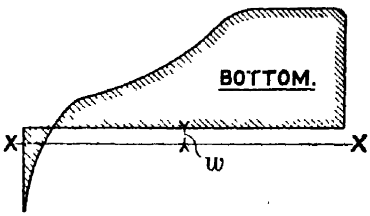


FIG. 491.

creased by an amount w for the down stroke, and the effort diagram is altered, as shown in Fig. 490, by lowering the base line from XX a distance equal to w on the force scale, and the effort diagram for the up stroke is corrected, as shown in Fig. 491, by raising the base line from XX an equal amount.

Frequently half the weight of the connecting-rod is reckoned as belonging to the reciprocating parts.

274. Forces due to Inertia of Reciprocating Parts.—The determination of the acceleration of the piston was fully discussed in Arts. 261 to 265. Let f denote the acceleration of the piston in feet per second per second, W the total weight of the reciprocating parts, in lbs., and P the force, in lbs., required to produce the acceleration f, then $\frac{P}{W}=\frac{f}{g}$, and $P=\frac{Wf}{g}$. It was shown in Art. 262 that, at the ends of the stroke, $f=\frac{V^2}{r}\left(1\pm\frac{1}{n}\right)$, where V is the velocity of the crank pin in feet per second, and n is the ratio of the length of the connecting-rod to r, the radius of the crank, the plus sign applying to the outer, and the minus sign to the inner, end of the stroke. Hence, at the ends of the stroke, $P=\frac{WV^2}{gr}\left(1\pm\frac{1}{n}\right)$. If w is the weight of the reciprocating parts, in lbs. per square inch of piston, and p is the accelerating force, in lbs. per

square inch of piston, then $p = \frac{wf}{g}$, and at the ends of the stroke $p = \frac{wV^2}{gr}\left(1 \pm \frac{1}{n}\right)$. After the point of zero acceleration is passed, the acceleration is of course negative, or the accelerating force reverses.

275. Correction of Piston Effort Diagrams for Inertia Forces.—From the beginning of the stroke of the piston up to the point of maximum velocity, or zero acceleration, part of the effort on the piston, as determined in preceding Articles, is required to accelerate the piston and the other reciprocating parts, and that part is therefore not available at the cross-head for transmission to the crank pin. The work done by that part of the steam pressure which is not transmitted to the cross-head is stored up in the reciprocating parts as kinetic energy. After the piston has reached its point of maximum velocity its velocity diminishes, and the kinetic energy in the reciprocating parts is given out, appearing as work done at the cross-head. During the latter part of the stroke, therefore, the effort due to the steam pressure is supplemented by the effort due to the retarding or negative accelerating force.

The necessary correction of the piston effort diagram due to the inertia forces is made as shown in Figs. 492 and 493, where the full line diagram

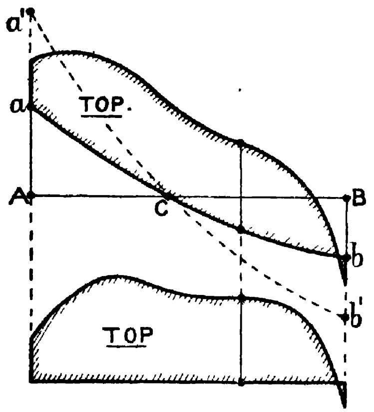


FIG. 492.

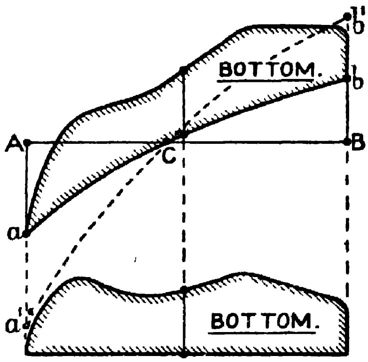


FIG. 493.

on the straight base AB is the piston effort diagram due to the steam pressure, and AaCbB is the accelerating force diagram on the same base AB. The curve a'Cb' is the curve got, say, by Klein's construction (Art. 261), and the curve aCb is obtained by altering the ordinates in the ratio of Aa' to Aa. The length Aa is measured with the pressure or effort scale to represent $p = \frac{wV^2}{gr}\left(1 + \frac{1}{n}\right)$, the accelerating force per square inch of piston at the beginning of the forward or "in" stroke. The curve aCb is the new base of the effort diagram. The corrected diagrams are shown constructed on straight bases below the others.

It is evident that the forces due to the inertia of the reciprocating parts do not affect either the work done or the mean effort during a complete stroke.

276. Crank Effort.—Referring to Fig. 494, if P is the effort on the

cross-head, Q, the thrust or pull on the connecting-rod, is equal to $P/\cos\phi$. At the crank pin the force Q produces a thrust or pull on the crank and a force T tangential to the path of the crank pin equal to $Q\sin(\theta+\phi)$.

Hence, $$T = \frac{P\sin(\theta+\phi)}{\cos\phi} = P(\sin\theta + \cos\theta\tan\phi).$$

If n is the ratio of the length of the connecting-rod to the radius of the crank, then $\tan\phi = \frac{\sin\theta}{\sqrt{n^2-\sin^2\theta}}$, and therefore

$$T = P\left\{\sin\theta + \frac{\sin\theta\cos\theta}{\sqrt{n^2-\sin^2\theta}}\right\} = P\left\{\sin\theta + \frac{\sin 2\theta}{2\sqrt{n^2-\sin^2\theta}}\right\}.$$

T is called the *crank pin effort*. The moment of T about C, namely, Tr, where r is the radius of the crank, is called the *crank effort*, but as r is constant, it follows that the crank effort is proportional to T. If Cb be made equal to P, and bd be drawn parallel to AB to meet Cd at d, where Cd is perpendicular to the line of stroke, then

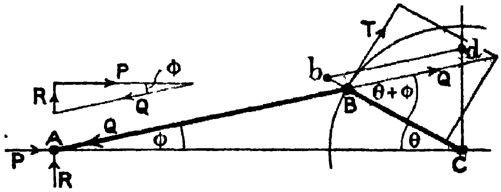


FIG. 494.

$$\frac{Cd}{Cb} = \frac{\sin(\theta+\phi)}{\sin(90^\circ-\phi)} = \frac{\sin(\theta+\phi)}{\cos\phi},$$

but $\frac{T}{P} = \frac{\sin(\theta+\phi)}{\cos\phi}$, therefore $\frac{Cd}{Cb} = \frac{T}{P}$. Hence, since C$b$ is equal to P, Cd must be equal to T. The construction for determining the crank effort from the piston or cross-head effort is therefore extremely simple, and if it be compared with the construction proved in Art. 260 for finding the piston velocity from the crank pin velocity, it will be seen that the constructions are identical. In fact, the construction and formula for the crank effort may be deduced at once from the construction and formula for piston velocity by the principle of work.

277. Crank Effort Diagrams.—The construction of diagrams which shall show the crank effort for any position of the crank will be readily understood by reference to Fig. 495. In the polar curves of crank effort, the effort found by the construction or by the formula of the preceding Article is marked off, either on the crank from the centre of the crank shaft, or on the crank produced from the path of the crank pin. The most useful crank effort diagram is the "rectangular diagram," in which the base is a straight line representing the circumference of the circle described by the crank pin, and the ordinates, perpendicular to that base, represent the crank effort.

If the base of the rectangular diagram of crank effort be made equal to the circumference of the circle described by the crank pin, then, friction being neglected, the area of the crank effort diagram for one revolution will be equal to the sum of the areas of the piston effort diagrams, but practically all that is to be learned from the rectangular crank effort diagram can be learned from it, whatever be the length taken for the base.

The principal use of the rectangular crank effort diagram is to show

the fluctuation of energy, which is discussed in the next Article, and for this the length of the base is immaterial.

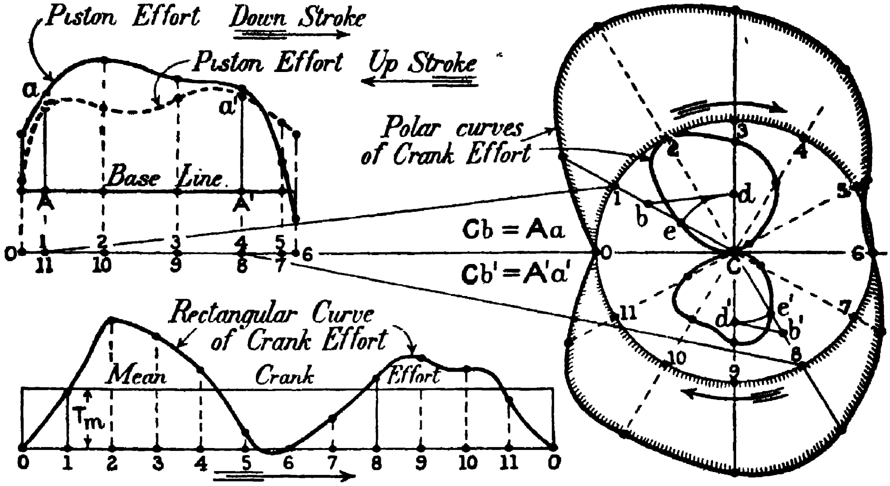


FIG. 495.

The maximum crank effort can evidently be found from either the polar or rectangular curves. The maximum crank effort is also the maximum torque on the crank shaft, and this is of great importance in designing the shaft.

If T_m is the mean effort on the crank pin and P_m is the mean effort on the piston, during one revolution, then, since the work done at the crank pin is equal to the work done on the piston in the same time, friction being neglected,

$$2\pi r T_m = 2P_m \times 2r, \text{ or } T_m = \frac{2P_m}{\pi}.$$

When there are two or more cranks on a shaft, the total turning effort on the shaft at any instant is the sum of the turning efforts on the separate cranks at that instant, and the total effort may be considered as acting on any one of the cranks. Hence a diagram of total turning effort may be constructed by adding to the ordinates of the effort diagram for one crank the corresponding ordinates of the effort diagrams for the other cranks, corresponding ordinates being those which show the efforts on the separate cranks at the same instant.

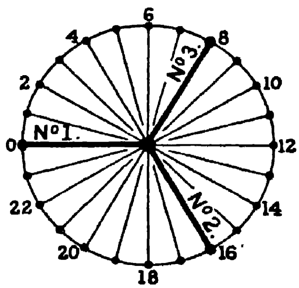


FIG. 496.

Fig. 496 shows the relative positions of three cranks on the same shaft, and Fig. 497 shows how the rectangular crank effort diagrams for these three cranks may be combined to give a total turning effort diagram. It will be observed in Fig. 497, that in order to bring the corresponding ordinates together the effort diagrams for cranks No. 2 and No. 3 have been moved forward distances corresponding to the respective angles which these cranks would have to move through to overtake No. 1 crank. It is obvious that the crank

effort diagrams for the separate cranks must be to the same effort scale before they can be combined into one effort diagram in the manner

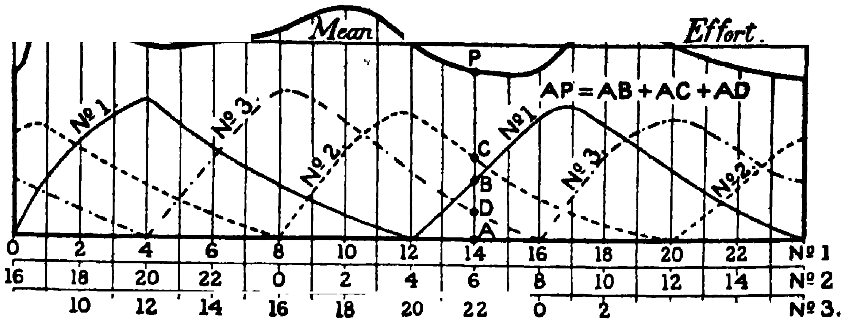


FIG. 497.

shown in Fig. 497. The mean total effort is of course equal to the sum of the mean efforts for the separate cranks.

278. Fluctuation of Energy.—When the direct-acting engine mechanism is used to transmit the work done on a piston to a shaft, the turning effort on the shaft is very variable when only one crank is used, and when two or more cranks, inclined to one another, and connected to different pistons are used, the turning effort on the shaft, although much more nearly uniform, is still variable. This want of uniformity in the turning effort on the crank shaft is a characteristic of all heat engines having reciprocating pistons, and the result of this is that, except in the very improbable case in which the moment of the resistance to the turning of the shaft varies so that at every instant it is equal to the turning moment, the supply of energy to the shaft over certain intervals must be greater, while over other intervals the supply must be less than that required by the resistance.

In most cases in practice the resistance to the rotation of the crank shaft of an engine may be considered to be uniform during a complete period or cycle, and the resistance reduced to the crank pin may therefore be considered as equal to the mean effort on the crank pin during a period or cycle.

Fig. 498 shows a rectangular diagram of crank effort on a base OX,

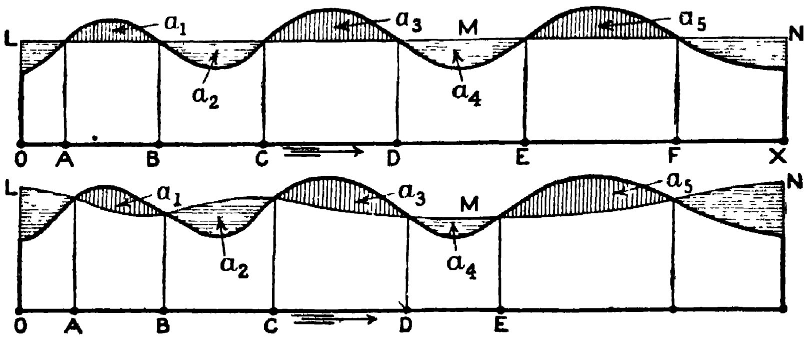


FIG. 498.

representing the path of the crank pin, and the ordinates of the line LMN

represent the resistance reduced to the crank pin. In the upper part of Fig. 498, LMN is a straight line parallel to OX, while in the lower part LMN is a curved line. In each case the work done by the effort is represented by the area between the effort curve and the base, and the work done on the resistance is represented by the area between the resistance line and the base. It will be noticed that the points A, B, C, D, E, and F are the points where the effort is equal to the resistance.

Let K denote the kinetic energy in the moving parts when the crank pin is at A, then while the crank pin moves from A to B the work done by the effort is greater than that required by the resistance by the amount represented by the area a_1, and therefore the kinetic energy in the moving parts when the crank pin reaches B is $K + a_1$. Again, while the crank pin moves from B to C the work done by the effort is less than that required by the resistance by the amount represented by the area a_2, and therefore the kinetic energy in the moving parts when the crank pin reaches C is $K + a_1 - a_2$. Similarly, the values of the kinetic energy in the moving parts when the crank pin reaches D, E, and F are, $K + a_1 - a_2 + a_3$, $K + a_1 - a_2 + a_3 - a_4$, and $K + a_1 - a_2 + a_3 - a_4 + a_5$ respectively. Between O and X the velocity of the crank pin will be a maximum at that point where the kinetic energy of the moving parts is greatest, and the velocity will be a minimum at that point where the kinetic energy is least.

The difference between the kinetic energy of the moving parts at the points of maximum and minimum speed is called the *fluctuation of energy*. The ratio which the fluctuation of energy bears to the work done per cycle is called the *coefficient of fluctuation of energy*. In an ordinary steam-engine the cycle takes place in one revolution, while in an internal combustion engine working on the Otto cycle, the cycle covers two revolutions of the crank shaft.

Referring to Fig. 498, suppose that OX represents the distance travelled by the crank pin during one cycle, and suppose that F is the point of maximum speed, and C the point of minimum speed Let the area between the effort curve and the base equal a, then the fluctuation of energy is represented by $a_3 - a_4 + a_5$, and the coefficient of fluctuation of energy is equal to $\dfrac{a_3 - a_4 + a_5}{a}$.

279. Fluctuation of Energy in Gas-Engines.—In a single-cylinder, single-acting gas-engine working on the "Otto cycle," the operations performed during a cycle are as follows:—

First Stroke.—The piston moves outwards, and draws in the charge of air and gas. This is the charging or suction stroke.

Second Stroke.—The piston moves inwards and compresses the charge. This is the compression stroke.

Third Stroke.—The compressed charge is ignited, an explosion takes place, and the piston is driven outwards by the expansive force of the products of combustion This is the working stroke.

Fourth Stroke.—The piston moves inwards and expels the products of combustion. This is the exhaust stroke.

The indicator diagram is shown in Fig. 499, but the suction and exhaust pressures are shown exaggerated for the sake of clearness. Fig. 500 shows the diagram as continuous on a four-stroke base.

All the work delivered to the crank shaft during a cycle is delivered during the working stroke, and the work done in the cylinder during the other strokes comes from the fly-wheel.

The fluctuation of energy is obtained from the rectangular crank

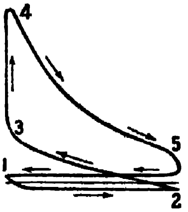


FIG. 499.

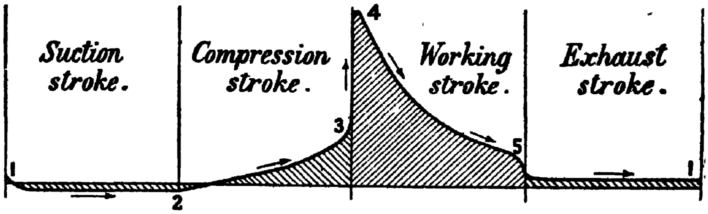


FIG. 500.

effort diagram as in a steam-engine, but the diagram must be constructed for a complete cycle. The net work done on the useful resistance at the crank shaft and on the friction of the engine is represented by the shaded area in the working stroke in Fig. 500, minus the shaded areas in the other strokes. The maximum speed of the crank shaft is approximately at the end of the working stroke, and the minimum speed is approximately at the beginning of that stroke. Hence the fluctuation of energy is approximately equal to the work done during the working stroke, minus 1-nth of the net work done during a cycle, where n is the number of strokes during a cycle.

If the engine is governed on the "hit or miss" principle, the governor acts by cutting off the gas, and there is no explosion and no effective work done for at least two revolutions after the completion of an effective cycle. The complete cycle then takes a number of revolutions, which is a simple multiple of two.

280. Fly-wheels.—The function of a fly-wheel is to reduce the fluctuation of speed due to the fluctuation of energy during the period or cycle of the working of a machine. If over an interval the supply of energy to a machine is greater than the resistance requires, the moving parts increase in speed, and their kinetic energy therefore increases by an amount equal to the surplus energy: and if over another interval the supply of energy is less than the resistance requires, the moving parts decrease in speed, and their kinetic energy therefore decreases by an amount equal to the deficiency in the supply of energy. In most cases where a fly-wheel is used it is usual to neglect the kinetic energy of all the moving parts other than the fly-wheel, so that over any interval the difference between the energy supplied and the energy required is equal to the change in the kinetic energy of the fly-wheel.

If R is the radius of gyration of the fly-wheel, in feet; v the velocity, in feet per second, of a point at a distance R from the axis; ω the angular velocity in radians per second; N the speed in revolutions per minute; W the weight of the wheel in lbs.; and K, its kinetic energy in ft.-lbs., then

$$K = \frac{Wv^2}{2g} = M_1 v^2 = \frac{WR^2\omega^2}{2g} = M_2\omega^2 = \frac{W \times 4\pi^2 R^2 N^2}{2 \times 60^2 g} = MN^2,$$

where M_1, M_2, and M are constants for a given wheel. The kinetic

energy of a given wheel is therefore equal to the square of the speed, in whatever way that speed may be stated, multiplied by a constant, and for certain problems this simple rule is useful.

If I is the moment of inertia of the wheel, in lb. and foot units, then $I = WR^2$, and from this and the foregoing formulæ the following formulæ are readily deduced, namely, $K = \frac{Iv^2}{2gR^2} = \frac{I\omega^2}{2g} = \frac{2\pi^2 IN^2}{60^2 g}$.

If during a period or cycle of the working of a machine the minimum and maximum speeds of the fly-wheel are N_1 and N_2 revolutions per minute respectively, then the fluctuation of energy is $\frac{2\pi^2 I}{60^2 g}\left(N_2^2 - N_1^2\right)$.

The difference between the maximum and minimum speeds is called the *fluctuation of speed*, and the ratio of the fluctuation of speed to the mean speed is called the *coefficient of fluctuation of speed*. If N is the mean speed in revolutions per minute, and c is the coefficient of fluctuation of speed, then $c = \frac{N_2 - N_1}{N}$. It is usual to assume that the mean speed is the arithmetical mean of the maximum and minimum speeds, so that $2N = N_2 + N_1$. Hence $N_2^2 - N_1^2 = (N_2 + N_1)(N_2 - N_1) = 2cN^2$.

If U denotes the work done per period or cycle in ft.-lbs., and k denotes the coefficient of fluctuation of energy, then the fluctuation of energy is kU, and $kU = \frac{4\pi^2 WR^2 cN^2}{60^2 g} = \frac{4\pi^2 IcN^2}{60^2 g}$.

If H is the horse-power of an engine, then the work per revolution is $\frac{33000H}{N}$, where N is the speed in revolutions per minute.

The following are some values of c, the coefficient of fluctuation of speed, found in practice :—

Pumps, and shearing and punching machines	0·05 to 0·03
Flour-mills	0·04 to 0·03
Looms, paper-making machines, and ordinary machine tools	0·03 to 0·025
Spinning machinery	0·02 to 0·01
Dynamos	0·007

Exercises XIX.

1. The piston of a steam-engine is 30 inches in diameter, and the stroke is 40 inches. Instead of a piston-rod there is a trunk 12 inches in diameter which works through the front end of the cylinder. The indicator diagrams for this engine are given in Fig. 501. The full line diagram is from the back end, and the dotted line diagram from the front end of the cylinder. The pressures marked are in lbs. per square inch. Reproduce these diagrams, making the length 5 inches, and the pressure scale 1 inch to 20 lbs. per square inch. Reconstruct the diagrams on a straight base to show *effective pressure* on the piston, *in lbs. per square inch of the larger face of the piston.*

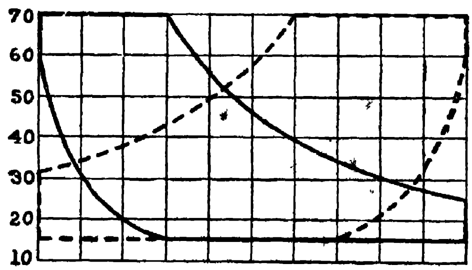


FIG. 501.

What is the effective pressure in lbs. per square inch of the larger face of the

piston at the middle of the forward stroke? Compute the horse-power of this engine when the speed is 80 revolutions per minute.

2. Indicator diagrams from the cylinders of a horizontal tandem compound steam-engine are given in Fig. 502. Diameter of H.P. cylinder, 24 inches. Diameter of L.P. cylinder, 46 inches. Stroke of pistons, 6 feet. Diameter of

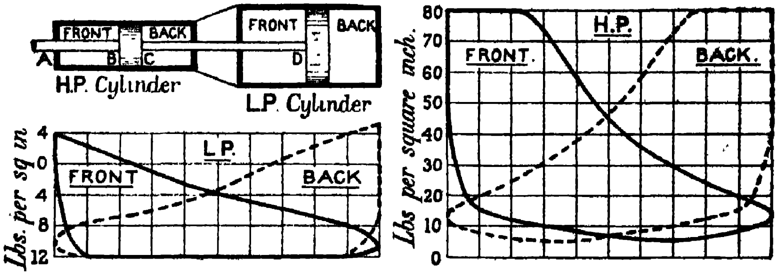


FIG. 502.

piston-rod AB, $5\frac{1}{4}$ inches. Diameter of piston-rod CD, $4\frac{3}{4}$ inches. Construct on a straight base 4 inches long the combined piston effort diagrams for the forward and return strokes, showing the combined effort on the two pistons per square inch of the back of the low-pressure piston. Effort scale, 1 inch to 30 lbs. Compute the horse-power of this engine when the speed is 50 revolutions per minute.

3. The piston of an engine, and all the parts rigidly connected to it, weigh 400 lbs., and the stroke is 20 inches. The crank shaft makes 150 revolutions per minute. Assuming an infinite connecting-rod, determine the difference between the total effective pressure on the piston and the thrust on the cross-head pin, (a) at the beginning of the stroke, (b) at 5 inches from the beginning of the stroke.

4. In a steam-engine the piston at the beginning of its stroke is exposed to a total pressure of 2000 lbs, but the inertia is such that the thrust of the piston-rod at the cross-head is only 1600 lbs. The speed of the engine is now raised until it becomes half as great again as before, while the pressure is unchanged; what is the thrust of the piston-rod? [Inst.C.E.]

5. In the engine referred to in Exercise 2, the total weight of the reciprocating parts is 6700 lbs. The length of the connecting-rod is 15 feet, and the speed of the crank shaft 50 revolutions per minute. Construct on a stroke base 4 inches long the diagram of accelerating force per square inch of the back of the low-pressure piston, the force scale to be 1 inch to 30 lbs.

6. In a direct-acting steam-engine the stroke is 2 feet. the connecting-rod 4 feet long, the piston 14 inches diameter, the weight of the reciprocating parts 300 lbs., and the revolutions 180 per minute. At the commencement of the down stroke the difference of pressure per square inch on the two sides of the piston is 40 lbs. (acting downwards); at the end of the down stroke the difference is 10 lbs. (acting upwards). Find the effective pressure transmitted to the crank pin in these positions. If the steam pressure remained unaltered, at what speed would the engine have to run in order to make the effective pressure at the end of the stroke zero, and what would then be the effective pressure at the commencement of the stroke? [U.L.]

7. Construct the polar and rectangular diagrams of crank effort for a direct-acting steam-engine in which the effective pressure on the piston is 50 lbs. per square inch throughout each stroke, and determine the coefficient of fluctuation of energy, (a) assuming an infinite connecting-rod, (b) taking the length of the connecting-rod 5 times the length of the crank.

8. To the left of Fig. 503 are shown the piston effort diagrams for a direct-acting steam-engine, the pressures being in lbs. per square inch. Construct the polar and rectangular diagrams of crank effort, and find the coefficient of fluctuation of energy, also the ratio of the maximum torque to the mean torque

on the crank shaft, (*a*) with infinite connecting rod, (*b*) with connecting-rod 45 inches long.

9. Referring to Fig. 503, calculate T, the effort on the crank pin per square inch of piston when $\theta=75°$, when $\theta=135°$, and when $x=5$ inches, assuming an infinite connecting-rod.

10. Same as preceding exercise for both forward and return strokes, but taking the connecting-rod 45 inches long.

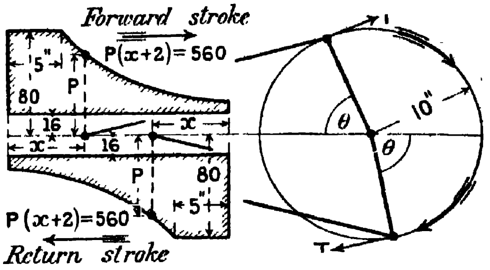


FIG. 503.

11. The cylinder of a vertical steam-engine is 45 inches diameter, and the stroke is 4 feet. The connecting-rod is 8 feet long, and the effective weight of the reciprocating parts is 10,000 lbs. The speed is 100 revolutions per minute. When the crank is 30 degrees from the top dead point the steam pressure on the top of the piston is 190 lbs. per square inch, and on the bottom 85 lbs. per square inch. Find the effective force transmitted along the piston-rod and the turning moment on the crank shaft when the crank is in the above position. [U.L.]

12. Construct the rectangular diagram of combined crank effort for a two-cylinder engine, the cylinders being of equal size, and the cranks at right angles to one another. The piston effort diagrams are given in Fig 503, and the connecting-rods are 45 inches long Find the coefficient of fluctuation of energy for this engine under these conditions.

13. Same as Exercise 12, except that the inertia of the reciprocating parts is to be taken into account, the weight of these parts being 3 lbs. per square inch of piston. The engine is a horizontal one, running at 150 revolutions per minute. Stroke of piston, 20 inches.

14. Considering Fig 503 to refer to a vertical engine in which the weight of the reciprocating parts is 3 lbs. per square inch of piston. Construct the rectangular diagram of crank effort, taking into account the weight and inertia of the reciprocating parts, and find the coefficient of fluctuation of energy. Length of connecting-rod, 45 inches. Speed, 130 revolutions per minute.

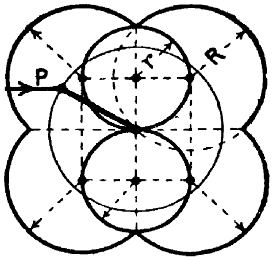
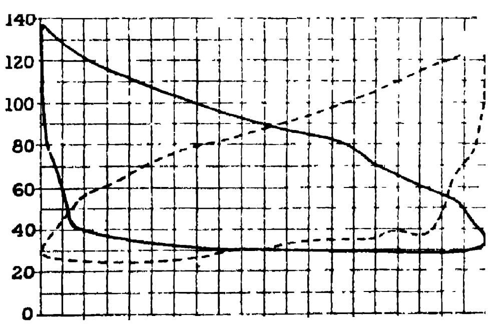


FIG. 504.

15. Show, (*a*) that with constant pressure P on the piston and infinite connecting-rod the polar crank effort diagrams for one revolution are two circles of radii $r=\frac{1}{2}P$, as shown in Fig. 501; (*b*) that with two cylinders of the same size constant pressure P on each piston, infinite connecting-rods, and two cranks at right angles, the polar diagram of combined crank effort for the two cranks for one revolution is bounded by four arcs of circles of radii $R=r\sqrt{2}$, the centres of the circles of radii R being situated at the corners of a square of side $=2r$, as shown in Fig. 504.

16. The following particulars * relate to a vertical triple expansion steam-engine: Diameters of cylinders, 18, 27, and 44 inches. Diameter of piston-rods, 4·75 inches. Stroke of pistons, 16 inches. Length



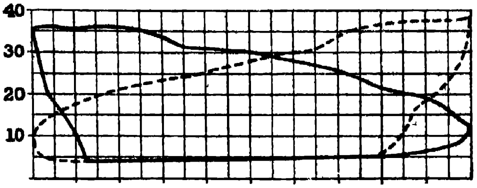
High Pressure.

FIG. 505.

* Kindly supplied by the makers of the engine, Messrs. W. H. Allen, Son, and Co., Bedford.

of connecting-rods, 48 inches. Weight of reciprocating parts (including piston, piston-rod, and cross-head), high pressure, 1082 lbs.; intermediate pressure, 1068 lbs.; low pressure, 1523 lbs. Weight of each connecting-rod, 953 lbs. Angles between cranks, 120°. Sequence of cranks, (1) high pressure, (2) intermediate pressure, (3) low pressure. Speed, 275 revolutions per minute.

Indicator diagrams taken from the engine at ¾ load are given in Figs. 505, 506, and 507. The pressures are in lbs. per square inch above or below the pressure of the atmosphere. The dotted line diagrams are for the under sides of the pistons.

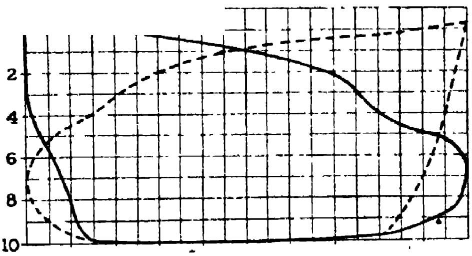


Intermediate Pressure.

FIG. 506.

(*a*) Carefully enlarge the indicator diagrams, making the length of each 4 inches, and take for pressure scales, 1 inch to 30 lbs. per square inch for the high pressure, 1 inch to 10 lbs. per square inch for the intermediate pressure, and 1 inch to 5 lbs. per square inch for the low pressure diagrams.

(*b*) Reconstruct all the diagrams except that for the top of the high pressure piston to show pressures per square inch of the top of the high pressure piston, to a scale of 1 inch to 30 lbs. per square inch. [For example, the area of the bottom of the intermediate pressure piston is 2·18 times the area of the top of the high pressure piston, therefore the heights of the diagram for the bottom of the intermediate pressure piston must be enlarged 2·18 times to correct for area of piston, and they must be reduced in the ratio of 30 to 10 to correct for pressure scale. The height of the resulting diagram will therefore be $2{\cdot}18 \div 3$, or 0·73 of the heights of the corresponding diagram in (*a*).]



Low Pressure.

FIG. 507.

(*c*) Reconstruct the diagrams in (*b*) on a straight base to show *effective pressures* on the respective pistons.

(*d*) Correct the diagrams in (*c*) for the weight and inertia of the reciprocating parts, reduced to per square inch of the top of the high pressure piston, including in the weight of the reciprocating parts half the weight of connecting-rod.

(*e*) Draw the polar and rectangular diagrams of crank effort for each crank.

(*f*) Draw the polar and rectangular diagrams of combined crank effort.

(*g*) Determine the mean combined crank effort in lbs. per square inch of the top of the high pressure piston.

(*h*) Determine the positions of the high pressure crank, measured in degrees in direction of motion from the top dead centre, for minimum and maximum speeds.

(*i*) Determine the coefficient of fluctuation of energy for this engine under the given conditions.

17. The following particulars * refer to a 400 horse-power Crossley gas-engine. There are two cylinders, with their open ends facing one another and their connecting-rods working on a crank common to both. Diameter of cylinders, 26 inches. Stroke of pistons, 3 feet. Length of connecting-rods, 6·707 feet. Speed of crank shaft, 150 revolutions per minute.

Total weight of fly-wheel and accessories, two crank slabs, two balance weights, crank pin, equivalent rotating part for two connecting-rods, and engine

* The particulars for this exercise are taken from the *Proceedings of the Institution of Mechanical Engineers*, 1901.

shaft and armature of dynamo, 87,638 lbs. Moment of inertia of all rotating parts (units in lbs. and feet), 62,654.

Weight of reciprocating parts, including one piston, cross-head pin, and equivalent part for one connecting-rod, 2080 lbs.

Indicator diagrams are given in Fig. 508.

(1) Re-draw and enlarge the indicator diagrams, making the length of each, say, 4 inches, and take for the pressure scale, say, 1 inch to 80 lbs. per square inch.

(2) Reconstruct the enlarged indicator diagram of the "A" cylinder on a four-stroke base, and add the inertia force curves, as shown at (*a*), Fig. 509.

(3) From (2) construct the indicator diagram corrected for inertia forces, as shown at (*b*), Fig. 509.

(4) Do the same as in (2) and (3) for the indicator diagram of the "B" cylinder, but observe that the diagram of the "B" cylinder must be moved one stroke forward in advance of that of the "A" cylinder, since the explosion in the "B" cylinder takes place one stroke in advance of that in the "A" cylinder.

(5) Construct, on a base of equal angles of crank motion, the combined

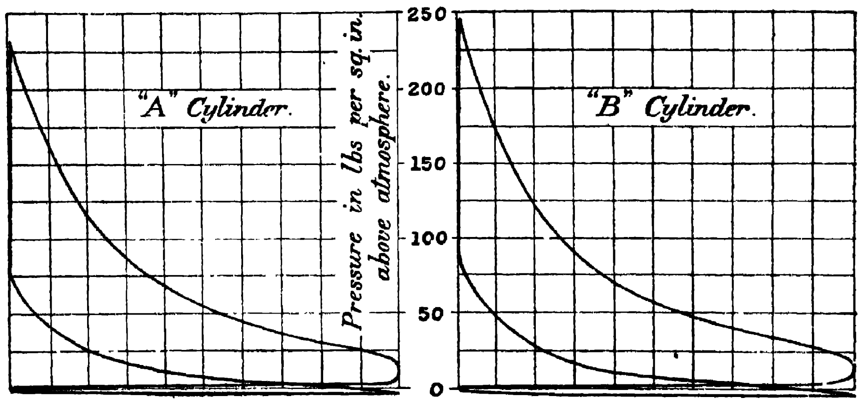


FIG. 508.

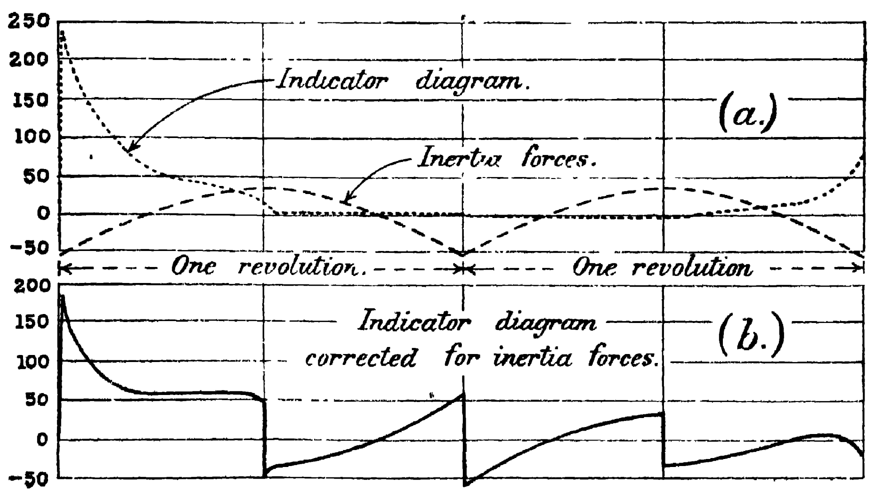


FIG. 509.

twisting moment diagram, as in Fig. 510, in pound-feet per square inch of piston area.

(6) Determine the coefficient of fluctuation of energy and the coefficient of fluctuation of speed.

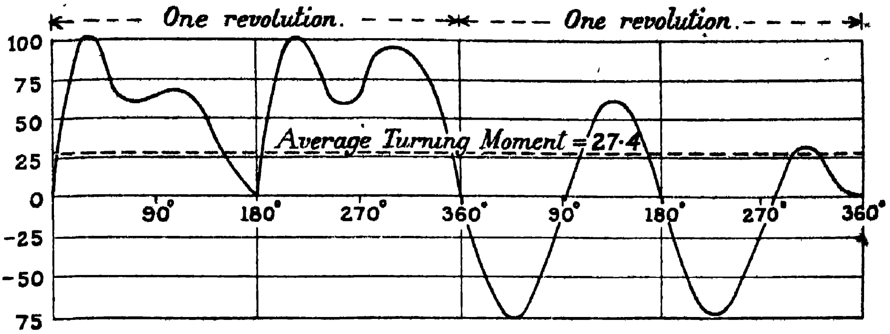


FIG. 510.

18. A fly-wheel, whose radius of gyration is 5 feet, weighs 4 tons. How many ft.-lbs. of energy will this wheel take up in changing its speed from 99 to 101 revolutions per minute?

19. A certain fly-wheel gives out 6500 ft.-lbs. of kinetic energy in changing its speed from 170 to 168 revolutions per minute. What is the kinetic energy of this wheel when its speed is 172 revolutions per minute?

20. What must be the weight, in lbs., of a fly-wheel, 15 feet in diameter, whose mean speed is 120 revolutions per minute, if the total fluctuation of speed is 7 per cent. of the mean speed, and the energy taken up between the minimum and maximum speeds is 12 foot-tons?

21. A steam-engine indicates 10 horse-power. The fluctuation of speed is $\frac{1}{30}$ of the mean speed, and the mean speed is 100 revolutions per minute. The fluctuation of energy is $\frac{1}{10}$ of the work per revolution. What must be the weight of the fly-wheel for this engine, assuming that all the weight is concentrated at a distance of 2 feet 3 inches from the axis of the wheel?

22. Calculate the moment of inertia of a fly-wheel (in ton and foot units) which will give up 20,000 ft.-lbs. of energy as its speed changes from 130 to 128 revolutions per minute.

23. The mean speed of a fly-wheel is 85 revolutions per minute, and the coefficient of fluctuation of speed is $\frac{1}{300}$. What are the minimum and maximum speeds? If the coefficient of fluctuation of energy is 0·07 and the indicated horse-power of the engine is 1600, what must be the moment of inertia of the fly-wheel (in ton and foot units)?

24. A cast-iron fly-wheel is in the form of a disc 6 inches thick and 4 feet 6 inches in diameter. Taking the weight of a cubic foot of cast-iron as 450 lbs., what is the kinetic energy of this wheel in foot-tons when it is running at 200 revolutions per minute?

25. A fly-wheel weighing 60 tons has a radius of gyration of 15 feet. The indicated horse-power of the engine is 3000, the mean speed is 75 revolutions per minute, and the coefficient of fluctuation of energy is 0·06. What is the coefficient of fluctuation of speed?

26. A 1000 horse-power engine, running at 240 revolutions per minute, has a wire-wound fly-wheel whose mass of 70 tons may be considered as concentrated at a radius of 10 feet. Express the energy stored in this fly-wheel in terms of the work done per revolution. Steam being shut off, find the moment of resistance which will reduce the speed from 240 to 120 revolutions in two minutes. [U.L.]

27. An engine developing 80 horse-power has a fly-wheel 10 feet mean diameter, weighing 4000 lbs., and making 120 revolutions per minute. The load on the engine is reduced to 60 horse-power. Assuming that the governor fails to act, that the speed increases at a uniform rate, that the horse-power developed

in the cylinder is proportional to the speed, and that all the surplus energy is stored in the fly-wheel, find the horse-power developed and the speed at the end of one minute. [U.L.]

28. A punching-machine needs 4 horse-power; a fly-wheel upon the machine fluctuates in speed between 100 and 110 revolutions per minute; a hole is punched every three seconds, and this requires five-sixths of the total energy given to the machine during the three seconds. Find the M and the I of this fly-wheel. "M" is the kinetic energy of the wheel at one revolution per minute. [B.E.]

29. In a gas-engine using the Otto cycle the indicated horse-power is 8 and the speed is 264 revolutions per minute. Treating each fourth single stroke as effective and the resistance as uniform, find how many foot-pounds of energy must be stored in the fly-wheel, at mean speed, in order that the speed shall not vary by more than one-fortieth of its mean value. [Inst.C.E.]

30. A gas-engine is provided with two fly-wheels, each weighing $11\frac{1}{2}$ cwts., and the radius of gyration of each is 1·87 feet. There is one working stroke in each four strokes. The diameter of the cylinder being $7\frac{1}{2}$ inches, the stroke 9 inches, and the mean revolutions per minute 250. The mean pressure during the firing stroke is 88·7 lbs. per square inch, during the compression stroke 15·1 lbs., during the exhaust stroke 4·4 lbs., and during the suction stroke atmospheric. If the resistance overcome is constant, find the percentage variation of speed of the engine. [U.L.]

31. A gas-engine drives a number of machines in a workshop. The work done on the piston during the working stroke is $\frac{4}{3}$ times the work done during the four strokes which make a complete cycle. The engine works for some time at 60 horse-power, and at a mean speed of 200 revolutions per minute. Immediately after an explosion in the working stroke has taken place, machines, which absorb 20 horse-power, are cut off, the speed at the instant being equal to the mean speed. Find the moment of inertia of the fly-wheel so that the change in velocity during the working stroke is not more than 4 per cent., and then find the number of revolutions per minute at the end of the fourth stroke. [U.L.]

CHAPTER XX

GOVERNORS

281. Function of a Governor.—The function of a governor is to regulate the mean speed of a machine or prime mover, or to keep the mean speed within certain limits, the limits of variation depending on the nature of the work which the machine or prime mover has to do. The limits of variation of mean speed will also depend on the sensitiveness of the governor used.

The function of the governor differs from that of the fly-wheel. The fly-wheel limits the variation of speed, during a cycle, which may be performed during a fraction of a revolution or during several revolutions, but the function of the fly-wheel is not to regulate the speed when a permanent change takes place in the load, or when the change in the load lasts for more than a cycle of operations of the machine or prime mover; this is the function of the governor, which should regulate the supply of power to the demand. For example, in a steam-engine the fly-wheel controls the variation of speed due to the difference between the effort on the crank pin and the resistance at the crank pin due to the load when the work done by the effort, during a cycle, is equal to the work done on the resistance.

A change in the average resistance should be accompanied or followed as soon as possible by a corresponding change in the average effort which is effected by the governor altering the point of cut off, or altering the initial pressure by operating a throttle valve. The governor of a reciprocating steam-engine can only act during the period of admission of steam to the cylinder, and if a permanent change in the load occurs between the periods of admission, the fly-wheel exerts a controlling influence on the speed until the governor can act.

282. Revolving Pendulum.—In its simplest form the revolving pendulum consists of a small body A revolving about a vertical axis OY, and suspended from a point B by a thread or slender rod. In Fig. 511 the point B is on the axis OY, while in Figs. 512 and 513 B is at some fixed distance from OY. When B is outside OY it rotates about OY with the same angular velocity as A by being on an arm fixed to a rotating spindle,

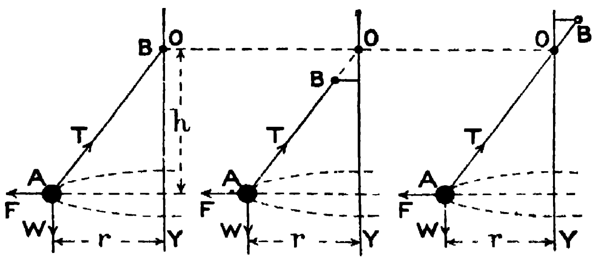


FIG. 511. FIG. 512. FIG. 513.

of which OY is the axis. If AB is a rod, there is a joint at B which permits of the free angular movement of AB about B in the plane AOY.

As A revolves at a steady speed, AB describes the surface of a cone whose vertex is at O, where AB intersects OY, whose height is h, and whose base has a radius r. The forces acting on A in the plane AOY are, its weight W, the centrifugal force F, and the tension T in AB, and for steady motion these must balance one another. Hence, taking moments about O, $Fh = Wr$. But $F = \frac{W\omega^2 r}{g}$, where ω is the angular velocity of A about OY, therefore $\frac{W\omega^2 rh}{g} = Wr$, and $h = \frac{g}{\omega^2}$.

If g is in feet per second per second, and ω is in radians per second, then h is in feet. If A makes n revolutions per second, or N revolutions per minute, then $h = \frac{g}{4\pi^2 n^2} = \frac{60^2 g}{4\pi^2 N^2}$.

Referring to Figs. 512 and 513, where the point of suspension B is not on the axis OY, if the speed of rotation is given, the height h is found as above, but there is no simple formula for calculating r, nor is there, so far as the writer is aware, any direct geometrical construction for fixing the position of AB. AB must therefore be fixed either by trial or by using a locus curve. Several locus curves may be used, but the one shown in Fig. 514 is probably the simplest. With centre B and radius equal to AB draw the arc DE, which must contain the point A. Draw PBp, QBq, etc., several positions of the axis of the arm AB, meeting the axis YY at p, q, etc. Make p1, q2, etc., each equal to h. Draw horizontal lines through 1, 2, etc., to meet PB, QB, etc., respectively. A fair curve drawn through the points thus determined will cut the arc DE at a point which is the position of A corresponding to the height h.

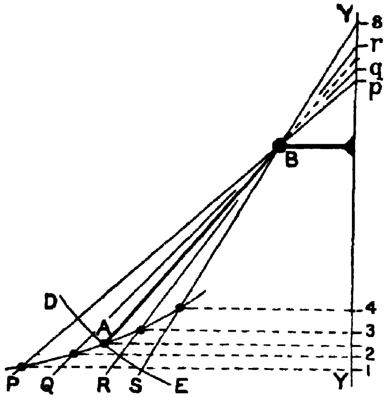


FIG. 514.

283. Effect of Mass of Arm in Revolving Pendulum.—In obtaining the result $h = g/\omega^2$ in the preceding Article, the weight and centrifugal force of the arm were neglected. The effect of these will now be considered.

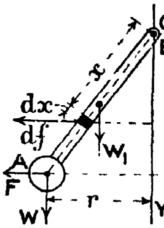


FIG. 515.

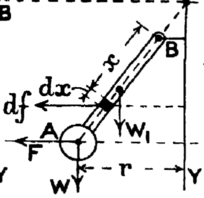


FIG. 516.

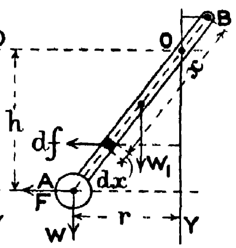


FIG. 517.

The arm AB will be assumed to be of uniform cross section, and to weigh w lbs. per foot of length. Its length a will be

measured from the centre of the ball A to the axis of the joint at B. The total weight of the arm is $aw = W_1$, and the centre of gravity of the arm will be taken at the middle of the length a. O is the point where the axis of the arm AB intersects the vertical axis of revolution OY. In Fig. 515, B and O coincide. In Figs. 516 and 517, the joint B is on an arm fixed to the vertical spindle, the axis of the joint B being at a horizontal distance c from OY. In Fig. 515, c is therefore $= 0$.

For simplicity, in what follows attention will be directed in the first instance to Fig. 516.

Consider an indefinitely small length dx of the arm at a distance x from B. The centrifugal force df of this small length of arm is $\dfrac{wdx\omega^2(x\sin\theta + c)}{g}$, where θ is the inclination of AB to OY. The moment of this centrifugal force about B is

$$x\cos\theta df = \frac{wdx\omega^2(x\sin\theta + c)x\cos\theta}{g} = \frac{w\omega^2\cos\theta}{g}(\sin\theta x^2dx + cxdx),$$

and the resultant moment about B of the centrifugal force of the whole arm is

$$\frac{w\omega^2\cos\theta}{g}\left(\sin\theta\int_0^a x^2dx + c\int_0^a xdx\right) = \frac{w\omega^2\cos\theta}{g}\left(\frac{a^3}{3}\sin\theta + \frac{a^2c}{2}\right)$$
$$= \frac{W_1\omega^2a\cos\theta}{g}\left(\frac{a}{3}\sin\theta + \frac{c}{2}\right).$$

Considering now all the forces acting on A and AB, and taking moments about B,

$$\frac{W\omega^2r}{g}(h - c\cot\theta) + \frac{W_1\omega^2a\cos\theta}{g}\left(\frac{a}{3}\sin\theta + \frac{c}{2}\right) = W(r-c) + W_1\frac{r-c}{2}.$$

Inserting $\cot\theta = \dfrac{h}{r}$, $\sin\theta = \dfrac{r-c}{a}$, and $\cos\theta = \dfrac{h(r-c)}{ar}$, the equation of equilibrium reduces to $\dfrac{\omega^2h}{g}\left\{W + \dfrac{W_1}{3}\left(1 + \dfrac{c}{2r}\right)\right\} = W + \dfrac{W_1}{2}$. To make this apply to Fig. 517, it is only necessary to change c to $-c$. Hence the general equation is $\dfrac{\omega^2h}{g}\left\{W + \dfrac{W_1}{3}\left(1 \pm \dfrac{c}{2r}\right)\right\} = W + \dfrac{W_1}{2}$.

If $c = 0$ (Fig. 515), then $\dfrac{\omega^2h}{g}\left(W + \dfrac{W_1}{3}\right) = W + \dfrac{W_1}{2}$, and $h = \dfrac{W + \dfrac{W_1}{2}}{W + \dfrac{W_1}{3}} \cdot \dfrac{g}{\omega^2}$

Since $\dfrac{c}{2r}$ will generally be comparatively small, the equations for Fig. 515 may be taken as applying to Figs. 516 and 517 also.

It will be seen that the effect of the mass of the arm AB is equivalent to increasing the centrifugal force of A by an amount due to an increase in its weight of $\dfrac{W_1}{3}$, and increasing the downward pull at A by an amount equal to $\dfrac{W_1}{2}$.

284. The Simple Conical Pendulum Governor.—One of the earliest forms of the simple conical pendulum governor, as applied to a steam-engine, is shown in Fig. 518. ABC and A′B′C′ are the arms, jointed together and to the vertical spindle HK at BB′. Links CD and C′D′ connect the arms to the sleeve E, which, while it rotates with the spindle HK, can slide up or down on it when the balls A and A′ fall and rise with changes of speed. The sleeve E has a groove turned on it to receive the forked end of a lever, through which, and through other levers and links if necessary, the sliding motion of the sleeve is transmitted and converted into the motion of the throttle valve. The vertical spindle HK is driven by the engine which the governor has to control. To reduce the strain on the joint at BB′, caused by the inertia of the balls when the

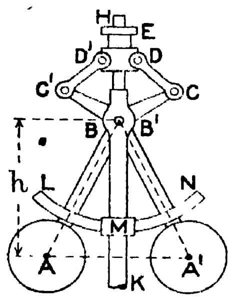


FIG. 518.

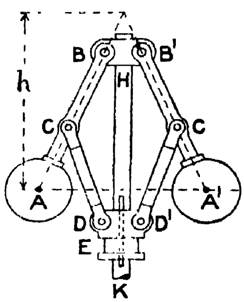


FIG. 519.

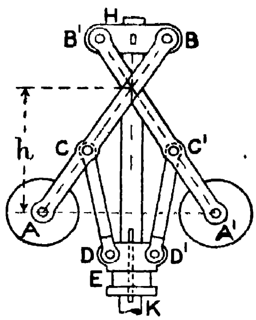


FIG. 520.

angular velocity of the spindle changes, the arms AB and A′B′ work in slots in the curved arms ML and MN, which are fixed to the spindle at M.

A later and more common form of the simple governor is that shown in Fig. 519, and to this the description just given will apply, except that the arms ML and MN are dispensed with, but the sleeve E is driven by a key on the spindle HK, which, however, does not interfere with the vertical sliding of the sleeve on the spindle.

A modification of the design shown in Fig. 519, which makes the governor more sensitive, is that in which the axes of the joints at B and B′ are made to coincide and intersect the axis of the vertical spindle, as in Fig. 518. A still more sensitive form is that shown in Fig. 520, which is known as a *crossed arm governor*. The three designs shown in Figs. 518, 519, and 520 correspond to the three forms of the simple conical pendulum shown in Figs. 511, 512, and 513, p. 329.

Neglecting friction and the effects of the mass of the arms and sleeve, the formulæ connecting the speed with the height h for the governors described in this Article are the same as for the simple conical pendulum, namely,

$$h = \frac{g}{\omega^2} = \frac{g}{4\pi^2 n^2} = \frac{60^2 g}{4\pi^2 N^2}.$$

The following results are useful in connection with calculations on governors :—

$$g = 32{\cdot}2. \qquad \sqrt{g} = 5{\cdot}6745. \qquad \frac{g}{4\pi^2} = 0{\cdot}8156.$$

$$\sqrt{\frac{g}{4\pi^2}} = 0{\cdot}9031. \qquad \frac{60^2 g}{4\pi^2} = 2936{\cdot}3. \qquad \sqrt{\frac{60^2 g}{4\pi^2}} = 54{\cdot}187.$$

285. Loaded Governors.—The simple governor is improved, particularly as regards its power of overcoming frictional resistances, by

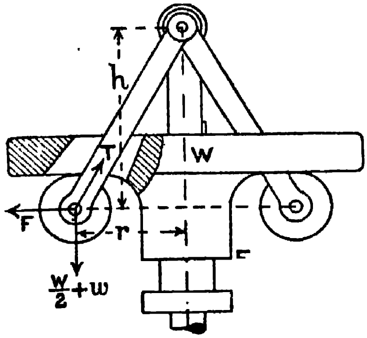


FIG. 521.

adding a central weight, which increases the downward pull on the revolving balls without increasing their centrifugal force. Fig. 521 shows a simple form of loaded governor. The central weight or load W is in the form of a disc with a central boss, which corresponds to the sleeve E in the illustrations of the preceding Article. The masses at the lower ends of the revolving arms, or pendulum weights, are in this case in the form of rollers, upon which the disc part of the central load rests, there being slots in the disc through which the revolving arms' pass, as shown.

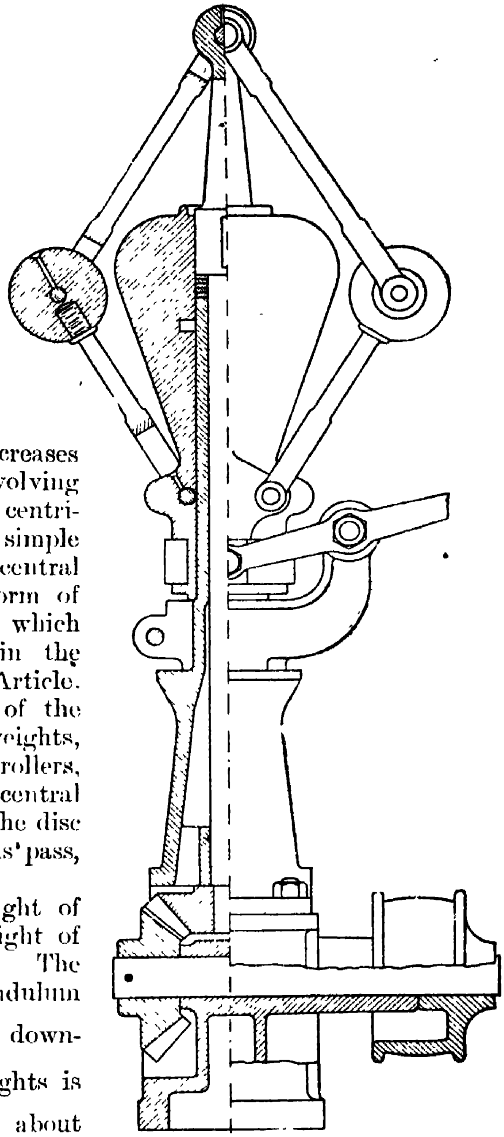


FIG. 522.

Let W equal the total weight of the central load, and w the weight of each of the pendulum weights. The centrifugal force F of each pendulum weight is equal to $\frac{w\omega^2 r}{g}$, and the downward pull on each of these weights is $\frac{W}{2}+w$, hence, taking moments about the point of suspension of the arms,

$$\left(\frac{W}{2}+w\right)r = Fh = \frac{w\omega^2 rh}{g}, \text{ and therefore } h = \left(\frac{W+2w}{2w}\right)\frac{g}{\omega^2}$$

Comparing this with the corresponding result for the simple governor, it is seen that for the same speed the height of this loaded governor is greater than that of the simple governor in the ratio of $W+2w : 2w$.

More frequently the central load is suspended from the pendulum weights by links, as in Fig. 522, which shows the *Porter governor*, so called from the name of its inventor. The particular governor shown in Fig. 522 is one made by Messrs. Tangyes of Birmingham.

To determine the relation between the height and speed in the Porter governor, consider the diagram Fig. 523. Let W equal the total weight of the central load, and w the weight of each revolving ball. The central load will cause a tension in each suspension link equal to $\frac{W}{2\cos\theta}$. This tension may be resolved at the centre of each ball into a vertical component $\frac{W}{2}$, and a horizontal component Q equal to $\frac{W}{2}\tan\theta$. Taking moments about B, the point of suspension of the pendulum arms,

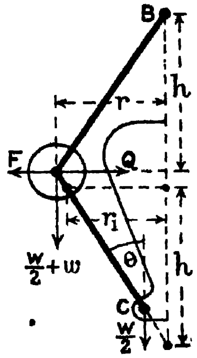


Fig. 523.

$$\left(\frac{W}{2}+w\right)r+\frac{W}{2}h\tan\theta=Fh=\frac{w\omega^2rh}{g}.$$

Let $\tan\theta=\frac{r_1}{h}=\frac{qr}{h}$, then $\left(\frac{W}{2}+w\right)r+\frac{Whqr}{2h}=\frac{w\omega^2rh}{g}$,

and therefore $$h=\left\{\frac{\frac{W}{2}(1+q)+w}{w}\right\}\frac{g}{\omega^2}.$$

If $r_1=r$, then $q=1$, and $h=\frac{W+w}{w}\cdot\frac{g}{\omega^2}$.

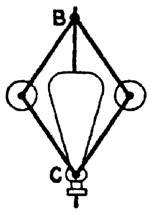


Fig. 524.

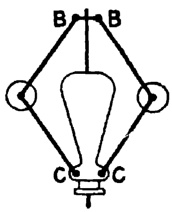


Fig. 525.

When the pendulum arms and the suspension links are of equal length, and the axes of the joints at B and C either intersect the main axis (Fig 524) or are at equal distances from that axis (Fig. 525), then q is equal to 1. In other cases, the value of q is best found by measuring r and r_1 on a diagram to scale. It should be noted that when q is not equal to 1, its value alters as the height h changes.

286. Effect of Friction on Governors.—The frictional resistances of the various joints of the governor itself, and of the gear which the governor has to operate, may be reduced to a single force R acting on the sleeve in a direction opposite to that of its motion. When the sleeve is rising, and the speed of the governor therefore increasing, R will act downwards, and in a loaded governor this will be equivalent to altering the central load from W to W + R. Again, when the sleeve is descending R will act upwards, and this will be equivalent to altering the central load from W to W − R. Hence for a loaded governor of the type shown in Fig. 521, $h=\frac{W\pm R+2w}{2w}\cdot\frac{g}{\omega^2}$, the plus (+) sign being used for increasing speed, and the minus (−) sign for decreasing speed.

For the Porter governor, $h=\frac{\frac{1}{2}(W\pm R)(1+q)+w}{w}\cdot\frac{g}{\omega^2}$ and when $q=1$, $h=\frac{W\pm R+w}{w}\cdot\frac{g}{\omega^2}$

The formulæ just given for the Porter governor will also apply to the simple governor when the upper joints of the suspension links are at the centres of the pendulum weights, but W will then be the weight of the sleeve. If the suspension links are jointed to the pendulum arms, as shown in Fig. 526, then $W\pm R$ must be changed to $(W\pm R)\frac{a}{l}$. The reason for the foregoing statement will be obvious from the following considerations. Draw AC parallel to A′C′. Let T′ be the tension in the suspension link when it is at A′C′, and let T be the tension in that link when it is transferred to AC. Then since the moment of T′ about B has to balance the moments of F and w about B, also since the moment of T about B has to balance the moments of F and w about B, it follows that $T'a$ must be equal to Tl, or $T=\frac{aT'}{l}$, hence $W\pm R$ at C′ must become $(W\pm R)\frac{a}{l}$ at C.

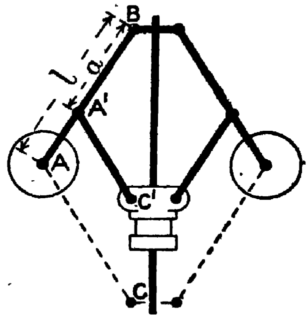


FIG. 526.

If the speed of a governor and the lift of the sleeve, or the lift of the pendulum weights, be plotted, (1) neglecting friction, and (2) taking the friction into account, instructive curves, such as are shown in Fig. 527, are obtained. HK is the lift of the sleeve When the sleeve is at Y the speed, say in revolutions per minute, is YL when friction is neglected, YL_1 when friction is considered and the sleeve is descending, and YL_2 when friction is considered and the sleeve is ascending. Preferably the speeds are measured from a vertical axis some distance to the left of HK in order that a larger scale may be used for the speeds, and so cause the points L_1, L, and L_2 to be further apart. The abscissæ and ordinates of the curve ALB represent the speed and lift respectively when friction is neglected. The abscissæ and ordinates of the curve $A_1L_1B_1$ represent the speed and lift respectively when friction is considered and the sleeve is descending. Lastly, the abscissæ and ordinates of the curve $A_2L_2B_2$ represent the speed and lift respectively when friction is considered and the sleeve is ascending.

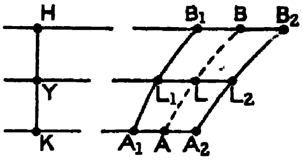


FIG. 527.

287. Sensitiveness of Governors.—The greater the change in the level of the revolving balls of a governor for a given *percentage* or fractional change in speed the greater is its sensitiveness, and the sensitiveness may be defined as the change in level of the revolving balls, due to a change of speed of, say, 1 per cent.

Consider the case where the axis of the top joint intersects the main

axis (Figs. 518, 521, 522, 523, and 524), and let the friction be neglected. In this case the change in the level of the revolving balls is the same as the change in the height h of the governor. If n is the speed of the governor in revolutions per second when the height is h, then for the simple governor and also for the loaded governor h and n are connected by an equation of the form $h=\frac{c}{n^2}$, where c is a constant depending on the type of governor and the various weights. Also, when friction is considered, c has one value for increasing speeds, and another value for decreasing speeds. Let the speed increase from n to xn. (If the increase in speed is 1 per cent., $x=1{\cdot}01$.) The height h will decrease to h_1

where $h_1=\frac{c}{x^2n^2}=\frac{h}{x^2}$. Hence, $h-h_1=\Delta h=h-\frac{h}{x^2}=h\left(\frac{x^2-1}{x^2}\right)$.

This shows that the sensitiveness of the governor is directly proportional to the height h, and it follows from this investigation that *the sensitiveness of the loaded governor is the same as that of the simple governor when friction is neglected.*

Many writers of note state that, friction being neglected, the loaded governor is more sensitive than the unloaded governor, and it is therefore necessary that this point should be considered more fully. Take a simple governor in which the revolving balls each have a weight w, and let this governor be converted into a loaded governor, say of the Porter type, by adding a central load of weight W, and for simplicity let the factor q (Art. 285) equal 1; then for the unloaded governor $h=\frac{g}{4\pi^2n^2}$, and for the loaded governor $h=\frac{W+w}{w}\cdot\frac{g}{4\pi^2n^2}$. Now if these governors are run at the *same speed*, the height of the loaded governor will be $\frac{W+w}{w}$ times the height of the unloaded governor, and under these circumstances the loaded governor would be $\frac{W+w}{w}$ times as sensitive as the unloaded governor; but what really happens in practice is that when the simple governor is replaced by a loaded governor the height h is about the same for both, and consequently the loaded governor is run about $\sqrt{\frac{W+w}{w}}$ times as fast as the unloaded governor, and the one governor is then no more sensitive than the other when friction is neglected.

Consider now the effect of friction on the sensitiveness of the governor. For the Porter governor, in its simplest form, it has been shown that $n^2=\frac{W\pm R+w}{w}\cdot\frac{g}{4\pi^2h}$, where R is the force required at the sleeve to overcome the friction of the governor and the gear which it has to operate. For a given value of h there are evidently two values of n,

namely, $n_1=\sqrt{\frac{W-R+w}{w}\cdot\frac{g}{4\pi^2h}}$, and $n_2=\sqrt{\frac{W+R+w}{w}\cdot\frac{g}{4\pi^2h}}$.

Referring to the diagram Fig. 527, if n_1 is represented by the point L_1, then n_2 is represented by the point L_2. If the sleeve is at the level Y, it must

have reached that level either by falling from a higher level, or by rising from a lower level. Suppose that the sleeve reached the level Y by falling from a higher level, due to a diminution in speed, then its speed must be n_1. Now suppose that the speed diminishes still further, the sleeve will again fall, but *the friction will not affect the sensitiveness of the governor;* the sensitiveness will be simply proportional to h, as has already been shown. Next, suppose that instead of the speed diminishing to less than n_1 it begins to increase after coming down to n_1, then there can be no change in the level of the sleeve until the speed has increased to n_2. If after the speed has increased to n_2 it goes on increasing, the sleeve will continue to rise, and *the sensitiveness will again be unaffected by the friction.*

If n is the mean speed $= \frac{1}{2}(n_1 + n_2)$, and represented by the point L (Fig. 527), then $\frac{n_2 - n_1}{n}$ is the coefficient of fluctuation of speed of the governor *when the direction of the motion of the sleeve is reversed,* and the smaller this coefficient is, the more sensitive is the governor.

The value of the expression $\frac{n_2 - n_1}{n}$ for a Porter governor of the simplest form is found as follows:—

$$n_2 = \sqrt{\frac{W + R + w}{w} \cdot \frac{g}{4\pi^2 h}}, \quad n_1 = \sqrt{\frac{W - R + w}{w} \cdot \frac{g}{4\pi^2 h}}, \text{ and}$$

$$n = \sqrt{\frac{W + w}{w} \cdot \frac{g}{4\pi^2 h}} \text{ (see footnote), hence}$$

$$\frac{n_2 - n_1}{n} = \frac{\sqrt{W + R + w} - \sqrt{W - R + w}}{\sqrt{W + w}} = \sqrt{1 + \frac{R}{W + w}} - \sqrt{1 - \frac{R}{W + w}}.$$

If W be increased, the term $\sqrt{1 + \frac{R}{W + w}}$ decreases, and the term $\sqrt{1 - \frac{R}{W + w}}$ increases, therefore the value of $\frac{n_2 - n_1}{n}$ decreases as W increases. Hence, considering the effect of friction on a loaded governor, the sensitiveness is greater the heavier the central load, and consequently the loaded governor is more sensitive than the unloaded governor when the pendulum weights are the same in both. But the unloaded governor may be made as sensitive as the loaded governor by increasing the pendulum weights. Let w_1 = weight of each ball of a simple or unloaded governor, w = weight of each ball of a loaded governor, and W = weight of central load. Then for the unloaded governor W = 0, and

$$\frac{n_2 - n_1}{n} = \sqrt{1 + \frac{R}{w_1}} - \sqrt{1 - \frac{R}{w_1}}.$$

For the loaded governor $\quad \frac{n_2 - n_1}{n} = \sqrt{1 + \frac{R}{W + w}} - \sqrt{1 - \frac{R}{W + w}}.$

Hence if $\frac{n_2 - n_1}{n}$ is the same for both governors, $w_1 = W + w$.

Note.—The mean of the rising and falling speeds for a given level of sleeve, and for a given value of R, is not quite the same as the speed for the same level when R = 0, but as R is generally small compared with W + w, the error introduced by taking n as above may be neglected.

288. Effort of Governors.—By the *effort* of a governor is meant the force which it is capable of exerting at the sleeve for a given percentage or fractional change of speed.

Consider first the effort of the Porter governor for which $\omega^2 = \frac{W+w}{w} \cdot \frac{g}{h}$. Let the speed increase from ω to $x\omega$, and let a force Q be applied at the sleeve in a downward direction, Q being just sufficient to prevent the sleeve rising. This will evidently be equivalent to increasing the central load from W to W + Q.

Let F = centrifugal force for two balls at the speed ω.
F_1 = centrifugal force for two balls at the speed $x\omega$.

$$F = \frac{2w\omega^2 r}{g} = 2(W+w)\frac{r}{h}. \quad F_1 = \frac{2wx^2\omega^2 r}{g} = 2(W+Q+w)\frac{r}{h}.$$

$$F_1 - F = \frac{2w\omega^2 r}{g}(x^2-1) = \frac{2Qr}{h}.$$

Therefore $Q = \frac{w\omega^2 h}{g}(x^2-1) = (W+w)\ (x^2-1)$. If the force Q be gradually diminished to zero, the sleeve will rise until $h = \frac{W+w}{w} \cdot \frac{g}{x^2\omega^2}$, and the average value of the effort on the sleeve during the rise will be $\frac{1}{2}$Q, or $\frac{1}{2}(W+w)(x^2-1) = P$, and this is the resistance at the sleeve which this governor is capable of overcoming with an increase of speed from ω to $x\omega$. For a decrease in speed from ω to $x\omega$ it follows that P, now acting in the opposite direction, is equal to $\frac{1}{2}(W+w)(1-x^2)$. For a change of speed of 1 per cent. $P = 0{\cdot}01(W+w)$.

Converting the Porter governor into an unloaded governor by removing the central load or making W = 0, it follows from the foregoing proof that $P = \frac{1}{2}w(x^2-1)$, or $\frac{1}{2}w(1-x^2) = 0{\cdot}01w$, for a change of speed of 1 per cent. If in the unloaded governor the sleeve is suspended, as in Fig. 526, then P, as just given, must be increased in the ratio of $l : a$, supposing that the suspension links and the pendulum arms are equally inclined to the main axis.

It is evident that in order that the unloaded governor may have the same power as the loaded governor of the Porter type, $\frac{lw_1}{a} = W + w$, where w_1 is the weight of each ball of the unloaded governor.

For the loaded governor of the type shown in Fig. 521,

$$P = \tfrac{1}{2}(W+2w)(x^2-1).$$

289. Power of Governors.—By the *power* of a governor is meant the amount of work which it is capable of doing at the sleeve for a given percentage or fractional change of speed. The work done at the sleeve is equal to the mean effort of the governor multiplied by the distance through which the sleeve moves for the given change of speed. Thus if P = mean effort, k = lift of sleeve, and U = the power of the governor, then U = Pk.

For the Porter governor (Fig. 524), $P = \frac{1}{2}(W+w)(x^2-1)$,

$$\Delta h = \left(\frac{x^2-1}{x^2}\right)h, \quad k = 2\Delta h = 2\left(\frac{x^2-1}{x^2}\right)h, \quad U = Pk = (W+w)\left(\frac{x^2-1}{x}\right)^2 h.$$

For the direct loaded governor (Fig. 521), $P = \frac{1}{2}(W + 2w)(x^2 - 1)$,

$$\Delta h = \left(\frac{x^2 - 1}{x^2}\right)h = k, \quad U = Pk = \tfrac{1}{2}(W + 2w)\left(\frac{x^2 - 1}{x}\right)^2 h.$$

For the simple governor, (Fig. 526), except that B and C′ are on the main axis, $P = \frac{1}{2}w(x^2 - 1)\frac{l}{a}$, $\Delta h = \left(\frac{x^2 - 1}{x^2}\right)h$, $k = 2\Delta h\frac{a}{l} = 2\left(\frac{x^2 - 1}{x^2}\right)\frac{ha}{l}$,

$$U = Pk = w\left(\frac{x^2 - 1}{x}\right)^2 h.$$

For a change in speed of 1 per cent. $\left(\frac{x^2 - 1}{x}\right)^2 = 0{\cdot}0004$.

For a change in speed of 10 per cent. $\left(\frac{x^2 - 1}{x}\right)^2 = 0{\cdot}0364$.

290. Diagrams of Governor Effort and Power.—The formulæ obtained in the two preceding Articles will perhaps be better understood by reference to the diagrams of effort and power now to be described. Taking first the simplest form of governor, namely, the simple conical pendulum, OA_1 (Fig. 528) is one position of the pendulum, and OA_2 is a higher position. Let N_1 denote the normal speed of the governor in revolutions per minute for the position OA_1, and N_2 the normal speed for the position OA_2. Let $H_1A_1 = r_1$, and $H_2A_2 = r_2$, and let w = weight of one ball.

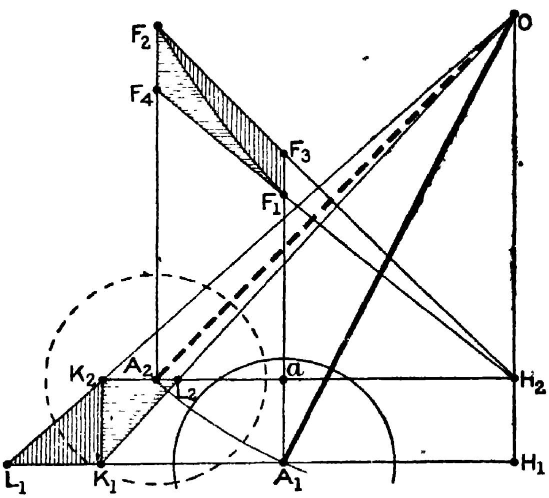


FIG. 528.

Make the vertical line aF_1 = centrifugal force of ball when its speed is N_1 and position A_1. $aF_1 = cwN_1^2 r_1$, where c is a constant.

Make A_2F_2 = centrifugal force of ball when its speed is N_2 and position A_2. $A_2F_2 = cwN_2^2 r_2$. Let also the centrifugal force for intermediate positions be plotted in the same way, and a fair curve F_1F_2 drawn through the points thus obtained.

The work done by the centrifugal force while the ball moves from A_1 to A_2 is evidently represented by the area of the figure $aF_1F_2A_2$. But as the speed has been assumed to be normal for each position of the ball, all the work done by the centrifugal force must have been spent in raising the ball against gravity. Draw the horizontal line H_1K_1 to represent w to the same scale as was used in representing the centrifugal force. Complete the rectangle $H_1K_1K_2H_2$. The area of this rectangle will represent the work done in raising the ball through the height H_1H_2.

Hence the area of the rectangle $H_1K_1K_2H_2$ is equal to the area of the figure $aF_1F_2A_2$.

Now suppose that when the ball is at A_1 the speed suddenly increases to N_2, and suppose that the ball is prevented from rising by a force S acting vertically downwards at A_1. The centrifugal force will now be $aF_3 = cwN_2^2r_1$. Make $K_1L_1 = S$, then $H_1L_1 = w + S$, the total downward force at A_1. For equilibrium it is obvious that $aF_3 \times OH_1 = H_1L_1 \times A_1H_1$. Next suppose that the force S is diminished so as to allow the ball to rise to A_2, then remembering that the speed during this change is N_2, the centrifugal force will be directly proportional to the radius, and will therefore be represented by the ordinates of the straight line F_2F_3, which when produced passes through H_2. In order that there may be equilibrium in each position of the ball as it rises $Fh = (w+s)r$, where s is the vertical effort at the centre of the ball when its distance from the axis is r, hence $\frac{F}{r} = \frac{w+s}{h}$, but $\frac{F}{r}$ is constant, therefore $\frac{w+s}{h}$ is constant, and $w+s$ will be represented by the abscissæ of the straight line L_1K_2, which when produced passes through O. The work done on the force s as the ball rises from A_1 to A_2 is therefore represented by the area of the triangle $K_1L_1K_2$. In the same time the work done by the centrifugal force is represented by the area of the figure $aF_3F_2A_2$, but the part of this, $aF_1F_2A_2$, represents the work done in raising w, therefore the external work done is represented by the area $F_1F_2F_3$.

If straight lines OL_2K_1 and $H_2F_1F_4$ be drawn, it is easy to show that the external work which the governor is capable of doing as the ball descends from A_2 to A_1 is represented by either the area of the triangle $K_1L_2K_2$ or the area $F_1F_2F_4$.

If H_1H_2 is the maximum or total lift of the balls, then for each ball the *maximum power* of the governor is represented by the area of the triangle $K_1L_1K_2$ when the ball is ascending, and by the area of the triangle $K_1L_2K_2$ when the ball is descending. But if the ascent from H_1 to H_2 is made in three steps (Fig. 529) instead of one, the external work done for each ball will only be that represented by the sum of the areas of the triangles shaded with *vertical* lines, and if the descent from H_2 to H_1 is made in three steps instead of one, the external work done for each ball will only be that represented by the sum of the areas of the triangles shaded with *horizontal* lines.

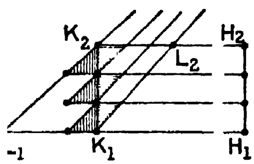


FIG. 529.

The horizontal widths of the triangles $K_1L_1K_2$ and $K_1L_2K_2$ at any given level measures the vertical effort s for each ball of the governor *at the centre of the ball* at that level, the width of the triangle $K_1L_1K_2$ being the effort during ascent, and the width of the triangle $K_1L_2K_2$ being the effort during descent. The effort *at the sleeve* is got by multiplying the effort at the balls by the ratio of the vertical motion of the balls to that of the sleeve.

For the direct loaded governor (Fig. 521), the length H_1K_1 (Fig. 528) is made equal to $\frac{W}{2} + w$.

For the Porter governor, H_1K_1 (Fig. 521) is made equal to $\frac{mW}{2}+w$, where m is the ratio of the motion of the sleeve to the vertical motion of the balls.

The ratio of the vertical motion of the balls to that of the sleeve is easily found from a diagram such as that shown in Fig. 530, where OA is the pendulum, and AC the link connecting it to the sleeve. Produce OA to meet the horizontal line through C at O_1, then O_1 is the instantaneous centre for the link AC in the given position, and if V_1 = the velocity of A in the direction at right angles to OA, and V_2 = the velocity of C in the vertical direction, then $\frac{V_1}{V_2}=\frac{O_1A}{O_1C}$. Draw the vertical line AD; then if V_1' is the vertical component of V_1, it is easily proved that $\frac{V_1'}{V_2}=\frac{O_1D}{O_1C}$. If the link occupies the position A′C′ where A′C′ is parallel to AC, then $\frac{V_1'}{V_2}=\frac{O_1D}{O_1C}\times\frac{a}{l}$, where V_2 now denotes the vertical velocity of C′.

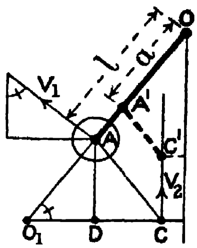


FIG. 530.

Exercises XX.

1. Plot on squared paper the height, in inches, and revolutions per minute, for a simple conical pendulum, from 30 to 120 revolutions per minute. Scales.—1 inch to 10 inches, and 1 inch to 20 revolutions per minute.

2. If the arm of a simple conical pendulum is $15\frac{1}{2}$ inches long, what will be its inclination to the axis when running at 50 revolutions per minute, and what will it be at 60 revolutions per minute? Also, what will its speed be, in revolutions per minute, when its inclination to the axis is 30°, and what will it be when the inclination is 45°?

3. A simple conical pendulum is running at 60 revolutions per minute; what is the decrease in height if the speed is increased 5 per cent., and what is the increase in height if the speed is decreased 5 per cent.?

4. Add to the diagram drawn in answer to Exercise 1 the curve showing the relation of height to speed for a conical pendulum when the weight of the arm is half the weight of the ball. Assume weight of arm per inch of length to be uniform.

5. Referring to Figs. 511, 512, and 513, p. 329, the length of the arm AB is 10 inches for Fig. 511, 8 inches for Fig. 512, and 12 inches for Fig. 513. The distance of B from the axis OY is 1 inch for Figs. 512 and 513. Starting in each case from the position in which the arm is inclined at 30° to the axis, calculate the percentage increase in speed for a rise of 1 inch in level of the balls. Draw the figures half full size for each position.

6. Find the answers to the preceding exercise when the weight of the arm is taken into account. The weight of the ball in each case is 6 lbs., and the weight of the arm is 1·75 lbs. for Fig. 511, 1·5 lbs. for Fig. 512, and 2 lbs. for Fig. 513.

7. Draw the speed curves, as in Fig. 527, p. 335, for the pendulums of Exercise 5, (1) neglecting friction, (2) taking friction into account, the amount of the friction being equivalent to a vertical force of 1 lb. at the centre of each ball. The weight of each ball is 6 lbs.

8. In a direct loaded governor (Fig. 521, p. 333) the arms are 10 inches long. Each ball weighs 4 lbs., and the load is 75 lbs. The sleeve is in its lowest position when the arms are inclined at 27° to the axis. The lift of the sleeve is 1 inch. What is the force of friction at the sleeve if the speed at the beginning

of the ascent from the lowest position is equal to the speed at the beginning of the descent from the highest position? Also, what is the range of speed for this governor under these conditions?

9. Referring to the preceding exercise, if the friction is 4 lbs. at the sleeve, what must be the lift if the maximum descending speed is equal to the minimum ascending speed?

10. In a Porter governor the arms and links are each 10 inches long, and the axes of the top and bottom joints intersect the main axis. Each ball weighs 5 lbs., and the central load is 50 lbs. R, the force of friction at the sleeve, is 5 lbs. The inclination of the arms to the vertical is 30° and 45° in the lowest and highest positions respectively. Calculate the following: (1) The travel of the sleeve, in inches. (2) The speeds at the bottom, middle, and top of the travel of the sleeve, neglecting friction. (3) The speeds at the bottom, middle, and top of the upward travel of the sleeve, allowing for friction. (4) The speeds at the top, middle, and bottom of the downward travel of the sleeve, allowing for friction.

Speeds to be in revolutions per minute. Plot the results as in Fig. 527, p. 335.

11. The arms and links of a Porter governor are all 9 inches long, and the axes of the top and bottom joints are at a distance of 1 inch from the main axis. The balls weigh 5 lbs. each, and the central load is 55 lbs. The friction is equivalent to a force of 4 lbs. at the sleeve. The sleeve is in its lowest position when the arms are inclined at 30° to the vertical. Find the lift of the sleeve, in inches, when the speed at the beginning of the ascent from the lowest position is equal to the speed at the beginning of the descent from the highest position.

12. Referring to the governor of the preceding exercise, if r is the radius of the circle described by the centres of the balls as they revolve, what are the extreme speeds in revolutions per minute corresponding to $r=5$ inches, and what is the range of speed between $r=5$ inches and $r=6$ inches?

13. The balls of a Porter governor weigh 4 lbs. each, the load on the governor is 40 lbs., and the arms intersect on the axis. What height will this governor run at if it revolves at the rate of 240 revolutions per minute? If the speed of the balls suddenly increases $2\frac{1}{2}$ per cent., what pull will be exerted on the gearing attached to the governor? If the friction of the regulating apparatus is equal to a dead load on the governor of 5 lbs., by how much will the speed increase before the balls rise? [U.L.]

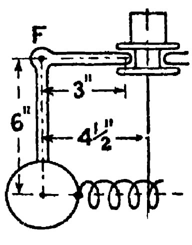


FIG. 531.

14. A spring-controlled governor is as shown in the sketch (Fig. 531), the fixed fulcrum of the arm being at F, and the weight of each ball being 5 lbs. There is no tension in the spring when the balls are at a radius of 3 inches. Neglecting the controlling effect of the balls and arms, draw the curve of controlling force. Find the speed at which the governor runs when the balls are at 5 inches radius, and find also the force on the sleeve if when the balls are in that position the speed is 10 per cent. higher. The spring extends 1 inch for 30 lbs. Show on your curve, roughly, the controlling effect exercised by the balls. [U.L.]

15. A spring-loaded governor is placed horizontally, as shown in Fig. 532. Let W be the weight of each of the balls in lbs.; r the radius of the path of the balls; l the length of each of the four arms; ω the angular velocity in radians per second. When the radius is zero, the tension in the spring is T lbs., and the force required to elongate the spring unit length is Q lbs. Show that

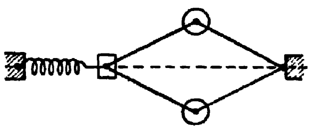


FIG. 532.

$$\omega^2 = \frac{g\{T+2Q(l-\sqrt{l^2-r^2})\}}{W\sqrt{l^2-r^2}}.$$

If the rate of change of ω with respect to r is to be 80 when ω is 26 radians per second, r is 0·25 foot, and l is 1 foot, and the weight of each ball is 3 lbs., find the values of T and Q. [U.L.]

16. A Wilson-Hartnell spring loaded governor is shown in Fig. 533. The maximum and minimum distances of the centres of the balls from the axis of the governor are 7 and 3·5 inches respectively. r_1 and r_2, the lengths of the arms of the bell crank levers, are 4·5 and 3·5 inches respectively. Each ball weighs

6 lbs. The maximum load on the spring is 258 lbs. Neglecting the moment of the weight of each ball, and assuming that the ball and roller ends of the levers

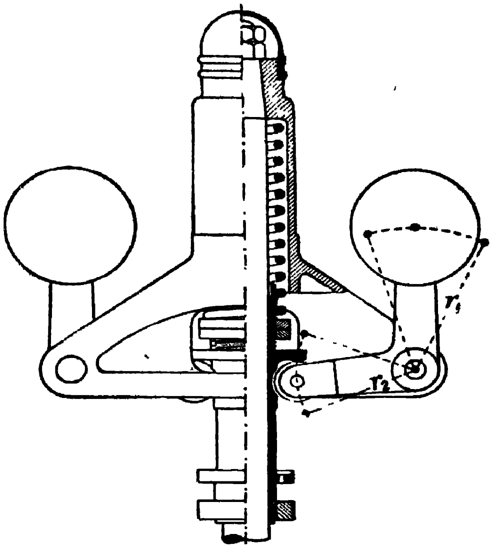


FIG. 533.

move in horizontal and vertical lines respectively, find (*a*) the maximum speed of the governor in revolutions per minute, and (*b*) the load on the spring at minimum speed, which is 270 revolutions per minute.

CHAPTER XXI

BRAKES AND DYNAMOMETERS

291. Brakes.—A brake is an instrument for introducing an artificial resistance to the motion of a machine or moving body. The object of introducing the artificial resistance is either to stop the machine or retard it, or prevent its speed increasing. In acting, the brake converts work into heat by means of friction. The friction may be between solids, or between solids and a fluid, or it may be partly between solids and a fluid, and partly between the particles of the fluid.

292. Band Brakes.—In a band brake, a band, generally of metal, embraces a portion of the circumference of a wheel, as shown at (*a*), Fig. 534. One end of the band is jointed to one arm of a lever, and the other end is either jointed to another arm of the same lever, or it is jointed to a pin fixed on the frame of the machine.

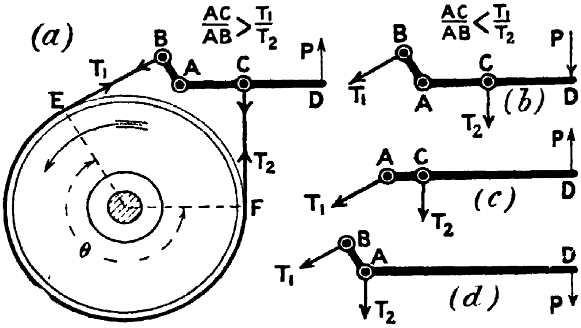


FIG. 534.

The required resistance is produced by the friction between the band and the rim of the wheel, and when the brake is in action there are tensions T_1 and T_2 in the straight parts BE and CF of the band respectively, of which T_1 is the greater, and the turning action of these forces on the lever is balanced by a force P applied to the lever at D.

By Art. 243, p. 277, $\frac{T_1}{T_2} = e^{\mu\theta}$.

The resisting torque due to the action of the band on the wheel is $(T_1 - T_2)R$, where R is the effective radius of the wheel, that is, the radius measured to the middle of the thickness of the band.

Referring to (*a*), Fig. 534, and assuming that the arms AB and AC of the lever are perpendicular to BE and CF respectively, then, taking moments about A, the fulcrum of the lever, $P \times AD = T_2 \times AC - T_1 \times AB$. If $\frac{AC}{AB}$ is made equal to $\frac{T_1}{T_2}$, then $P = 0$, which means that once the brake is in action it will remain in action without the application of any further effort on the lever. Again, if $\frac{AC}{AB}$ is nearly equal to $\frac{T_1}{T_2}$, only a small

effort will be required on the lever at D to keep the brake in full action.

It is obvious that if AC is greater than AB, a downward motion of the end D of the lever will loosen the band from the wheel.

If the ratio $\frac{AC}{AB}$ is made less than $\frac{T_1}{T_2}$, then, when the brake is in action, the force P must act downwards, as shown at (*b*), Fig. 534; but if this force be increased beyond what is sufficient to balance the turning action of T_1 and T_2 on the lever, the end D will drop, and the band will be disengaged from the wheel.

At (*c*), Fig. 534, the tight end of the band is shown anchored at A, the fulcrum of the lever. Here $P\ AD = T_2 \times AC$. At (*d*) the slack end of the band is shown anchored at A. Here $P \times AD = T_1 \times AB$. Comparing the arrangements at (*c*) and (*d*), it is obvious that, for the same effort P, the latter arrangement will require a larger leverage for P to produce the same resistance at the circumference of the wheel.

293. Band and Block Brakes.—By lining the band of the brake discussed in the preceding Article with wood blocks, as shown in Fig. 535, a higher coefficient of friction is introduced, and the wear is confined to the wood blocks, which may easily be renewed from time to time. The ratio of the tensions T_0 and T_n at the ends of the band is obtained as follows. Let there be n blocks, each subtending an angle 2θ at the centre of the wheel. The first block at the tightest end is shown separately at (*a*). The forces acting on this block are T_0 and T_1, the tensions in the band where it leaves the block, and R, the reaction of the wheel on the block; the latter force is inclined to the normal at an angle ϕ, as shown, when slipping is taking place, ϕ being the friction angle. The triangle of forces for the block under consideration is shown at (*b*).

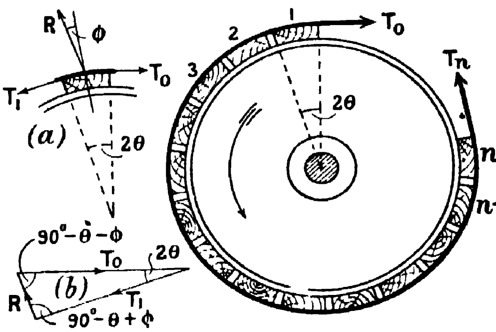


FIG. 535.

From the triangle of forces it follows that

$$\frac{T_0}{T_1} = \frac{\sin\{(90-\theta)+\phi\}}{\sin\{(90-\theta)-\phi\}}, \text{ which reduces to } \frac{T_0}{T_1} = \frac{1+\mu\tan\theta}{1-\mu\tan\theta},$$

where $\mu = \tan\phi$ is the coefficient of friction between the block and the wheel.

In like manner for the second block $\frac{T_1}{T_2} = \frac{1+\mu\tan\theta}{1-\mu\tan\theta} = \frac{T_0}{T_1}$, and for all the blocks $\frac{T_0}{T_1} = \frac{T_1}{T_2} = \frac{T_2}{T_3} = \ldots\ldots \frac{T_{n-1}}{T_n} = \frac{1+\mu\tan\theta}{1-\mu\tan\theta}$.

Hence $\frac{T_0}{T_n} = \left\{\frac{1+\mu\tan\theta}{1-\mu\tan\theta}\right\}^n$.

294. Block Brakes.—In a block brake a block is pressed against the rim of a revolving wheel. This is the type of brake which is nearly

always used on railway trains, tram-cars, and vehicles on common roads. The block is made of a softer material than the rim or tyre of the wheel against which it is pressed, so that the wear is confined mainly to the block, which is easily renewed. When the block is made of wood, a hard and strong wood, such as elm, oak, or beech, should be used. For heavy and rapid running vehicles, cast-iron brake blocks are the most common.

If P is the normal force pressing the brake block on the wheel whose radius is R, then the resisting torque set up by the brake on the wheel is μPR when the wheel is rotating.

When a brake block is applied to a rolling wheel an additional load is thrown on the bearing or journal of the wheel or axle, but if two blocks are applied at opposite ends of a diameter of the wheel, there is no such additional load. The braking action is also doubled by the use of two blocks, and the two blocks may be operated by practically the same force which will operate one.

Fig. 536 shows the Milnes-Daimler differential block brake as used on motor omnibuses.* A is the brake drum fixed to the final driving shaft, BB are the brake blocks, C is a bracket fixed to the frame of the chassis, D is the operating lever, E is the pull-off spring, and F is the brake adjusting screw.

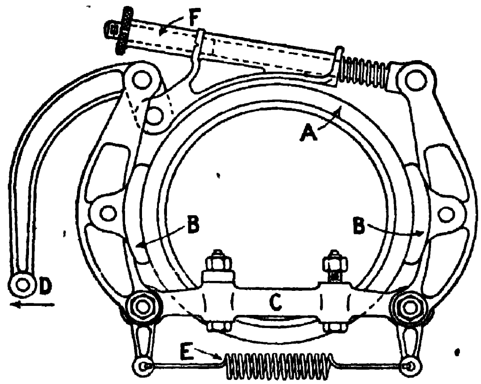


FIG. 536.

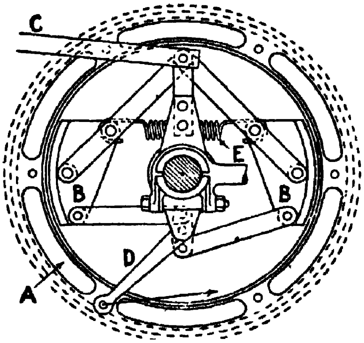


FIG. 537.

Fig. 537 shows the James and Browne block brake, as used on motor cars.† A is the sprocket wheel and brake drum, BB are the brake blocks which act on the inside of the rim of the brake drum, C is an arm jointed to the frame of the chassis, D is the operating lever, and E is the pull-off spring.

295. Action of Railway Brakes.—The block brakes used on the wheels of railway vehicles are found to be most effective when the forces pressing the brake blocks on the wheels just prevent the wheels from skidding on the rails. The explanation of this is that the coefficient of sliding friction between the wheels and the rails is less than that between the brake blocks and the wheels, and also that the coefficient of friction between the wheels and the rails just before skidding begins is greater than the coefficient of friction of skidding.

For example, at a speed of 50 miles per hour the coefficient of sliding

* *Proceedings of the Institution of Mechanical Engineers*, 1907, p 432.

† *Ibid.*, 1902, p. 730.

friction between a wheel and the rail may be, say, 0·05, while the coefficient of friction between the brake block and the wheel when the wheel is not skidding may be, say, 0·08. Let the weight on the wheel be 10,000 lbs., and let the braking force on the block be 9000 lbs. If the wheel is skidding the resisting force is $10{,}000 \times 0{\cdot}05 = 500$ lbs., and the work absorbed per foot of travel of train is 500 ft.-lbs. for this wheel. If, however, the wheel is not skidding the resisting force is $9000 \times 0{\cdot}08 = 720$ lbs., and the work absorbed per foot of travel of train is 720 ft.-lbs. for this wheel. But in order that the resistance of 720 lbs. due to the sliding of the rim of the wheel on the brake block may not lock the wheel and make it skid, there must be a resistance to sliding of the wheel on the rail of not less than 720 lbs., and this force is greater than the 500 lbs. which is the resistance to sliding when the wheel skids. Now, when the wheel is rolling, the part of the wheel in contact with the rail is for the instant at rest on the rail, and the coefficient of friction between the wheel and the rail before skidding commences may be, say, 0·15, and therefore the resistance before skidding commences must be $10{,}000 \times 0{\cdot}15 = 1500$ lbs., which gives an ample margin.

The coefficients of sliding friction between the wheels and rails when the wheels skid, and between the wheels and the brake blocks when the wheels roll, are found to vary with the speed, being least at high speeds, and they increase as the speed decreases, as shown approximately in the following table :—

Speed of sliding in miles per hour	60	50	40	30	20	10	0
μ between wheels and rails	0·04	0·05	0·06	0·07	0·09	0·11	0·15
μ between wheels and brake blocks	0·06	0·08	0·10	0·13	0·17	0·21	0·25

296. Dynamometers.—A dynamometer is an instrument for measuring the effort or torque exerted by or on a machine. The work done in a given time by the effort or torque is found by multiplying the effort by the distance moved in the given time by the point at which the effort acts, or by multiplying the torque by the circular measure of the angle described in the given time by the piece on which it acts.

Dynamometers may be divided into two principal classes, namely, *absorption dynamometers* and *transmission dynamometers*. In an absorption dynamometer the work done by the effort or torque is wasted by being converted into heat by means of friction. In a transmission dynamometer the work done by the effort or torque is transmitted through the dynamometer with only a small waste necessary to operate the instrument.

297. Block Brake Dynamometer.—What is generally known as the *Prony brake* is a simple form of absorption dynamometer. In its simplest form the Prony brake consists of two blocks of wood clamped together with a pulley between them, the pulley being fixed to a revolving shaft; one of the blocks has a lever attached to it, which carries a weight at its

outer end, the magnitude of the weight being adjusted so that its moment balances the moment of the friction between the blocks and the pulley.

Fig. 538 shows a form of Prony brake designed by the author for testing small high-speed motors. A and B are wood blocks clamped together and embracing the small cast-iron pulley C, which is keyed to the shaft D. The blocks are clamped by means of two bolts with nuts.

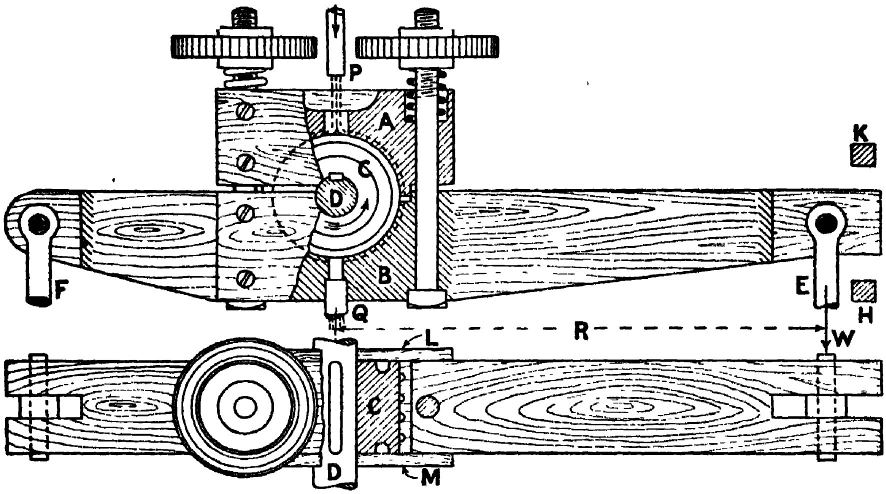


FIG. 538.

The nuts may be of the ordinary form, to be turned with a spanner, or they may be small hand-wheels, as shown. Between the nuts and the block A, and partly recessed in the block, are stiff helical springs, which serve to keep the pressure between the blocks and the pulley constant. The lower block B is extended to right and left to form a two armed lever. A rod E, suspended from the right-hand end of the lever, carries the load W. A rod F, suspended from the other end of the lever, carries a weight, which balances the brake when unloaded. The rod F may also be extended and carry a piston to work in a dash-pot, to be presently described. H and K are stops to limit the motion of the lever.

If R is the horizontal distance of the axis of the rod E from the axis of the shaft in feet, W the load in lbs., and N the speed of the shaft in revolutions per minute, then the horse-power absorbed by the dynamometer is $\frac{2\pi RWN}{33000}$. It should be observed that it is not necessary to know the diameter of the pulley or the coefficient of friction between the brake blocks and the pulley in order to compute the horse-power absorbed.

The great defect of this type of brake is that it is liable to violent oscillations when the driving torque on the shaft is not uniform, and even when the driving torque is uniform, as in a steam turbine, variations in the coefficient of friction between the blocks and the pulley often cause great unsteadiness in the lever.

To keep the coefficient of friction constant, the brake should be kept well lubricated with a stream of soapy water. The stream of water also serves to carry away the heat and keep the pulley and brake blocks cool.

In the brake shown in Fig. 538, the lubricating and cooling water enters at the top from the pipe P, and is distributed over the surface of the pulley by grooves formed in the rubbing surfaces of the blocks, and leaves by the pipe Q at the bottom. Wooden shrouds L and M on the sides of the blocks prevent the water coming out at the sides.

Oscillations of the brake may be damped by means of a dash-pot, such as that shown in Fig. 539. This dash-pot is a cylinder containing oil or water, and a piston, which is attached to a rod suspended from and jointed freely to the lever of the brake. The piston may be about 1-16th inch smaller in diameter than the bore of the cylinder, and it should be thin at its edge. The oscillations of the lever are communicated to the piston, but the motion of the piston is retarded by the liquid in the cylinder, a portion of which must move from one side of the piston to the other through the narrow passage round the edge of the piston as the latter moves. In this way the amplitude of the oscillations of the brake is considerably reduced.

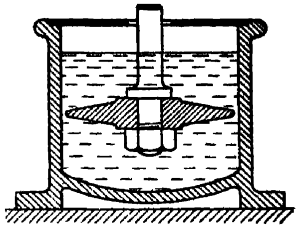


FIG. 539.

The brake should be balanced, when unloaded, with the piston immersed in the liquid in the dash-pot.

Instead of hanging the brake load on the end of the lever, the brake may be turned "end for end," and the load end of the lever be made to rest on a pedestal placed on the platform of a weighing-machine. In many cases this is a very convenient arrangement.

298. Use of Compensating Lever on Dynamometer.—When a band-block brake is used as a dynamometer, a *compensating lever* is often added. This lever provides a means of automatically adjusting the tension in the band to suit variations in the coefficient of friction between the brake blocks and the wheel.

Referring to Fig. 540, ABC is the compensating lever jointed to the ends of the band at A and B, and D is the point of suspension of the brake load W. The band is tightened by the screw at E, so that when the brake is in action the compensating lever is horizontal and in line with D.

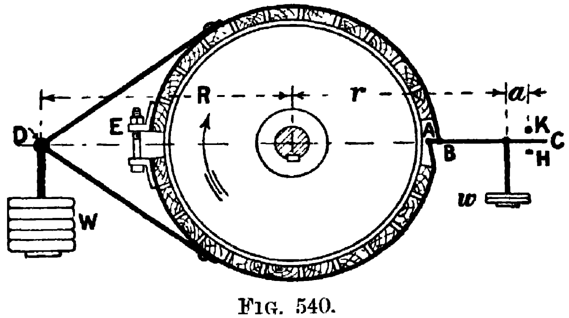


FIG. 540.

The action of the compensating lever is as follows. Suppose that the coefficient of friction between the brake blocks and the wheel should increase, this will cause the wheel to carry the brake and its levers round with it until the lever ABC strikes the stop H. The points A and B will continue to move round with the wheel, but A will move faster than B, because the outer end of the lever is resting on the stop H; the effect of this is obviously to slacken the brake strap and diminish the resist-

ance of friction, and so compensate for the increase in the coefficient of friction. A diminution in the coefficient of friction will produce the opposite action, the compensating lever will strike the upper stop K, and the band will be tightened.

When the compensating lever is floating between the stops, a load w, in addition to the weight of the lever ABC, may be required to balance the vertical components of the tensions in the band at A and B. The horizontal components of these tensions balance one another.

If W is the load at D over and above that required to produce static balance when w is removed, and the whole brake is free to move about its axis, which is the axis of the wheel, then when the brake is in action, with the compensating lever floating and the loads W and w on, the driving torque is $WR - wr$. If, however, the compensating lever is not floating, but rests against one of the stops, the driving torque is $WR - wr \pm p(r+a)$, where p is the reaction of the stop on the compensating lever. The + sign is to be taken when the lever is against the lower stop, and the − sign when the lever is against the upper stop.

If N is the speed of the wheel in revolutions per minute, and if the forces are measured in lbs. and distances in feet, then the horse-power absorbed is $\frac{2\pi\{WR - wr \pm p(r+a)\}N}{33000}$. If the brake is carefully adjusted the force p should be small, and may then be neglected.

299. Pullen's Friction Brake Dynamometer. — A form of dyna-

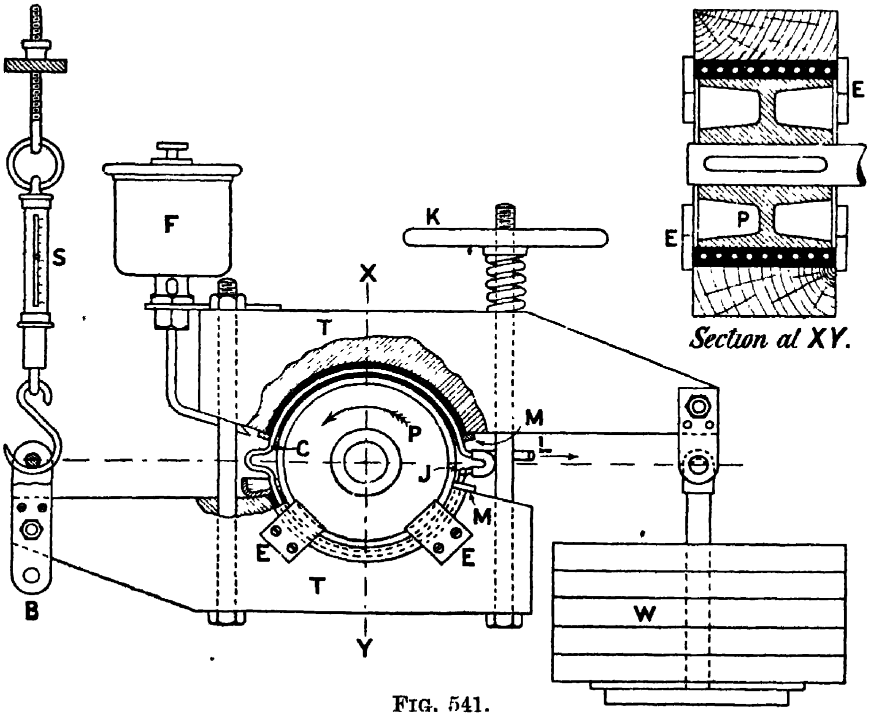


FIG. 541.

mometer suitable for testing small high-speed motors, such as petrol

engines, is shown in Fig. 541. This is a design by Prof. W. W. F. Pullen.* P is an ordinary cast-iron pulley, 6 inches in diameter, mounted on the shaft of the motor to be tested. Small solid-drawn copper tubes C are bent to the shape shown. Small plates of brass M having holes drilled in them are threaded over the copper tubes, and brazed on to them in the positions shown. The copper tubes are placed in position on the pulley, and are then embedded in white metal, plaster of Paris being used for moulds. The white metal is shown black in the sections. Two timber levers T are fitted over the white metal, and are held together by two bolts, one of which has a hand-wheel K for a nut, and between this hand-wheel and the upper lever there is a helical spring, which enables a practically constant pressure to be maintained between the white metal and the pulley when the dynamometer is in use. The rubbing surface of the pulley is lubricated with oil from the sight-feed lubricator F. Water is circulated through the copper tubes, entering at J and leaving at L. The brake is retained in position on the pulley by small wooden ear-pieces E. The spring balance S is useful for measuring small variations of torque, but most of the load is put on by dead weights W. A dash-pot and any weight required to balance the parts may be connected at B. This brake easily absorbed 8 horse-power at 1000 revolutions per minute without undue heating.

300. Rope Brake Dynamometer.- The simplest and most reliable form of absorption dynamometer is probably the rope brake, shown in Fig. 542. One, two, or more lengths of rope are passed once round the rim of the fly-wheel or the rim of a pulley fixed on the shaft. The different lengths of rope are kept in position by blocks of wood, as shown, the blocks being laced to the rope. The upper ends of the several lengths of rope are united and attached to a spring balance B, while the other ends are united and attached to the weight W. Let

W = hanging weight, in lbs., including portion of rope, hook, etc., hanging from A.

S = tension registered by spring balance, less the weight of the rope, etc., between A and the balance, in lbs.

R = effective radius of wheel = nominal radius of wheel + radius of rope, in feet.

N = number of revolutions of wheel per minute.

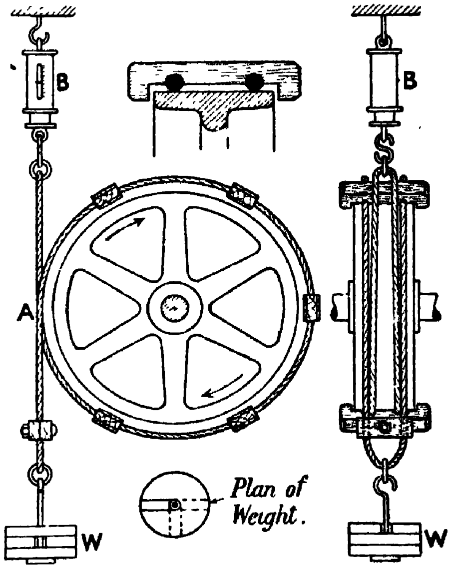


FIG. 542.

The effective resistance at radius R is W − S, and the brake horse-power is therefore $= \dfrac{2\pi RN(W-S)}{33000}$.

* *Transactions of the Civil and Mechanical Engineers Society* (London), 1907.

The whole of the work is converted into heat at the rubbing surfaces between the rope and the wheel. *For small powers or for short trials the air in contact with the revolving wheel will carry away enough heat to keep the wheel sufficiently cool. For larger powers and long trials,* however, it is necessary to cool the wheel rim with water. For water cooling it is best to make the rim of the wheel of channel section, as shown in Fig. 543. The water is held in by centrifrugal force so long as the speed of rotation is not less than a certain critical speed. The centrifrugal force of a small mass of water weighing w lbs. at a radius R feet and revolving at N revolutions per minute is $F=\frac{w}{gR}\left(\frac{2\pi RN}{60}\right)^2$. When this mass of water is in its highest position the resultant force holding it to the rim of the wheel is $F-w$, and the minimum speed at which the water will remain in contact with the wheel in its highest position is found by putting $F=w$, then $N=\frac{30}{\pi}\sqrt{\frac{g}{R}}$.

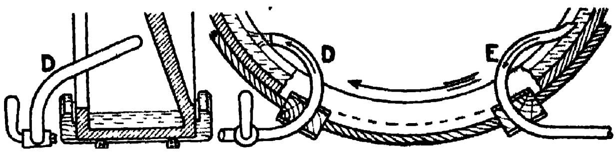


FIG. 543.

During a long trial it is necessary to renew the cooling water. This may be done by using two pipes, as shown in Fig. 543. One pipe D supplies cold water to the channel in the rim, and the other E scoops water out and discharges it. Just before stopping a trial the water supply is cut off, and the pipe E is turned over slightly so as to nearly touch the bottom of the channel and collect and discharge sufficient water to prevent an overflow from the channel when the wheel stops. The collecting end of the pipe E is flattened out so as to present a narrow slit to the water.

301. Fan Brake Dynamometer.—One of the simplest and most convenient of dynamometers for testing the output of small high-speed motors is a simple form of fan. This was first used by M. Renard, whose design for small powers consisted of a rectangular bar of ash, to which were

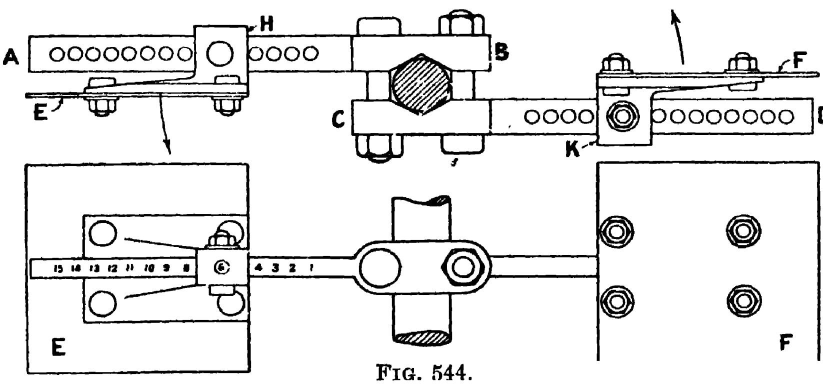


FIG. 544.

bolted two rectangular aluminium plates, the wooden bar being mounted on the shaft of the motor at right angles to its axis. Fig. 544 shows the

brake as made by Mr. W. G. Walker, and which the writer has used successfully since the beginning of 1905 for testing petrol motors.

With the exception of the plates E and F, which are made of aluminium, all the parts are made of steel. The arms AB and CD, of rectangular cross section, are clamped to the shaft of the motor by two bolts, as shown. The plates E and F are bolted to brackets H and K respectively, which are carried by the arms. The plates are equally distant from the axis of the shaft, but the distance may be varied according to the speed and power of the motor. Different sizes of plates may also be used. In the brake used by the author there are three sets of plates, the smallest being 6 inches × 6 inches, the intermediate size 8½ inches × 8½ inches, and the largest 17 inches × 8½ inches. With the smallest plates the distance of the outer edges from the axis of the shaft may be varied from 8 inches to 15 inches; with the intermediate size plates this distance is from 10½ inches to 17½ inches; and for the largest size, 19 inches to 26 inches.

The resistance is, of course, the pressure of the air on the plates and exposed parts of the arms, and this produces a pure torque, and there is no bending action on the shaft, except that due to the weight of the instrument, which is only about 9¾ lbs. with the smallest plates, and 14¼ lbs. with the largest. The parts are also perfectly balanced.

Once the instrument is calibrated it is only necessary to know the speed to determine the horse-power, since the power to drive the fan is proportional to the cube of the speed. For example, when the intermediate size plates are used, and the bracket pins are in the sixth holes from the inside, the brake horse-power is given by the formula,

$$\text{B.H.P.} = 0{\cdot}000{,}000{,}0038N^3 = 38 \times 10^{-10}N^3,$$

where N is the speed in revolutions per minute.

The resistance to the motion of the plates is proportional to the density and viscosity of the air. Now the density is proportional to the pressure, and inversely proportional to the absolute temperature, while the viscosity, according to some authorities, is proportional to the absolute temperature. Hence the resistance varies with the pressure only, and if the coefficient c in the equation B.H.P. $= cN^3$ is obtained experimentally when the height of the barometer is h, then if, when a test is made, the height of the barometer is h', the coefficient c should be multiplied by h'/h.

This dynamometer may be run for any length of time, as there is no heating effect on the instrument, the heat being carried away by the circulating air.

The dynamometer having the dimensions given above may be used for powers up to 20 horse-power.

302. Eddy Current Brake Dynamometer.—In the *eddy current brake dynamometer* the resisting torque is obtained without actual contact between the revolving and the floating elements. A system of field magnets, with alternate poles, is mounted on the floating portion of the brake, while the motor under test drives one or two copper discs past the pole faces, whose magnetic flux induces very large circulating currents, which, by their magnetic action, tend to retard the motor and absorb its energy in heating the discs.

Brakes on this principle were constructed by Pasqualini in 1892, by Grau in 1900, by Siemens and Halske in 1901, and by others. These

brakes, though convenient in laboratory use, did not become extensively employed owing to the skill required in fixing to the motor, and also to the small horse-power absorbed for a given size, and cost of apparatus. The great accuracy and convenience of the electrical control of the resisting torque, and also the cleanliness arising from the use of air instead of water to get rid of the heat, was, however, early recognised.

Morris & Lister, in 1905,* gave the theory of this apparatus, and showed how the design might be greatly improved and simplified. They showed also that there was a certain thickness of copper which ought to be used on the discs, and that it was just as bad to use too much as too little copper.

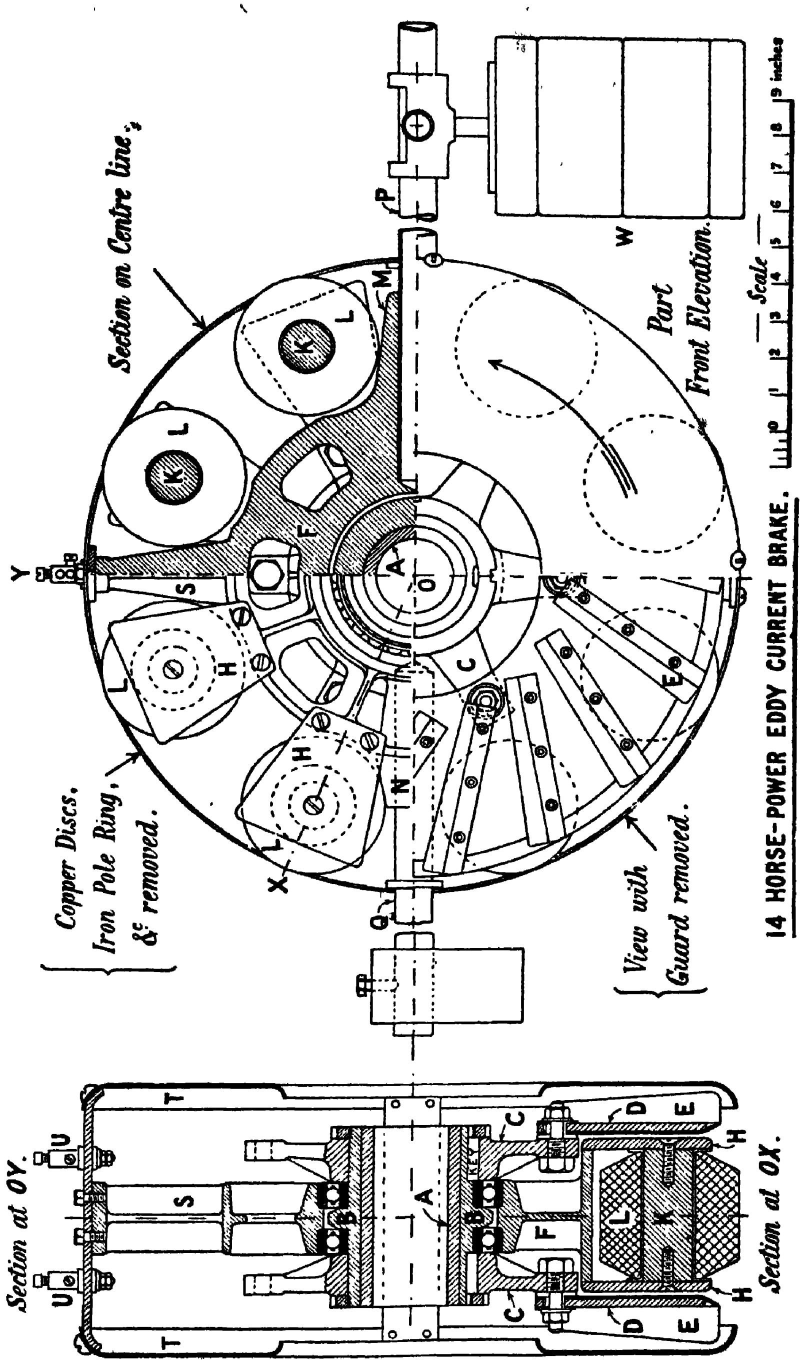
Fig. 545 shows the Morris & Lister eddy current brake.† The brake may be mounted on its own shaft in a frame, independent of the motor to be tested. In some cases, however, especially with electric motors, it is simpler to mount the brake direct on the shaft of the motor in place of the driving pulley, or, in the case of a petrol motor, in place of the fly-wheel. A is a central cast-iron bush, to be secured to the shaft of the motor to be tested. On this bush are fitted two ball-bearings B, and two strong aluminium spiders C, carrying stout iron discs D, faced with sheet copper on the inner sides, and provided with cooling vanes E on the outer sides. On the ball-bearings and between the discs floats a strong aluminium casting F, formed for carrying conveniently a number of flat iron pole pieces H, arranged in pairs opposite one another. Between each pair of poles is a stout iron core K, on which a coil L is slipped. These coils are so connected that the poles present alternately magnetised faces to each disc. The central casting F has also two strong bosses M and N at the ends of its horizontal diameter. Into the boss M is inserted the main graduated lever P, on which slides the carrier for the weights W. Into the boss N is inserted the short counterpoise lever Q. The long lever is sometimes replaced by a short one provided with a hook for a spring balance. Further projections S from the central casting F at the top and bottom support the outside guards T and the insulated terminals U, by which current is led into the windings from a source of continuous current supply.

When the magnets are excited, the magnetic flux from each pole is compelled to cross the revolving copper disc in order to reach the iron disc behind it, and to return by the adjacent magnetic poles. The flux has then to cut the other copper disc twice in a similar manner before the magnetic circuit is completed. In this way large eddy or Foucault currents are generated in the copper discs, which then exert a resisting torque, the power corresponding to which is converted into heat in the discs. This heat is got rid of by means of the vanes E, which are set so as to induce a strong current of air.

The discs have to be so supported on the spider as to prevent the passage of heat to the arms, and so to the framework of the brake or motor, and at the same time to allow them to expand while still keeping true. This is done by mica washers and slotted holes in the discs. The spiders also must be able to resist the attractive force on the discs, which is large, although it diminishes as the speed increases.

* *Journal of the Institution of Electrical Engineers*, vol. xxxv. p. 445.

† Made by Messrs. Morris & Lister, Ltd., Coventry.



14 HORSE-POWER EDDY CURRENT BRAKE.

FIG. 545.

By means of a special regulator, or other suitable regulating appliance, the exciting current can be increased until the lever floats, the weights having been previously placed to correspond with the required torque. The amount of electric power required for exciting the magnets is quite small. Thus a torque of 73 ft.-lb., which gives 14 horse-power at 1000 revolutions per minute, requires less than $\frac{1}{4}$ kilowatt in the coils. The exciting current need not be measured.

The power is computed by means of the formula used for the Prony brake, namely, $\text{B.H.P.} = \frac{2\pi WRN}{33000}$.

A good feature of the eddy current brake is that the resisting torque is practically constant over a considerable range of speed (10 per cent. above or below normal). Hence when testing petrol or other motors in which the effort fluctuates, the lever does not oscillate but floats steadily, regardless of periodic fluctuations of speed. This constancy of torque at or near its rated speed occurs in a similar way to the maximum torque in an induction motor, and constitutes not the least of the advantages of this convenient and accurate type of absorption brake dynamometer.

A small amount of power is used in overcoming the air resistance at the vanes E; a portion of this resistance is communicated to the guards T on the floating element, but there remains a certain amount which is not communicated to the floating element, and is therefore not measured, but, if necessary, this may be allowed for by using a constant determined by experiment. The unmeasured resistance is, however, only a fraction of 1 per cent. of the total resistance.

303. Epicyclic-Train Dynamometer.—One form of transmission dynamometer is shown in Fig. 546. AB is a lever, which may turn round the fixed axle CD. Mounted on the lever, and turning freely on it, are two equal bevel wheels E and F, which gear with two equal bevel wheels H and K, mounted on the axle, and turning freely on it. A wheel or pulley is secured to the boss L of the wheel H, and another wheel or pulley is secured to the boss M of the wheel K. The torque to be measured is transmitted from L to M, or from M to L through the wheels E and F.

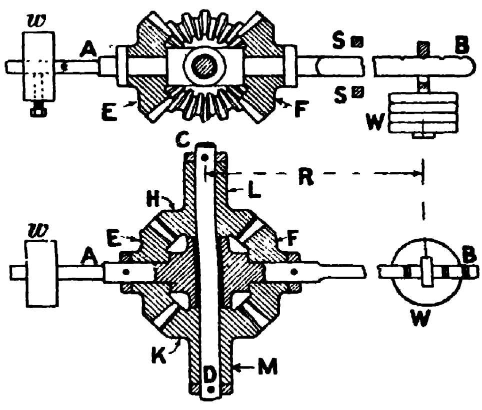


FIG. 546.

Let P be the effort exerted by the teeth of the wheel H on the teeth of the wheel E at a radius r from the axis of CD, and suppose P to act downwards. There will be an equal effort P at radius r from the axis of CD acting upwards on F from H. The torque on H is therefore $2Pr$. The wheel E in driving K will cause the latter to exert a downward force P at radius r from the axis of CD, and the wheel F in driving K will cause the latter to exert an upward force P at radius r from the axis

of CD. These four forces acting on the wheels E and F will produce a torque on the lever AB equal to $4Pr$. Hence if the weight W on the lever is at a radius R from the axis of CD, $WR = 4Pr$, and the torque transmitted from L to M is $\frac{1}{2}WR$.

If the wheels H and K run at a speed of N revolutions per minute, then the horse-power transmitted is $\frac{WR \times 2\pi N}{2 \times 33000} = \frac{\pi WRN}{33000}$.

When unloaded, the lever is balanced by the weight w.

Stops SS limit the motion of the lever.

304. Belt Dynamometers.—When a belt is transmitting power from one pulley to another the tangential effort on the driven pulley is equal to the difference between the tensions on the tight and slack sides of the belt. If T_1 and T_2 are these tensions in pounds, and V is the speed of the belt in feet per minute, then the horse-power transmitted is

$$\frac{(T_1 - T_2)V}{33000}.$$

Several forms of dynamometer have been introduced for measuring $T_1 - T_2$ directly while the belt is running; one form is shown in Fig. 551, p. 362, another is shown in Fig. 547. The latter illustration and the

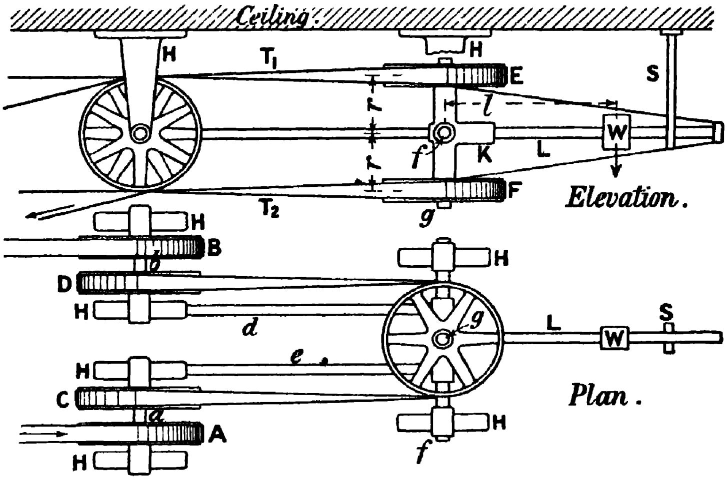


FIG. 547.

following description are taken from a paper by Mr. S. P. Watt in the *Transactions of the American Society of Mechanical Engineers*, 1891.

The dynamometer (Fig. 547) consists of a set of pulleys mounted on a suitable frame and disposed as follows. The pulleys A and C are fixed to the shaft a, and B and D are fixed to the shaft b. The pulleys E and F revolve freely as independent loose pulleys on the shaft g. The shaft g has a second shaft f, fixed to it midway between the pulleys E and F. Shaft f constitutes a pivoting axis, parallel to the shafts a and b, for the shaft g, together with the frame K and the weight lever L, all rigidly connected. Only enough motion of L is allowed to determine the direction of action. Instead of the weight of the lever L, the end of the lever could be connected to a small platform scale, and its tendency to rotate weighed. It

might be useful to make the frame between the pulleys adjustable at d and e in order to vary the tension of the dynamometer belt, or better still, it might be so constructed that the absolute tension could be noted at any time, whether working or at rest. The working of the apparatus is as follows. A driving belt from the source of power is put on the pulley A. The machine to be driven is belted from the pulley B. The dynamometer belt passes from the lower side of the pulley C to the pulley F, around F to D, around D to E, around E back to C. It will be seen that C is a driving pulley, and D a driven pulley. When the system is at rest, the four strands of the dynamometer belt have the same tension. If now C revolve and drive D, the tension T_1 of the belt from C around the loose pulley F to D will correspond to the tension of the taut side in a simple system of two pulleys, and the tension T_2 of the belt from the lower side of D around F back to the lower side of C will correspond to the slack side.

The difference of tension is the driving force P, and taking what actually occurs,

$$\frac{2T_1 - 2T_2}{2} = T_1 - T_2 = P.$$

Now P in pounds multiplied by the speed of the belt is foot-pounds developed or consumed, ignoring friction. Let r be the radius of the position of pulleys E and F from the pivot f. Let l be the distance of the weight W from f, to balance the tendency of the frame K to rotate about f when working, then

$$Wl = r(2T_1 - 2T_2), \text{ or } \frac{Wl}{2r} = T_1 - T_2 = P.$$

It is evident that should a Prony brake be put in place of the pulley B, the power developed by the motor to A could be determined. If a machine be driven from the pulley B, the power consumed could also be noted in the speed of belt and the position of the weight from the same formula. In the use of different belts as dynamometer belts the relative efficiency of such belts can readily be determined by the use of the brake attachment. It will also be seen that only one side of the belt comes in contact with the pulleys.

305. Torsion Meters.—The introduction of the steam turbine, particularly for the propulsion of ships, has created a demand for a means of measuring the power transmitted by the shaft, since there is no direct means of measuring the work done in the turbine. The power to be measured is generally very large, and the ordinary forms of dynamometers are not suitable or convenient. In the case of the reciprocating engine the power developed in the cylinders is readily determined from the indicator diagrams, and if these be taken at any given load and also at no load, the actual power transmitted by the shaft at the given load can be determined with sufficient accuracy for most practical purposes.

A number of instruments have been designed to measure the angle of twist of a given length of shaft transmitting power, and from this observation and the speed of the shaft, and certain particulars of the shaft itself, the power transmitted is readily computed.

From formulæ proved in Art. 95, p. 80, and Art. 96, p. 81, the angular deflection of a shaft in degrees is, $n = \frac{180 \times 32Tl}{\pi^2 Cd^4}$ for a solid shaft, and $n = \frac{180 \times 32Tl}{\pi^2 C(D^4 - d^4)}$ for a hollow shaft, where T is the twisting moment, l the length of shaft considered, C the modulus of rigidity, d the diameter of the solid shaft, D and d the external and internal diameters respectively of the hollow shaft. Evidently for a particular shaft $T = \frac{kn}{l}$, where k is a constant to be determined experimentally for the particular shaft. In the absence of direct experiment on the shaft itself, k may be computed by assuming a value for C based on experiments on shafts or rods of similar material, then $k = \frac{\pi^2 Cd^4}{180 \times 32}$ for a solid shaft, and $k = \frac{\pi^2 C(D^4 - d^4)}{180 \times 32}$ for a hollow shaft.

If force is in lbs. and linear dimensions in inches, then the horse-power transmitted at N revolutions per minute is,

$$H = \frac{2\pi TN}{12 \times 33000} = \frac{2\pi knN}{12 \times 33000l} = \frac{k_1 nN}{l}, \text{ where } k_1 = \frac{2\pi k}{12 \times 33000}$$

is a constant for the particular shaft.

Two types of torsion meters will now be illustrated and described, the particulars being taken from a paper by Mr. J. Hamilton Gibson, read before the North-East Coast Institution of Engineers and Shipbuilders in January 1908.

The main features of *Fottinger's torsion meter*, which is a purely mechanical contrivance, are shown in Fig. 548. A is a disc secured

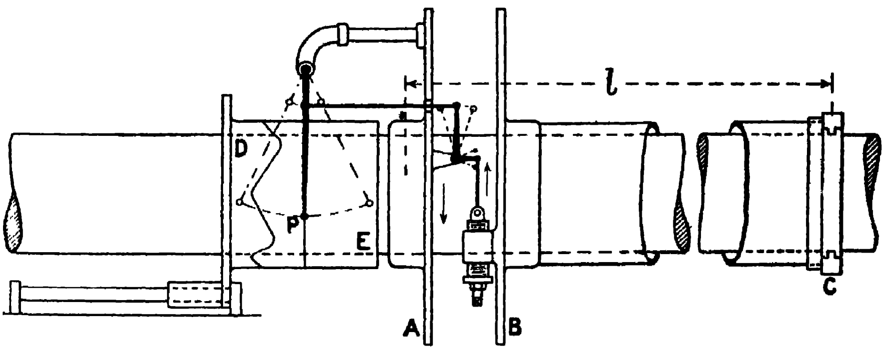


FIG. 548.

directly to the shaft. B is another disc secured to the shaft at a distant section C through a stiff tube coaxial with but clear of the shaft. The motion of these two discs will be the same as that of the points on the shaft at which the connections are made, and which are at a distance l apart, and the relative angular motion of these discs will be the angle of twist of the length l of the shaft.

The relative angular movement of the discs is magnified and recorded

by a pencil P, actuated by the system of levers shown, on paper placed round the fixed cylinder DE, which is coaxial with the shaft. When there is no torque on the shaft the pencil traces a line parallel to the ends of the cylinder carrying the paper, and this is the zero line of the diagram. When the shaft is transmitting power the pencil moves to the right or left of the zero line, depending on the direction of rotation of the shaft, and at the same is carried round with the shaft, describing a more or less wavy line whose ordinates represent the angular deflection of the shaft as it revolves. The ordinates of the line traced by the pencil are evidently arcs of circles, whose centres lie on the zero line, and whose radii are equal to the length of the pencil lever. The cylinder carrying the paper can be moved to the left clear of the pencil, and the diagram can then be taken off and the mean torque determined. It is of course the mean torque which must be used in computing the horse-power. In the case of a turbine-driven shaft the torque is practically uniform.

The *Bevis-Gibson flash-light torsion meter* depends on the facts that the velocity of light is practically infinite, and that light travels in straight lines through air of uniform density. Two blank discs A and B are fixed on the shaft at a convenient distance apart, as shown in the oblique elevation at (*a*), Fig. 549. Each disc has near its periphery a

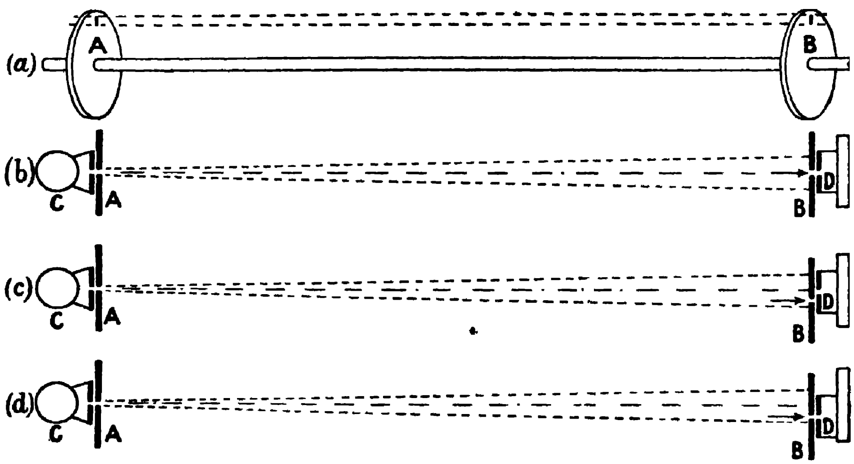


Fig. 549.

small radial slot, and these two slots are in the same radial plane when no power is being transmitted and there is no torque on the shaft. (*b*), (*c*), and (*d*), Fig. 549, are sectional plans, the planes of section going through the slots in the discs when in or near their highest positions. Behind the disc A is fixed on, say, one of the bearings of the shaft, a bright electric lamp C, masked, but having a slot cut in the mask directly opposite the slot in the disc A when the latter slot is in its highest position. At every revolution of the shaft a flash of light is projected through the slot in the disc A towards the disc B in a direction parallel to the shaft. Behind the disc B, on, say, another shaft bearing, is fitted the torque finder D, an instrument fitted with an eye-piece, and capable

of slight circumferential adjustment. The end of the eye-piece next the disc B is masked, except for a slot similar and opposite to the slot in the disc. When the four slots are set in line, as shown at (*b*), Fig. 549, a flash of light is seen at the eye-piece every revolution, and if the shaft revolves quickly enough the light will appear to be continuous. This effect is apparent at any speed over 100 revolutions per minute. At lower speeds the flash is seen to be intermittent, but this in nowise affects the accuracy and reliability of the result.

Suppose now that the shaft is transmitting power. One disc lags behind the other, and although the slots in C, A, and D are still in line, the light is cut off by the displacement of the slot in B, due to the lag just mentioned. This cutting off of the light is clearly shown at (*c*). Now if the torque finder D be moved round by an amount equal to the lag of the disc B the slot in D will then be opposite to the slot in B when the slot in A is opposite to the slot in C, and the flash will now be received by D, as shown at (*d*). The torque finder is moved by operating a micrometer spindle, and by means of a scale and vernier the angular movement can be measured to the $\frac{1}{100}$ of a degree.

The Bevis-Gibson torsion meter as just described will evidently give the twist of the shaft at one definite point of each revolution, and in the case of turbine shafts, where the torque is practically uniform, this is all that is required. For reciprocating engines, where the torque varies considerably during each revolution, a simple modification enables the operator to take several readings, usually twelve, at definite points of a revolution. The discs are perforated with slots arranged in the form of a spiral, as shown in Fig. 550. The lamp and torque finder must be moved radially so as to bring them into line with each corresponding pair of slots in the discs. Plotting the readings on squared paper, the actual twisting moment diagram can be drawn, and from this the mean torque is readily found.

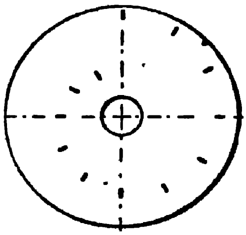


Fig. 550.

Exercises XXI.

1. The drum of a band brake is 18 inches in diameter. The band is $\frac{3}{16}$ inch thick, and it embraces three-quarters of the circumference of the drum. The band lever is arranged as shown at (*c*), Fig. 534, p. 344. AC = 3 inches, and AD = 18 inches. If the force P at the end of the lever is 40 lbs., and the co-efficient of friction between the band and the drum is 0·2, what is the resisting torque, in ft.-lbs., exerted on the brake drum?

2. In a band and block brake (Fig. 535, p. 345) the wheel is 24 inches in diameter, and the band is $\frac{1}{16}$ inch thick. There are twelve wood blocks, each 2 inches thick, and each subtending an angle of 18 degrees at the centre of the wheel. The coefficient of friction between the blocks and the wheel is 0·35. The brake is operated by means of a lever, arranged as shown at (*d*), Fig. 534, p. 344. AB = 4 inches, and AD = 24 inches. What force must be applied to the end D of the lever when a weight of 300 lbs. is being lowered at a uniform velocity, the weight being hung by a rope which is coiled round a barrel on the axle of the brake wheel, the effective diameter of the barrel being 20 inches?

3. A wheel 12 feet in diameter, rotating at the rate of one revolution in 2 seconds, is acted on by a brake which applies normal pressures of 1 cwt. each at opposite ends of a diameter. If the coefficient of friction be 0·6, find (in horse-power) the rate at which work is being absorbed? [Inst.C.E.]

4. A bicycle and rider, weighing together 180 lbs., are travelling at the rate of 10 miles per hour on the level. Supposing a brake is applied to the top of the front wheel, which is 30 inches in diameter, and this is the only resistance acting, how far will the bicycle travel before stopping if the pressure of the brake is 20 lbs., and the coefficient of friction is 0·5? [Inst.C.E.]

5. Continuous brakes are now capable of reducing the speed of a train 3¾ miles an hour every second, and take 2 seconds to be applied; show in a tabular form the length of an emergency stop at speeds of 3¾, 7½, 15, 30, 45, 60 miles per hour. Compare the retardation with gravity, and express the resisting force in lbs. per ton. [U.L.]

6. If the force available on the block of the brake on a wheel of a railway vehicle is 90 per cent. of the weight on the wheel, and if the coefficients of sliding friction between the block and the wheel and between the wheel and the rail are, at 60 miles per hour, 0·06 and 0·04 respectively, what is the maximum resistance to the motion of the wheel, in lbs. per ton, when the brake is applied, (*a*) when the wheel does not skid, (*b*) when the wheel skids, at the above speed?

7. To determine the brake horse-power of a small de Laval steam turbine a Prony brake was used. The brake was placed on a pulley on the second-motion shaft, whose speed was 2992 revolutions per minute. The brake load at 18 inches from the axis was 5 lbs. Calculate the brake horse-power.

8. The internal diameter of a fly-wheel rim, which is of channel section, is 5 feet. Find the minimum speed, in revolutions per minute, at which the wheel will hold, without spilling, a layer of water 1 inch deep.

9. The brake horse-power of a gas-engine is to be measured with a rope brake on the fly-wheel. The diameter of the wheel is 5 feet, and the diameter of the rope is ½ inch. At a speed of 183 revolutions per minute the hanging weight is 67½ lbs., and the spring balance indicates 4¾ lbs. What is the brake horse-power?

10. In an epicyclic-train dynamometer of the form shown in Fig. 546, p. 356, the wheels on the axle CD run at 100 revolutions per minute. The weight W on the lever is 60 lbs., and its distance from the axis of the axle is 24 inches. Calculate the horse-power transmitted through the dynamometer.

11. Fig. 551 shows a Froude and Thornycroft dynamometer for measuring the difference between the tensions on the tight and slack sides of a belt which is transmitting power from a pulley A to a pulley B. The T shaped lever has its fulcrum at D, and carries pulleys E and F. The diameters of the pulleys are such that the straight parts of the belt may be taken as horizontal. Neglecting the work lost in friction in the instrument, show that the horse-power transmitted is given by the formula, $HP = \frac{\pi d N W r}{33000 \times 2c}$, where d is the diameter of the pulley A, and N its speed in revolutions per minute, dimensions being in feet, and W in lbs.

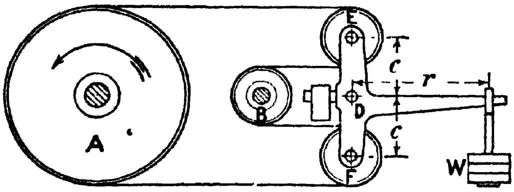


FIG. 551.

12. The horse-power of a marine steam turbine was found by observing that the angle of twist of a 20-feet length of the propeller shaft at 480 revolutions per minute was 1·75 degrees. The shaft, which was solid, had a diameter of 7 inches, and it was known that the modulus of rigidity of the material of the shaft was 12,000,000 lbs. per square inch. Neglecting the effect of the end thrust, calculate the horse-power.

CHAPTER XXII

BELT, ROPE, AND CHAIN GEARING

306. Velocity Ratio in Belt Gearing.—If motion be transmitted from one pulley to another by means of a thin inextensible belt, and if there is no slipping between the belt and the rims of the pulleys, every part of the belt will have the same velocity, and the outer surfaces of the rims of the pulleys will have the same velocity as the belt. Hence if d_1 and d_2 be the diameters of the driver and follower respectively, and if the driver makes N_1 revolutions in a given time, while the follower makes N_2 revolutions in the same time, then $\pi d_1 N_1 = \pi d_2 N_2$, and $\frac{N_2}{N_1} = \frac{d_1}{d_2}$. This formula is true, whether the belt is "open," as in Fig. 552, or "crossed," as in Fig. 553; but the direction of the rotation will not be the same in these two cases. With an open belt the pulleys rotate in the same direction, while with a crossed belt they rotate in opposite directions.

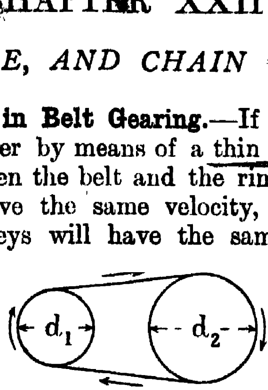


FIG. 552.

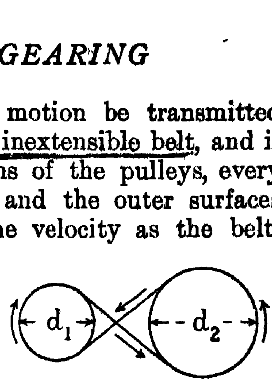


FIG. 553.

307. Effect of Thickness of Belt on Velocity Ratio.—When a thick belt is bent over a pulley its inner surface is compressed and its outer surface is stretched, but the surface midway between these two remains of the same length. It follows, therefore, that the velocity of the inner surface of the belt in contact with the pulley must be less than the velocity of the middle surface of the belt, and it is only the middle surface of the belt which has the same velocity at every point. The effective radius of a pulley is therefore its nominal radius plus half the thickness of the belt, and using the notation of the preceding Article, $\frac{N_2}{N_1} = \frac{d_1 + t}{d_2 + t}$, where t is the thickness of the belt.

308. Length of Belt connecting Two Pulleys.—Referring to Figs.

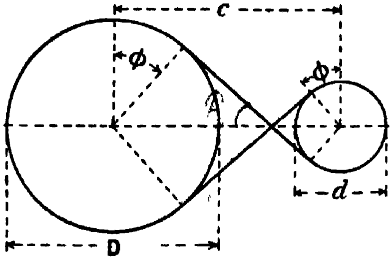


FIG. 554.

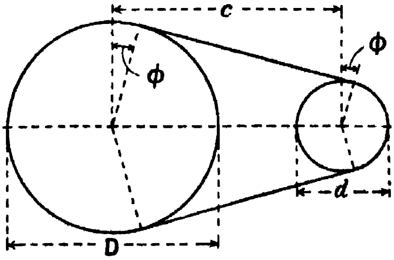


FIG. 555.

554 and 555, the length of belt in contact with the larger pulley is

$\frac{D}{2}(\pi + 2\phi) = D\left(\frac{\pi}{2} + \phi\right)$. The length of belt in contact with the smaller pulley is $\frac{d}{2}(\pi \pm 2\phi) = d\left(\frac{\pi}{2} \pm \phi\right)$, where the + sign applies to the crossed belt (Fig. 554), and the − sign to the open belt (Fig. 555). The inclination of the straight portions of the belt to the line of centres is ϕ, and the length of each straight portion is $c \cos \phi$. Hence if l is the total length of the belt,

$$l = D\left(\frac{\pi}{2} + \phi\right) + d\left(\frac{\pi}{2} \pm \phi\right) + 2c \cos \phi$$

$$= \frac{\pi}{2}(D + d) + \phi(D \pm d) + 2c \cos \phi$$

$$\sin \phi = \frac{D \pm d}{2c}, \quad \cos \phi = 1 - 2 \sin^2 \frac{\phi}{2}.$$

If ϕ is a small angle, then, approximately, $\phi = \sin \phi$, and

$$\tfrac{1}{2} \sin \phi = \sin \frac{\phi}{2}.$$

Hence approximately

$$l = \frac{\pi}{2}(D + d) + \sin \phi(D \pm d) + 2c\left(1 - 2 \sin^2 \frac{\phi}{2}\right)$$

$$= \frac{\pi}{2}(D + d) + \frac{(D \pm d)^2}{2c} + 2c - \frac{(D \pm d)^2}{4c}$$

$$= \frac{\pi}{2}(D + d) + \frac{(D \pm d)^2}{4c} + 2c,$$ where the + sign applies to the crossed belt, and the − sign to the open belt.

Referring to the crossed belt (Fig. 554), it is evident that if $D + d$ is constant, and c is fixed, ϕ will remain the same, and therefore l is constant.

309. Stepped Pulleys.—Two or more pulleys of different diameters placed side by side form a *stepped pulley*. A stepped pulley is, however, generally cast in one piece.

A pair of stepped pulleys and one belt form a common arrangement for driving a shaft or spindle at different speeds from a shaft rotating at a fixed speed. Fig. 556 shows a pair of stepped pulleys mounted on shafts A and B, whose axes are parallel and at a distance c apart. Let the speed of A be N revolutions per minute, and let it be required to make B rotate at N_1, N_2, or N_3 revolutions per minute as may be necessary. Each pulley will require three steps. Let the diameters of the pulley on A be D_1, D_2, and D_3, and let the diameters of the pulley on B be d_1, d_2, and d_3. The following equations can be stated at once, viz. $\frac{d_1}{D_1} = \frac{N}{N_1}$, $\frac{d_2}{D_2} = \frac{N}{N_2}$, and $\frac{d_3}{D_3} = \frac{N}{N_3}$. A value may now be selected for one

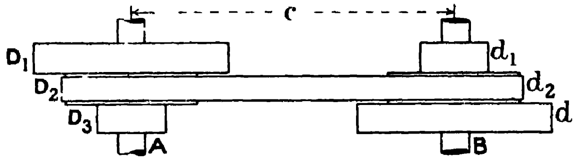


FIG. 556.

of the diameters, say D_1, then $d_1 = \frac{ND_1}{N_1}$. Having fixed D_1 and d_1, no other diameters can be selected arbitrarily. The other diameters must not only satisfy the equations $\frac{d_2}{D_2} = \frac{N}{N_2}$, and $\frac{d_3}{D_3} = \frac{N}{N_3}$, but they must be such that the same belt will be equally tight when placed on corresponding pairs of steps. This gives rise to two cases, (1) belt crossed, (2) belt open.

For a crossed belt it was shown in the preceding Article that if the sum of the diameters of the pulleys is not altered, the length of the belt will be the same. Therefore, for a crossed belt, having fixed D_1 and d_1, $D_1 + d_1$ is known, and $D_2 + d_2$, also $D_3 + d_3$ must be equal to $D_1 + d_1$. The sum of a pair of diameters being known, and also the ratio of the one to the other, the diameters can be easily found.

Coming now to the case where the belt is an open one, D_1 and d_1 are determined as before, then the length of the belt is

$$l = \frac{\pi}{2}(D_1 + d_1) + \frac{(D_1 - d_1)^2}{4c} + 2c.$$

Then for D_2 and d_2,

$$l = \frac{\pi}{2}(D_2 + d_2) + \frac{(D_2 - d_2)^2}{4c} + 2c,$$

but $d_2 = \frac{ND_2}{N_2}$, therefore

$$\frac{\pi}{2}D_2\left(1 + \frac{N}{N_2}\right) + \frac{D_2^2}{4c}\left(1 - \frac{N}{N_2}\right)^2 + 2c = l,$$

a quadratic equation from which

$$D_2 = \frac{2c}{(1 - n_2)^2}\left\{\sqrt{\left(\frac{\pi^2}{4}(1 + n_2)^2 + \frac{(l - 2c)}{c}(1 - n_2)^2\right)} - \frac{\pi}{2}(1 + n_2)\right\},$$

where $n_2 = \frac{N}{N_2}$.

If $N_2 = N$, then $D_2 = \frac{l - 2c}{\pi}$.

The solution of the quadratic equation may be avoided, and a result sufficiently accurate in most cases obtained, by first finding values for D_2 and d_2 on the assumption that the belt is crossed, that is, $D_2 + d_2 = D_1 + d_1$. Let the difference between the values of D_2 and d_2 thus found be equal to a, then approximately $\frac{\pi}{2}(D_2 + d_2) + \frac{a^2}{4c} + 2c = l$, a simple equation from which $D_2 + d_2$ can easily be found. Use this more exact value of $D_2 + d_2$, together with $\frac{d_2}{D_2} = \frac{N}{N_2}$, to find a more accurate value of $D_2 - d_2$, which is to be substituted in the equation $\frac{\pi}{2}(D_2 + d_2) + \frac{(D_2 - d_2)^2}{4c} + 2c = l$. The value of $D_2 + d_2$ found from this simple equation is then used, together with the equation $\frac{d_2}{D_2} = \frac{N}{N_2}$, to find D_2 and d_2.

310. Forms of Rims of Pulleys.—The rim of a pulley for a flat belt

is either straight or convex on the outside of its cross section. At first sight it would seem as if the belt would remain on the straight rim more readily than on the convex one, and that it would be still more secure from falling off if the section of the rim were concave on the outside, but experiment shows that a flat belt tends to run on the largest diameter of the pulley. Consider a belt on a conical pulley (Fig. 557). Each part of the belt as it approaches the pulley receives a set towards the base or larger end of the cone, so that each part of the belt as it passes

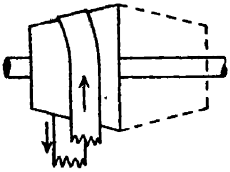


FIG. 557.

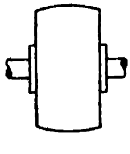


FIG. 558.

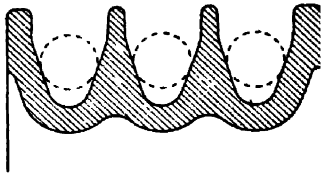


FIG. 559.

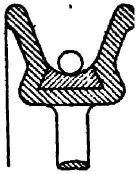


FIG. 560.

on to the pulley is a little nearer the base than the part in front of it; but once in contact with the pulley, it remains in contact without slipping until it leaves on the other side. The result is, therefore, that the belt ultimately reaches the highest part, and to prevent it falling off a similar pulley is placed, as shown by the dotted lines. From this the form shown in Fig. 558 is obtained.

The straight rim is used when it is necessary to move the belt from one part of the rim to another, as in the case where a pulley drives a pair of fast and loose pulleys.

It must be borne in mind, however, that the convex section of rim only helps to keep the belt on the pulley when the belt and pulley rotate together. If the belt should slip through the resistance being too great, it will fall off the pulley more readily if the rim is convex than if it is straight.

For hemp or cotton ropes the pulley rim has V shaped grooves, as shown in Fig. 559, the angle of the V being generally about 45°. The rope does not rest on the bottom of the groove, but on its sides only, so that it is wedged in, causing the resistance to slipping to be much greater.

For wire ropes the design is altered so that the rope rests on the bottom of the groove, as shown in Fig. 560. The resistance to slipping is increased, and the wear of the rope reduced by lining the bottom of the groove with some material softer than metal, such as leather, wood, or tarred oakum.

311. Fast and Loose Pulleys.—The motion of a shaft driven from another by belt gearing may be stopped or reversed without affecting the motion of the driving shaft by using a combination of fast and loose pulleys. A "fast" pulley is one which is fixed to the shaft, while a "loose" pulley is one which can turn freely on the shaft. Fig. 561 shows a pair of such pulleys, F being the fast pulley secured to the shaft by a key K, and L is a loose pulley. When the belt is on F the shaft A revolves, and when the belt is shifted to L the motion of A is stopped.

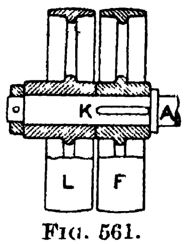


FIG. 561.

The belt is shifted from one pulley to the other by pressing on one edge

of that part of the belt which is advancing towards the pulley by means of a fork, which should be as close to the fast and loose pulleys as possible.

Examples of the use of fast and loose pulleys are shown in Figs. 562 and 563. These arrangements of belt gearing are very frequently found in connection with machine tools. AB is a driving shaft, and CD a counter-shaft. E is a broad pulley fixed on AB. L_1 and L_2 are loose pulleys, and F is a fast pulley. The machine is driven by a belt from the stepped pulley G. An open belt H and a crossed belt K pass from the pulley E to the pulleys on CD, as shown. MN is a rod carrying forks, by means of which the belts H and K may be shifted simultaneously. When the belts are in the position shown, CD is at rest. If MN be moved to the right K will remain on L_2, H will embrace F, and CD will rotate in the same direction as AB. By shifting the belts to the left of the position shown, CD is made to rotate in the opposite direction to AB. A modification of the arrangement just described, to give a quick return motion, is shown in Fig. 563. The latter arrangement may be used to get fast or slow motion as desired, in the same direction, by having both belts open or both crossed.

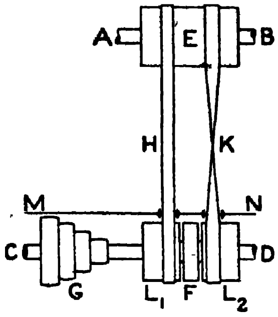


FIG 562.

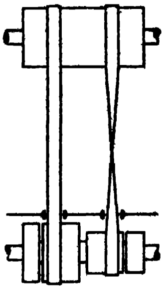


FIG. 563.

In the arrangements of fast and loose pulleys shown in Figs. 562 and 563, the loose pulleys have a width not less than twice the width of the belt. By using a suitable form of belt-shifting gear the loose pulleys may be of the ordinary width, and then not only is space saved on the shaft, but only one belt has to be shifted at a time. Fig. 564 shews such an arrangement. L_1 and L_2 are loose pulleys, while F_1 and F_2 are fast pulleys. The belt forks are attached to levers A_1B_1, and A_2B_2 mounted on fixed pins at C_1 and C_2. These levers are provided with projecting pins at B_1 and B_2, which enter into slots in a disc DE, keyed to a spindle F. The lower parts of the slots in DE are concentric with F, and the upper parts are more or less radial. In the position shown, both belts are on the loose pulleys. If the disc DE be turned in the direction of the arrow H the lever A_1B_1 remains at rest, because the pin at B_1 remains in that part of the slot which is concentric with F, but the pin at B_2 will be pushed to the right by the upper part of the right-hand slot, and the belt which was on L_2 will be shifted to F_2. By moving DE from the position shown in the direction of the arrow K the lever A_2B_2 will remain at rest, and the lever A_1B_1 will be moved so as to shift the belt which was on L_1 to F_1.

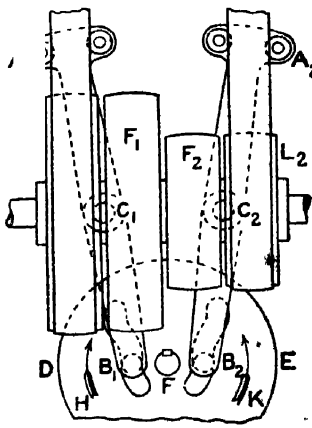


FIG. 564.

312. Belt Gearing for Non-Parallel Shafts.—Motion may be transmitted directly from one shaft to another when the axes are not in the same plane by means of a belt and two pulleys, one pulley on each shaft, provided that the pulleys are so arranged that the middle point of the width of the belt where it leaves one pulley is in the central plane of the other pulley. In other words, the centre line of each of the straight portions of the belt must be in the central plane of the pulley towards which it is travelling. An example is shown in Fig. 565, where the axes of the two shafts are at right angles to one another, but not in the same plane. The arrangement of the two pulleys mentioned above is only possible when the motion is in one direction; if the direction of the motion be reversed, the belt comes off the pulleys.

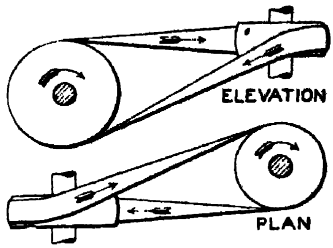


FIG. 565.

When it is not possible or convenient to arrange the pulleys on non-parallel shafts, so as to permit of the one driving the other directly, one or more guide pulleys may be introduced. The guide pulleys must be placed so that all the straight portions of the belt comply with the condition already stated. Fig. 566 shows two pulleys A and B, whose central planes intersect in the line CD. Any convenient points E and F are taken in CD, and tangents EH, EK, FL, and FM are drawn to A and B. Guide pulleys N and O, touching these tangents as shown, and having for their central planes the planes HEK and LFM respectively, will serve to guide the belt between the pulleys A and B. As arranged in Fig. 566, the belt may run in either direction.

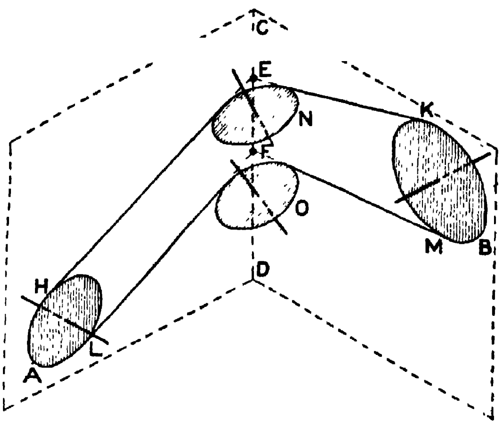


FIG. 566.

Another example of the use of guide pulleys is shown in Fig. 567.

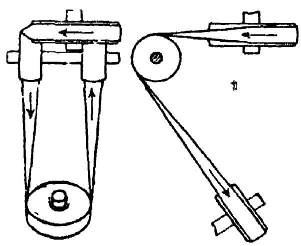


FIG. 567.

313. Straining or Jockey Pulleys.—A belt passing round two pulleys may be tightened without shortening it by placing a third pulley on the slack part of the belt, that is, the part which runs from the driving pulley to the following pulley, as shown in Fig. 568. This third pulley, which is called a *straining*, *tightening*, or *jockey pulley*, runs in bearings which are loaded with a weight to press the pulley on

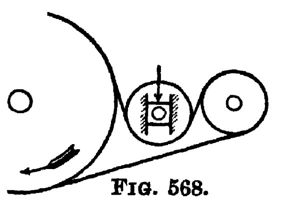


FIG. 568.

the belt, or the desired pressure may be obtained by means of an adjusting screw.

The use of a jockey pulley also permits of two pulleys, differing considerably in diameter, being placed much closer together by increasing the arc of contact of the belt on the smaller pulley.

314. Power Transmitted by Belts.—When a belt is transmitting power from one pulley AB (Fig. 569) to another CD, the motion being in the direction of the arrows, the tension in the portion BD is greater than the tension in the portion AC. BD is called the tight side, and AC the slack side of the belt. Let T_1 = the tension on the tight side, and T_2 = the tension on the slack side, then $T_1 - T_2 = P$ is the driving force at the rim of the pulley CD. If V is the velocity of the belt in feet per minute, v the velocity in feet per second, and H the horse-power transmitted, then $H = \frac{PV}{33000} = \frac{Pv}{550}$, where P is in lbs.

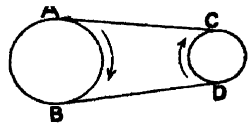


FIG. 569.

The ratio of T_1 to T_2 when the belt is on the point of slipping was discussed in Art. 243, p. 277. For most practical cases T_1 may be taken equal to $2T_2$.

If b is the breadth of the belt, and t its thickness, both in inches, and f the working stress in lbs. per square inch, then $T_1 = btf$, and if $T_2 = nT_1$, where n is a fraction, $P = (1-n)T_1 = (1-n)btf$. Hence $H = \frac{(1-n)btfv}{550}$.

For leather belts, f is generally from 200 to 350.

315. Effect of Centrifugal Tension on Power Transmitted by Belts.—In Art. 92, p. 76, it was shown that a thin hoop revolving has a tensile stress in it due to centrifugal force, and the demonstration there given is applicable to a belt running on a pulley. If f_1 is the stress on a belt, in lbs. per square inch, due to centrifugal force, w_1 the weight of a cubic inch of the belt in lbs., v its velocity in feet per second, then $f_1 = \frac{12w_1v^2}{g} = \frac{wv^2}{g}$, where w is the weight in lbs. of a portion of the belt 12 inches long and 1 square inch in section. For leather, w may be taken at 0·4 lb.

The total tension on the tight side of the belt is now $T_1 + btf_1 = btf$, and $T_1 = bt(f - f_1)$. The total tension on the slack side is $T_2 + btf_1$, and

$$P = T_1 - T_2 = (1-n)T_1 = bt(1-n)(f - f_1) = bt(1-n)\left(f - \frac{wv^2}{g}\right).$$

Hence $H = \frac{Pv}{550} = \frac{bt(1-n)}{550}\left(fv - \frac{wv^3}{g}\right)$; this is of the form $H = av - cv^3$, where $a = \frac{bt(1-n)f}{550}$, and $c = \frac{bt(1-n)w}{550g}$.

To find when H is a maximum, $\frac{dH}{dv} = a - 3cv^2$, and H will be a maximum when $\frac{dH}{dv} = 0$, that is, when $3cv^2 = a$, or $v = \sqrt{\frac{a}{3c}}$, putting in the values of a and c. H is a maximum when $v = \sqrt{\frac{fg}{3w}}$.

Inserting the value $v = \sqrt{\frac{fg}{3w}}$ in the formula for the horse-power,

$$H_{max} = \frac{bt(1-n)}{550} \cdot \tfrac{2}{3} f \sqrt{\frac{fg}{3w}}.$$

H will be zero when $fv = \frac{wv^3}{g}$, that is, when $v = 0$ or $v = \sqrt{\frac{fg}{w}}$.

316. Power Transmitted by Wire Ropes.—The principles involved in the determination of the power transmitted by a wire rope are the same as for a leather belt, but in the case of the wire rope it is necessary to allow for the stress due to the bending of the wires to the circle of the pulley.

Let d = diameter of each wire of the rope in inches.

D = diameter of pulley in inches.

v = velocity of rope in feet per second.

f = maximum working stress in wire in lbs. per square inch.

f_1 and f_2 = stresses in tight and slack portions of rope respectively in lbs. per square inch, neglecting the stresses due to centrifugal force and bending.

$f_2 = nf_1$, where $\frac{1}{n} = e^{\mu\theta}$ (see Art. 243, p. 277).

w = weight of 1 linear foot of rope per square inch net section in lbs.

E = modulus of elasticity of wire in lbs. per square inch.

H = horse-power transmitted per square inch of net section of rope.

Stress due to centrifugal force $= \frac{wv^2}{g}$.

Stress due to bending $= \frac{Ed}{D}$ (Art. 109, p. 103),

$$f = f_1 + \frac{Ed}{D} + \frac{wv^2}{g}, \text{ therefore } f_1 = f - \frac{Ed}{D} - \frac{wv^2}{g}.$$

Driving force per square inch net section of rope

$$= f_1 - f_2 = f_1(1-n) = \left(f - \frac{Ed}{D} - \frac{wv^2}{g}\right)(1-n).$$

Hence
$$H = \left(f - \frac{Ed}{D} - \frac{wv^2}{g}\right)\frac{(1-n)v}{550}.$$

The maximum working stress should not exceed 25,000 lbs. per square inch. w may be taken equal to 4·16, which makes the weight of the rope per foot of length equal to $3{\cdot}27d^2n_1$, where n_1 is the number of wires in the rope.

The maximum horse-power and the speed at which the horse-power is a maximum may be determined in the same way as for a leather belt (Art. 315).

It will generally be found that the speed at which the horse-power is a maximum is greater than the safe rim speed for the pulleys.

The tensions in the rope are regulated by the amount of sag given to the tight and slack portions of the rope between the pulleys.

317. Chain Gearing.—Motion may be transmitted from one shaft to another, the axes of the shafts being parallel, by means of a chain with

links of suitable form which embrace toothed wheels, called *sprocket wheels*, carried by the shafts. Fig. 570 shows a simple form of chain and the form of sprocket wheel to gear with it.

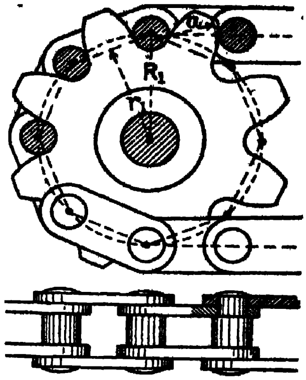


Fig. 570.

The first point to notice about chain gearing is that the pitch line of the sprocket wheel is a polygon, whose sides are equal to the pitch of the links of the chain. Generally the links are all of the same pitch, but in some forms of chain the links are alternately of long and short pitch, and the pitch polygon of the sprocket wheel has then an equal number of long and short sides alternating. It follows that the velocity ratio in chain gearing is not constant, and if R_1 and R_2 are the radii of the circumscribed circles of the pitch polygons of two sprocket wheels connected by a chain, and if r_1 and r_2 are the radii of the inscribed circles of the same polygons, then the velocity ratio may range from R_1/r_2 to r_1/R_2.

An inspection of Fig. 570 will show that if the outlines of the teeth of the sprocket wheel are arcs of circles described from the angular points of the pitch polygon with radii equal to the pitch of the chain less the radius of the pin, the pins will just touch the faces of the teeth as they come into or go out of gear, and if the outlines be arcs of circles of a slightly smaller radius, as shown at *a*, the pins will clear the teeth as they come into or go out of gear.

A second point to notice about chain gearing is that there is practically no tension on the slack portion of the chain, and therefore the work transmitted is equal to the tension on the tight or driving portion multiplied by the distance through which it travels.

A third point about chain gearing is that in general the full tension on the driving portion of the chain is supported by only one tooth at a time on each wheel. Although the chain may have exactly the same pitch as the teeth when new, the pitch of the chain soon increases because of the wear of the pins and their bearings in the links of the chain, and to a small extent by the permanent stretch of the links. To permit of the lengthening of the pitch of the chain the space between the teeth must be wider than the diameter of the pins, as shown in Fig. 571, which also shows that the load is carried by one tooth when the pitch of the chain is only slightly greater than the pitch of the teeth.

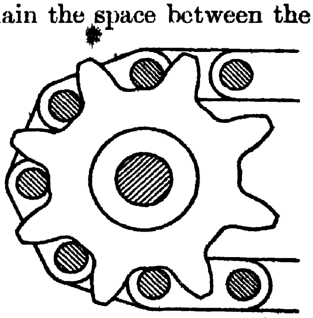


Fig. 571.

A consequence of the load being carried by one tooth at a time on each wheel is that there is considerable friction and probable jarring as each tooth in turn takes up the load. The friction may however be reduced by providing the pins with rollers.

The objection to ordinary chain gearing just mentioned is overcome in the *Renold's chain*, shown in

Fig. 572. The teeth of the wheels in this design are wedge-shaped, and the links have wedge-shaped projections which gear with the teeth, as shown. When the chain is new one edge of a wedge on one link and the opposite edge of the adjacent wedge on the next link bear on the front of one tooth and on the back of the next respectively, but when the pitch of the chain has increased through wear, the contact is as shown in Fig. 572. With this chain there is no rubbing of the links on the faces of the teeth as they come into or go out of gear.

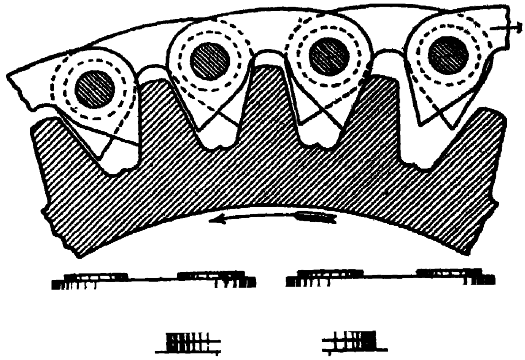


FIG. 572.

Exercises XXII.

1. A belt $\frac{7}{16}$ inch thick drives a pulley 15 inches diameter, which makes 800 revolutions per minute. Find the speed of the belt in feet per minute, (*a*) neglecting the thickness of the belt, (*b*) taking the thickness of the belt into account. Express the difference between the two results as a percentage of the second.

2. Two pulleys are connected by a belt The sum of the diameters of the pulleys is 36·5 inches and while the one makes 50 revolutions the other makes 200 revolutions. Find the diameters of the pulleys.

3. A train of pulleys is shown in Fig. 573. B and C are fixed on one shaft, and D and E are fixed on another shaft. The diameters of the pulleys, in inches, are, A = 30, B = 15, C = 50, D = 16, E = 35, and F = 12. A runs at 80 revolutions per minute. Find the speed of F, (*a*) neglecting thickness of belts, (*b*) taking thickness of belts as $\frac{1}{16}$ inch. Express the difference between the answers) and (*b*) as a percentage of answer (*b*).

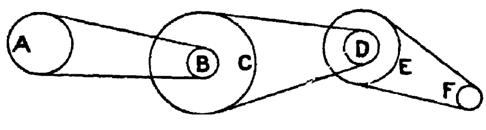


FIG. 573.

4. A shaft running at 200 revolutions per minute carries a pulley 50 inches diameter, which drives a dynamo at 1200 revolutions per minute by means of a belt $\frac{1}{4}$ inch thick. Allowing for the thickness of the belt and a slip of 4 per cent, determine the diameter of the pulley on the dynamo.

5. A pulley 36 inches diameter is connected to a pulley 18 inches diameter by a crossed belt. The distance between the axes of the pulleys is 48 inches. Calculate the length of the belt in inches, (*a*) by the exact formula, (*b*) by the approximate formula (Art. 308).

6. Same as preceding exercise, except that the belt is an open one.

7. A pulley 50 inches diameter is connected to a pulley 10 inches diameter by an open belt. The distance between the axes of the pulleys is 45 inches. Calculate the length of the belt in inches, (*a*) by the exact formula, (*b*) by the approximate formula (Art. 308). Express the difference between the two results as a percentage of the first. What will the results be for a crossed belt?

8. Referring to the stepped pulleys shown in Fig. 556, p. 364, $D_1 = 36$ inches, $d_1 = 12$ inches, $D_2 = d_2$, $d_3 = 4D_3$, $c = 48$ inches. The belt is an open one. Find D_2, D_3, and d_3, and also the length of the belt l, all in inches.

9. Referring to Fig. 556, p. 364. The shaft A runs at a constant speed of

200 revolutions per minute. The stepped pulleys are to be designed so that the shaft B may be driven at 600, 300, or 100 revolutions per minute as required. $D_1 = 30$ inches, $c = 50$ inches. Find the other diameters and l, the length of the belt, which is an open one.

10. Find the answers to the preceding exercise when a crossed belt is used.

11. A cone pulley AE (Fig. 574) drives the cone pulley *ae* by means of an open belt. Diameter at A = diameter at $e = 16$ inches. Diameter at a = diameter at E = 8 inches. The slant side of AE is straight. Find the diameters at b, c, and d, so that the belt may be equally tight in each position. Draw the pulley *ae* to scale, half size for diameters, and $\frac{1}{8}$th size for widths.

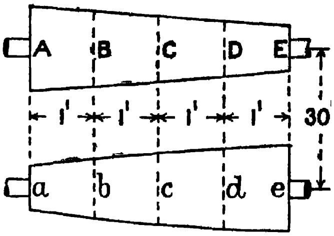


FIG. 574.

12. Taking the dimensions given in Fig. 574, and in the preceding exercise, except that the slant side of AE is no longer straight, determine the diameters at B, b, C, c, D, and d for an open belt, so that the speed ratios when the belt is at Aa, Bb, Cc, Dd, and Ee in turn may be in geometrical progression. Draw the pulley *ae* to scale, half size for diameters, and $\frac{1}{8}$th size for widths.

13. A belt drives a pulley 4 feet in diameter at 100 revolutions per minute, and transmits $3\frac{1}{2}$ horse-power. Assuming that the tension on the tight side is twice that on the slack side, find these tensions.

14. Given $H = 5$, $V = 3000$, and $T_1 = 1{\cdot}8T_2$, find T_1 and T_2.

15–20. In the exercises given in the annexed table H = horse-power transmitted by a belt. V = speed of belt in feet per minute. T_1 = tension on tight side. T_2 = tension on slack side. b = breadth of belt in inches. t = thickness of belt in inches f = working stress in lbs. per square inch. w the weight of 12 cubic inches of belt is to be taken at 0·4 lb. Find the unknown quantities in each case, (1) neglecting centrifugal tension, (2) taking centrifugal tension into account.

Exercise	15	16	17	18	19	20
H	...	...	25	100	20	50
V	3000	2800	2500	3000	3500	...
$T_1 \div T_2$	2	2	$1\frac{3}{4}$	2	2	2
b	5	9	...	...	5	10
t	$\frac{7}{32}$	$\frac{7}{16}$	$\frac{3}{8}$	$\frac{7}{16}$	$\frac{1}{4}$	$\frac{3}{8}$
f	300	350	400	[illegible]	...	350

21. Taking the data of Exercise 16, with the exception of V, determine the maximum horse-power which may be transmitted, taking into account the centrifugal tension.

22. Given $T_1 = 2T_2$, $f = 350$, and $w = 0{\cdot}4$, calculate the horse-power H, per square inch of belt section, taking into account the centrifugal tension, at intervals of 10 feet per second, between the values of v which make $H = 0$. Plot the results on squared paper. Scales.—1 inch = 5 horse-power, and 1 inch = 20 feet per second. Determine H and v for the highest point of the curve.

23. Show that when a belt is transmitting the maximum power, the centrifugal stress is one-third of the greatest stress.

24. A countershaft, which runs at 300 revolutions per minute, is required to transmit 10 horse-power from a main line shaft to a machine. The driving pulley of the machine shaft is 12 inches in diameter. The main shaft runs at 100 revolutions per minute, and the machine shaft at 900 revolutions per minute. The diameter of the main shaft pulley is 3 feet. Assuming the coefficient of friction between the belt and its pulley to be 0·3 in each case, and the belt $\frac{3}{8}$ inch thick, determine the width of each belt, taking account of the centrifugal tensions. The weight of a cubic inch of belt may be taken as 0·035 lb., and the tension per square inch as 350 lbs. Prove the formulæ you use. [U.L.]

25. Find the maximum horse-power which can be transmitted by a hemp rope 1 inch in diameter at a speed of 70 feet per second if the rope is broken with a pull of 5700 lbs., and it is desired to have a factor of safety of 30. The

angle of the groove in which the rope runs is 60°, and the coefficient of friction may be taken as 0·25, and the rope is in contact with the pulley for half the circumference. Find also the centrifugal tension in the rope if the fly-wheel is 10 feet in diameter, and the reduction in the horse-power transmitted due to this tension. Weight of rope for 1 foot of length = 0·28 lb. [B.E.]

26. Calculate the horse-power transmitted by a wire rope under the following conditions. The driving pulley is 12 feet in diameter, and it runs at 150 revolutions per minute. The rope consists of six strands, each strand having seven wires, and each wire having a diameter of 0·064 inch. The rope embraces half the circumference of the pulley, and the coefficient of friction between the rope and the pulley is 0·25. The weight of the rope is 0·56 lb. per foot of length. The working stress is 24,000 lbs. per square inch of wire section, and the modulus of elasticity E is 29,000,000 lbs. per square inch.

27. If the bending stress is not to exceed 12,000 lbs. per square inch, find the minimum diameter of the driving pulley for a wire rope made up of wires No. 15 I.S.W.G. (0·072 inch diameter), the modulus of elasticity of the wire being 29,000,000 lbs. per square inch.

28. Adhering to the conditions given in Exercise 26, except as regards the speed of the pulley, determine the velocity of the rope, in feet per second, when the horse-power transmitted is a maximum, and find the maximum horse-power.

29. A wire rope made up of 72 wires each 0·048 inch diameter is used to transmit power. Taking the maximum working stress in the wires at 25,000 lbs. per square inch, the bending stress at 13,000 lbs. per square inch, $w = 4{\cdot}16$, and $n = 0{\cdot}4$, plot the horse-power transmitted, and v the velocity of the rope in feet per second, between the limits $v = 0$. and $v = 200$. State the maximum horse-power and the corresponding value of v. Scales.—Horse-power, 1 inch to 40 horse-power ; velocity, 1 inch to 40 feet per second.

30. A chain of uniform pitch transmits motion from a sprocket wheel having 15 teeth to another having 10 teeth. What is the mean velocity ratio? Express the difference between the possible maximum and the possible minimum velocity ratio as a percentage of the mean.

CHAPTER XXIII

TOOTHED GEARING

318. Definitions Relating to Toothed Wheels.—The *pitch surfaces* of two toothed wheels which gear with one another are the surfaces of two imaginary friction wheels which have the same axes, and which would have the same relative angular velocities as the toothed wheels if one was to drive the other by rolling contact.

A section of a pitch surface by a plane at right angles to its axis is called a *pitch line*, or a *pitch-circle*, if the section should be a circle, which it is in most cases.

The *pitch* of the teeth is the distance from a point on one tooth to the corresponding point on the next, *measured along the pitch line.* In the

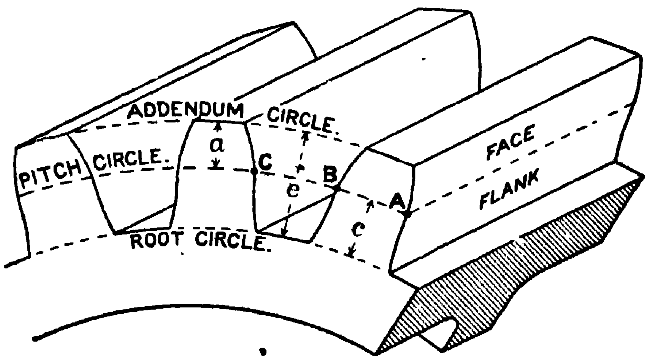


FIG 575.

case of a circular wheel whose pitch circle has a diameter d, and which has n teeth of pitch p, it is obvious that $np = \pi d$. The pitch just defined is the *circumferential* or *circular* pitch, and is equal to the circumference of the pitch circle divided by the number of teeth. If the diameter of the pitch circle be divided by the number of teeth, the result is called the *diametral pitch.* If p' denote the diametral pitch, then $np' = d$ and $p = \pi p'$. In the designing of machine-cut toothed wheels it is usual to arrange that p' is a simple fraction of the form $\frac{1}{m}$, where m is a whole number, then m is the number of teeth in the wheel *per inch of diameter*, and the number m is frequently called the diametral pitch.

When the term pitch is used without qualification, circular pitch is to be understood.

The part of a tooth beyond the pitch surface is called the *point* or *addendum*, and the part within the pitch surface is called the *root*. The

acting surface of the addendum is called the *face*, and the acting surface of the root is called the *flank*. Circles concentric with the pitch circles, and passing through the tops and bottoms of the teeth, are called the *addendum circle* and *root circle* respectively. In the case of an internal wheel the addendum is inside and the root is outside the pitch surface.

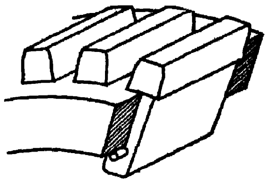


FIG. 576.

In a *mortice wheel* (Fig. 576) the teeth are made of wood, and have tenons formed on them which fit into mortices in the rim of the wheel. The teeth in this case are called cogs. The wood used is generally hornbeam or beech.

319. Ordinary Proportions of Teeth.—The following proportions represent average practice for cast-iron teeth. Pitch $= p =$ arc ABC (Fig. 575). Thickness $=$ AB $= 0{\cdot}47p$. Width of space $=$ BC $= 0{\cdot}53p$. Total height $= e = 0{\cdot}7p$. Height beyond pitch line $= a = 0{\cdot}3p$. Depth within pitch line $= c = 0{\cdot}4p$. Width $= 2p$ to $3p$. For heavy mill-gearing the width is sometimes as great as $5p$.

The cogs of mortice wheels have a thickness $= 0{\cdot}6p$, and the iron teeth which gear with them have a thickness $= 0{\cdot}4p$, so that there is no side clearance when the teeth are new.

320. Frequency of Contact of a Pair of Teeth.—If N_1 and N_2 be the numbers of teeth on two wheels A and B which gear with one another, then the ratio of their angular velocities is as N_2 is to N_1. Let n_1 and n_2 be the quotients got by dividing N_1 and N_2 respectively by their greatest common divisor, then if a particular tooth on A gears with a particular tooth on B, the same pair will again come in contact after n_2 revolutions of A and n_1 revolutions of B. Also one tooth on A will in turn gear with n_2 teeth on B, and one tooth on B will in turn gear with n_1 teeth on A. For example, if A has 60 teeth and B has 20, the same pair of teeth will come in contact after every revolution of A or after every three revolutions of B. Also a particular tooth on A will come in contact with only one particular tooth on B, and a particular tooth on B will come in contact in turn with three particular teeth on A. If the number of teeth on A be increased to 61, the velocity ratio will be altered to a small extent only, but the same pair of teeth will now only come in contact after 20 revolutions of A or 61 revolutions of B. Also each tooth on one wheel will now come in contact in turn with every tooth of the other wheel. The extra tooth added in this case is called a *hunting tooth* or *hunting cog*. The effect of the hunting cog is to cause the teeth to wear more uniformly.

321. Condition to be Fulfilled by the Curves of the Teeth of Wheels in order that they may Work correctly.—Two toothed wheels, in gear with one another, are said to work correctly when the ratio of their angular velocities is exactly the same *at every instant* as that of their pitch surfaces working in rolling contact without slipping.

In Fig. 577, O_1 and O_2 are the centres of two toothed wheels whose pitch lines PQ and PR are in contact at P. The shaded curves represent portions of two teeth, one on each wheel, which are in contact at the point *ab*, *a* being the point on the tooth of the one wheel which is in contact with the point *b* on the tooth of the other. In geometry it is

shown that when two curves touch one another they have a common normal. Let M_1M_2 be the common normal of the curves of the teeth at ab, and let O_1M_1 and O_2M_2 be the perpendiculars from the centres of the wheels on to this common normal. The point a moves in a circle aA

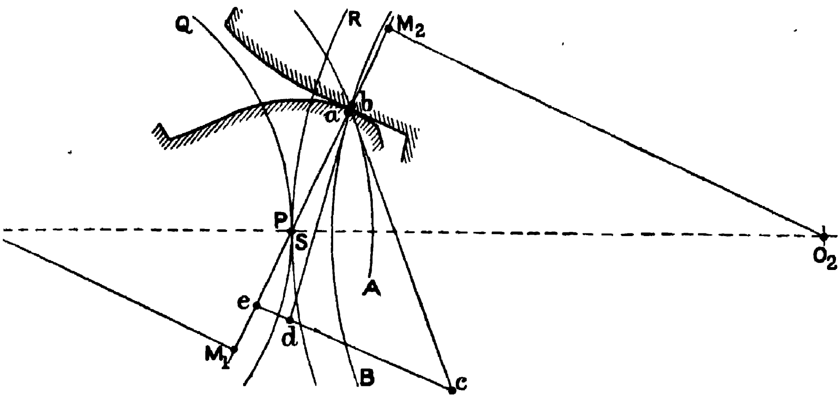


FIG. 577.

whose centre is O_1, and the point b moves in a circle bB whose centre is O_2. At the instant that the points a and b are in contact the point a is moving in the direction ac, the tangent to the circle aA at a, and the point b is moving in the direction ad, the tangent to the circle bB at b. Let v_1 and v_2 be the linear velocities of a and b respectively in the directions in which they are moving at the instant when they are in contact. Make $ac = v_1$ and $ad = v_2$. Now, although the points a and b are moving in different lines with different velocities, the components of these velocities in the direction M_1M_2 must be the same, otherwise the points a and b would move relatively to each other along the line M_1M_2, but for a small movement of the wheels so long as the teeth remain in contact the only possible relative motion of a and b is in a direction perpendicular to M_1M_2. Therefore if ce be drawn at right angles to M_1M_2 it will pass through d, and $ae = v$ will be the component velocity of a and also of b in the direction M_1M_2. Hence the ratio of the angular velocities of the two wheels must be

$$\frac{v}{O_1M_1} \div \frac{v}{O_2M_2} = \frac{O_2M_2}{O_1M_1} = \frac{O_2S}{O_1S},$$

where S is the point of intersection of the lines O_1O_2 and M_1M_2. But with rolling contact between the pitch lines PQ and PR the ratio of the angular velocities of the two wheels would be equal to $\frac{O_2P}{O_1P}$. Therefore if $\frac{O_2S}{O_1S} = \frac{O_2P}{O_1P}$, the point S must coincide with the point P.

The condition to be fulfilled by the curves of the teeth is therefore as follows. *The common normal to the curves of the teeth in contact must pass through the pitch point,* the pitch point being the point of contact of the pitch lines.

Another way of proving that the common normal to the curves of the teeth should pass through the pitch point is as follows. The relative

motion of two teeth in gear will not be altered if one of the pitch circles is considered to be at rest and the other pitch circle is supposed to roll on the first. Let the pitch circle PQ (Fig. 577) be at rest, and let the pitch circle PR roll on PQ. The direction of the motion of the point b is perpendicular to Pb, because, in the position considered, b is rotating about P, and in order that the pure rolling of PR on PQ may not be interfered with, and in order that the two teeth in gear may remain in contact, the direction of the motion of b must be tangential to the curves of the teeth at a or b. Therefore the common normal to the curves of the teeth passes through P.

322. Cycloidal Teeth.—Let APB and CPD (Fig. 578) be the pitch circles of two wheels. Let the outline of the flank of a tooth on APB be a portion of the hypocycloid aQb, described by the rolling of the circle PQR on the inside of the pitch circle APB. Let the outline of the face of a tooth on CPD be a portion of the epicycloid cQd, described by the rolling of the circle PQR on the outside of the pitch circle CPD. Next let the face cQ be brought round so as to touch the flank aQ, and let Q be the point of contact. The point Q must be on the rolling circle PQR when the latter touches both pitch circles, because the normal to the hypocycloid at Q must pass through the point of contact of the rolling circle and the circle APB when the former is describing that part of the hypocycloid at Q, also the normal to the epicycloid at Q must pass through the point of contact of the rolling circle and the circle CPD when the former is describing that part of the epicycloid at Q, therefore, since the two normals coincide, the rolling circle when it passes through Q must touch both pitch circles.

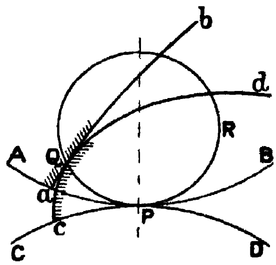


FIG. 578.

Since the common normal to the curves aQb and cQd, at their point of contact Q, passes through the pitch point P, the wheels will work correctly if the faces of the teeth on one are epicycloids and the flanks of the teeth on the other are hypocycloids, *described by the same rolling circle.*

It is evidently not necessary that the flanks of the teeth of two wheels which gear together be described by the same rolling circle, but the rolling circle which describes the flanks of the teeth on one wheel must be used to describe the faces of the teeth on the other.

Since the hypocycloid becomes a straight line passing through the centre of the pitch circle when the diameter of the rolling circle is equal to the radius of the pitch circle, it follows that the flanks of wheel teeth may be made radial.

If a number of wheels are to be interchangeable, that is, if any one of them is to be capable of working correctly with any of the others, it is obvious that the faces and flanks of the teeth on each must be described by the same rolling circle.

323. Path of Contact.—In the preceding Article it has been shown that the point of contact of two cycloidal teeth must be on one or other of the rolling circles when the latter are at the pitch point; it follows, therefore, that the path of contact of two teeth must be made up of arcs of these rolling circles.

If APB and CPD (Fig. 579) be the pitch circles of two wheels with cycloidal teeth in gear with one another, and if *ade* be the addendum circle of the teeth of the lower wheel, and *bfg* the addendum circle of the teeth of the upper wheel, then the rolling circles being at the pitch point as shown, the points *a* and *b* where the addendum circles cut the rolling circles are the extreme points of contact of the teeth, the upper wheel being the driver, and having its motion in the direction of the arrow. At the point *a* a point on the flank of a tooth on the driver will come in contact with the extreme point of the face of a tooth on the follower. As the motion proceeds, the flank of the tooth on the driver will slide on the face of the tooth on the follower until the point of contact, which moves along the arc *a*P, reaches the point P. The face of the tooth on the driver will then slide on the flank of the tooth on the follower until the point of contact, which moves along the arc P*b*, reaches the point *b*.

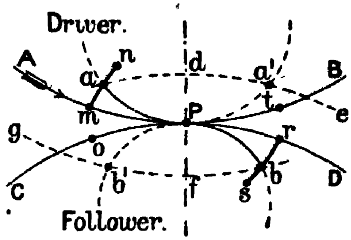


FIG. 579.

The arc *a*P is called the *path of approach*, and the arc P*b* the *path of recess*. If the driver move in the opposite direction, the path of contact will evidently be the line *a*′P*b*′.

If *man* be the flank of a tooth on the upper wheel just coming into contact with a tooth on the lower wheel, the point *m* on the pitch line APB will come into contact when it has travelled to P, and the arc *m*P is the *arc of approach*. Again, if *rbs* be the flank of a tooth on the lower wheel just going out of contact with a tooth on the upper wheel, the point *r* on the pitch line CPD will have travelled over the arc P*r* since being in contact, and the arc P*r* is the *arc of recess* If the arc P*o* be made equal to the arc P*m*, and the arc P*t* be made equal to the arc P*r*, then either of the arcs *o*P*r* or *m*P*t* is the *arc of contact*.

The arc of contact may also be defined as that part of the pitch line which passes the pitch point during the time of contact of a pair of teeth.

In order that one pair of teeth may always be in contact, the arc of contact must not be less than the pitch of the teeth. If possible the arc of contact should not be less than twice the pitch, so as to ensure that at least two pairs of teeth are always in contact. Generally the arc of contact is not less than 1·4 times the pitch.

324. Obliquity of Action and Effect of Friction.—Referring to Fig. 580, if a pair of teeth are in contact at *a*, and friction is neglected, the line of action of the pressure between the teeth is the straight line *ca*P, and the angle α which this line makes with the common tangent to the pitch circles is the angle of obliquity of action.

With cycloidal teeth the obliquity of action during approach is greatest at the beginning of the path of contact, and diminishes to nothing at the pitch point. During recess the obliquity of action is nothing at the pitch point, and increases to a maximum at the end of the path of recess.

The effect of friction during approach is to increase the angle of obliquity of action by the amount ϕ, where ϕ is the angle whose tangent

is equal to μ, the coefficient of sliding friction between the teeth. The driving force on the tooth of the lower wheel is now along the line *dae*, and the angle of obliquity of action is $\alpha + \phi$.

If b is a point of contact between a pair of teeth during recess, β is the angle of obliquity of action at b when friction is neglected. When friction is considered, the angle of obliquity of action at b is obviously $\beta - \phi$.

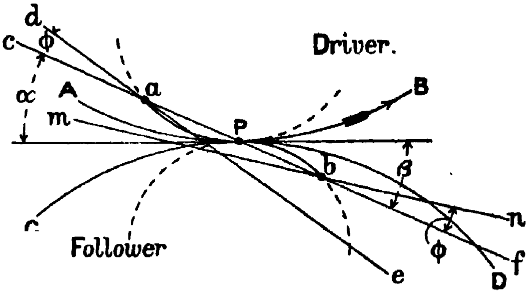


FIG. 580.

The effect of friction is to increase the obliquity of action during approach, and to diminish it during recess. Consequently friction is more objectionable during approach than during recess.

The effect of friction in altering the direction of the pressure between a pair of teeth in contact may be better understood by reference to Fig. 581. m and n are portions of a pair of teeth in contact, and the arrows show the direction of sliding of the one tooth on the other. R is the reaction of n on m, and T is the reaction of m on n. T is of course equal and opposite to R. The left-hand portion of Fig. 581 shows the conditions during approach, while the right-hand portion shows the conditions during recess, being on the driver and n on the follower.

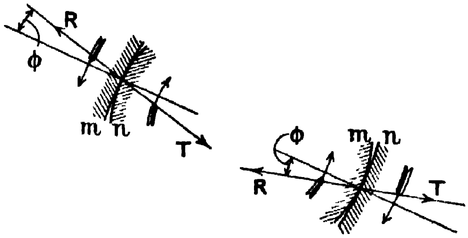


FIG. 581.

Referring further to Fig. 580, since the effect of friction is to divert the line of pressure between the teeth from the pitch point P, it is evident that during approach the length of the perpendicular from the centre of the driver to the line of pressure is diminished, and for a given turning moment on the driver at any instant the pressure on the teeth is increased by the action of the friction. During recess, however, the length of the perpendicular from the centre of the driver to the line of pressure is increased, and for a given turning moment on the driver at any instant the pressure on the teeth is diminished by the action of the friction. Hence friction is more injurious during approach than during recess.

Friction does not affect the accuracy of the working of the teeth so far as velocity ratio at any instant is concerned.

325. Involute Teeth.—Although the involute of a circle is a particular case of the epicycloid, being the epicycloid when the rolling circle is of infinite diameter, involute teeth are not considered as a special case of cycloidal teeth, because the involutes used are not involutes of the pitch circles, but are involutes of smaller circles, called the base circles.

Let APB and CPE (Fig. 582) be the pitch circles of two wheels with involute teeth in gear with one another. Let *a*T*b* and *c*T*e* be the outlines of the surfaces of two teeth in contact at T, these outlines being

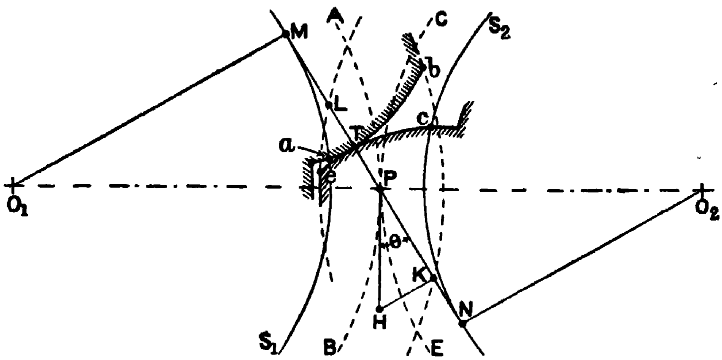


FIG. 582.

lines of the surfaces of two teeth in contact at T, these outlines being involutes of the base circles S_1 and S_2 respectively. Since a line drawn from any point on an involute to touch the base circle of that involute is a normal to the involute at that point, it follows that the common normal to the two involutes in contact at T must be a common tangent MN to the two base circles. Hence the point of contact is always on the line MN, and a portion of that line is the path of contact.

Comparing the similar triangles O_1PM and O_2PN, it is clear that if the ratio of the radii of the base circles be the same as the ratio of the radii of the pitch circles, the common normal to the curves of the teeth in contact must pass through the pitch point.

If the centres of the wheels be pushed closer together or further apart, the wheels will still work correctly, because this is equivalent to altering the radii of the pitch circles without altering their ratio. This is a special property of involute teeth, and is a valuable one in cases where the distance between the centres of the two wheels cannot be maintained constant. This property also makes it possible to regulate the amount of side clearance or back lash between the teeth. Altering the distance between the centres of the wheels obviously alters the inclination of the path of contact. The angle θ which the path of contact makes with the common tangent to the pitch circles is usually from $14\frac{1}{2}$ degrees to $15\frac{1}{2}$ degrees. In designing involute teeth the direction of the path of contact is first fixed, and the base circles are then drawn to touch it.

If on the tangent at P, PH be made equal to the pitch of the teeth, measured on the pitch circles, and if HK be drawn perpendicular to MN, then since $PK : PH :: O_1M : O_1P$, PK must be the pitch of the teeth measured on the base circles. The pitch PK is called the *normal pitch.* It is usual to make the parts of the path of contact on opposite sides of P equal to one another, then if two pairs of teeth are to be in contact, and PL be made equal to PK, KL will be the minimum length of the path of contact, and circles through K and L with centres at O_1 and O_2 respectively will be the minimum addendum circles. There should be a

small clearance between the root circle of one wheel and the addendum circle of the other.

If the parts of the path of contact on opposite sides of the pitch point are equal, and if there are two pairs of teeth always in contact, then PM or PN, whichever is least, will be the maximum value of the normal pitch of the teeth. Let r = radius of the smaller of the two base circles, p = maximum normal pitch, and n = the minimum number of teeth, then $2\pi r = np$, but $p = r \tan\theta$, therefore $nr \tan\theta = 2\pi r$, and $n = 2\pi/\tan\theta$. When $\theta = 15°$, $n = 24$. With only one pair of teeth in contact at a time, $n = \pi/\tan\theta$, or $n = 12$ when $\theta = 15°$.

When the pitch circle becomes of infinite diameter, as in a rack, the base circle will also become of infinite diameter, and the involute will become a straight line. Hence in a rack which gears with a wheel having involute teeth, the teeth are straight on face and flank, as shown in Fig. 583. The faces and flanks are perpendicular to the path of contact, and therefore make an angle of $90° - \theta°$ with the pitch line.

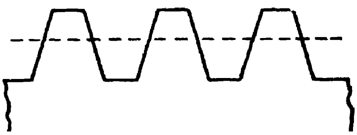


FIG. 583.

The essential condition that two wheels, or a wheel and rack, having involute teeth, may gear correctly together, is that the *teeth shall have the same normal pitch*. Two or more wheels having different numbers of involute teeth of the same normal pitch can be arranged to rotate about the same axis and gear correctly with one wheel or one rack. The base circles of the wheels on the same axis will of course be of different diameters, and the paths of contact will be inclined at different angles.*

326. Internal Teeth.—The theory of the forms and the methods of drawing the outlines of the teeth for internal or annular wheels in which the teeth are on the inside of the rim, as shown in Figs. 584 and 585, are the same as for external teeth.

In the case of cycloidal teeth (Fig. 584) the face *ab* is a hypocycloid of the pitch circle ABC described by the rolling circle which describes

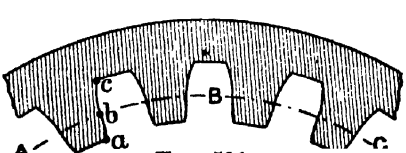


FIG. 584.

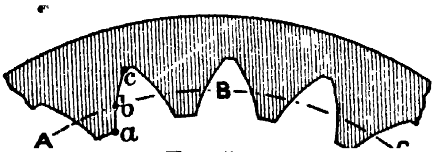


FIG. 585.

the hypocycloidal flanks of the teeth on the wheel which is to gear with ABC, and the flank *bc* is an epicycloid of the pitch circle ABC described by the rolling circle which describes the epicycloidal faces of the teeth of the other wheel.

In the case of involute teeth (Fig. 585) the curve *abc* is the involute of a base circle which must be concentric with the pitch circle ABC, and which must touch the straight line, which is the path of contact.

327. Pin Wheels.—When the rolling circle which describes the hypocycloidal flank of a tooth on a wheel A has a diameter equal to that

* Except when the single wheel becomes a rack, in which case the paths of contact are inclined at the same angle.

of the pitch circle the hypocycloid becomes a point, and no part of the tooth lies within the pitch circle. The face of a tooth on a wheel B which gears with A will be an epicycloid described by the pitch circle of A as rolling circle. If a rolling circle which describes the face of a tooth on A be diminished until it becomes a point no part of the tooth on A will lie outside the pitch circle, and as this rolling circle which has become a point must be used to describe the flanks of the teeth on B no part of a tooth on B will be inside the pitch circle. The teeth on A have thus become mere points, while the teeth on B will have epicycloidal outlines lying entirely outside the pitch circle. This is shown in the left-hand half of Fig. 586.

It is obvious that practically this is an impossible case, but if instead of mere points, cylindrical pins of sensible size be used, as shown in the right-hand half of Fig. 586, where the outlines of the teeth which gear with the pins are curves parallel to the epicycloids and at a distance from them equal to the radius of the pins, then the wheels will gear correctly, and either wheel will drive the other.

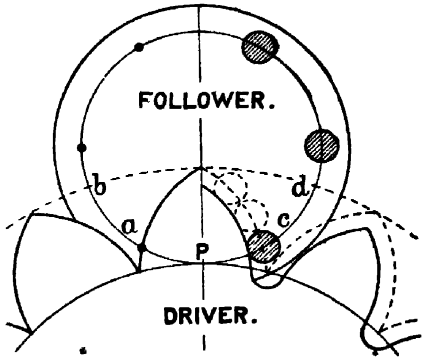


FIG. 586.

The path of contact will be either the arc P*ab* or the arc P*cd*.* If the pins are on the follower, contact will take place during recess only, and if the pins are on the driver, contact will take place during approach only. Since the friction is more serious during approach than during recess, it is best to put the pins on the follower.

Figs. 588 and 589 show wheels gearing internally, one of them having

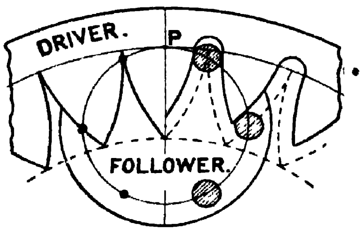


FIG. 588.

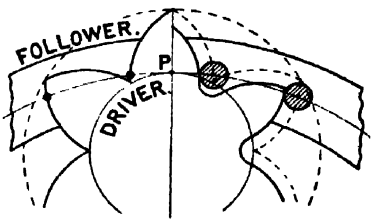


FIG. 589.

pins for teeth. Reasoning as for external contact, it is easy to show that the curves of the teeth in the left-hand half of Fig. 588, where the pins

* For practical purposes this may be taken as true when the pins are small, but the exact path of contact is a curve determined as shown in Fig. 587, where *a*P*d* is the pitch circle of the pin wheel, and P the pitch point. Take *c* any point on the arc P*cd*. Join *c*P. Make *cc'* equal to the radius of the pin; then *c'* is a point on the real path of contact. Repeating this construction a sufficient number of times and joining the points so obtained, the real path of contact P*c'd'* is determined.

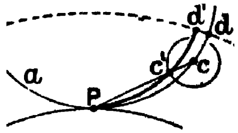


FIG. 587.

are mere points, and the dotted curves on the right must be hypocycloids described by the pitch circle of the pin wheel rolling inside the pitch circle of the other. The corresponding curves in Fig. 589 are epicycloids described by the pitch circle of the pin wheel rolling on the pitch circle of the other, the latter being inside the former.

An interesting case of the internal gearing shown in Fig. 588, is where the pitch circle of the pin wheel has a diameter equal to the radius of the pitch circle of the other. The faces of the teeth on the outside wheel now become the sides of parallel slots, the centre lines of which are radial lines of the larger pitch circle. Two examples of this case are shown in Figs. 590 and 591. In Fig. 590 the pin wheel has two teeth, while in Fig. 591 the pin wheel has four teeth. A peculiarity of this gearing is that the path of contact between a pair of teeth is the circumference of the pitch circle of the pin wheel excepting a small arc in the neighbourhood of the centre of the larger wheel. When a pin is in the neighbourhood of the centre of the larger wheel the obliquity of action is approaching a right angle and the driving effort is approaching zero, but when there are two or more pins on the pin wheel, only one pin will be in a disadvantageous position at a time. The path of approach is equal in length to the path of recess, and it is therefore immaterial which of the two wheels is the driver, except in the case where the pin wheel has only one tooth. When the pin wheel has only one tooth it must be the driver, otherwise motion of the follower would cease when the pin reached the centre of the larger wheel, unless it was carried past this dead centre by the inertia of the follower or the parts moving with it. Contact in the neighbourhood of the centre of the larger wheel can be insured, and larger bearing surfaces secured by making the slots wider and placing blocks on the pins, as shown in Fig. 592.

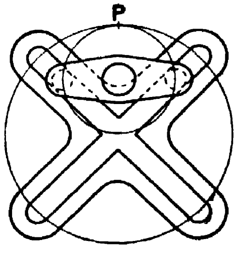


FIG. 590.

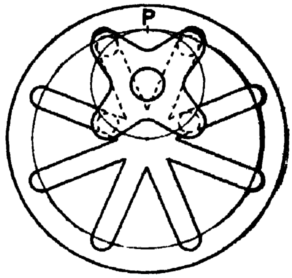


FIG. 591.

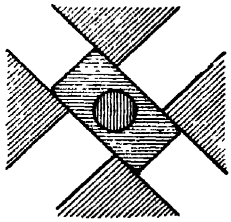


FIG. 592.

328. Bevel Wheels.—The pitch surfaces of bevel wheels in gear are frusta of cones whose vertices coincide, the axes of the cones being the axes of the wheels. Fig. 593 shows the pitch surfaces of two bevel wheels in gear, the one cone being external to the other. In this case the wheels are said to have external contact. Fig. 594 shows the pitch surfaces of two bevel wheels having internal contact, one cone being inside the other

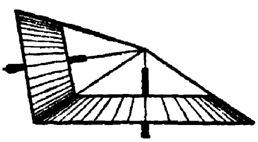


FIG. 593.

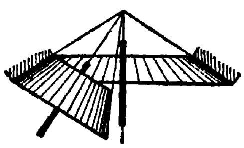


FIG. 594.

A *mitre wheel* is a bevel wheel whose pitch cone has a base angle of 45°.

To understand the theory of the forms of the teeth of bevel wheels, it is desirable to refer again to the way in which the forms of the teeth of spur wheels were derived. In Fig. 595, APB is the pitch circle of a spur wheel. aPR is the rolling circle which is used to describe the epicycloid ab, which is the profile of the face of a tooth on the wheel. The pitch surface of the wheel is a cylinder, and if the rolling circle aPR be taken as the end of another cylinder, the two cylinders, being of the same length, and having their axes parallel, the face aa_1b_1b of a tooth on the wheel is formed by the straight line aa_1 on the surface of the rolling cylinder as the latter rolls on the pitch surface of the wheel.

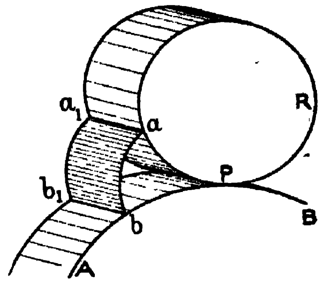


FIG. 595.

For a bevel wheel (Fig. 596), the pitch surface of which is the frustum ABB_1A_1 of the cone OAB, the rolling cylinder of Fig. 595 becomes the frustum of a rolling cone, and the curve ab becomes a spherical epicycloid. The face of a tooth on the wheel is formed by a straight line aa_1 on the surface of the rolling frustum as the latter rolls on the outside of the pitch surface of the wheel. The flank of a tooth is formed in like manner by a straight line on a rolling frustum when the latter rolls on the inside of the pitch surface.

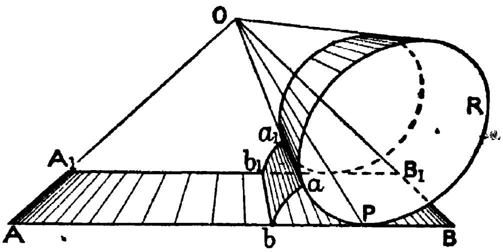


FIG. 596.

As the cone OaPR (Fig. 596) rolls on the cone OAB, the point a which describes the curve ab is always at a distance from O equal to the length of the slant side of the cone OaPB; the point a therefore moves on the surface of a sphere whose radius is OA, and whose centre is at O. The surface of the outer ends of the teeth formed in this way on a bevel wheel is therefore a portion of the surface of a sphere, and cannot be developed. If, however, a cone be taken enveloping the sphere and having for its circle of contact the pitch circle AB, this cone will cut the true face of a tooth in a curve which, when developed, will for all practical purposes in ordinary cases be an epicycloid. Hence the practical method, due to Tredgold, of designing the forms of bevel wheel teeth, shown in Fig. 597.

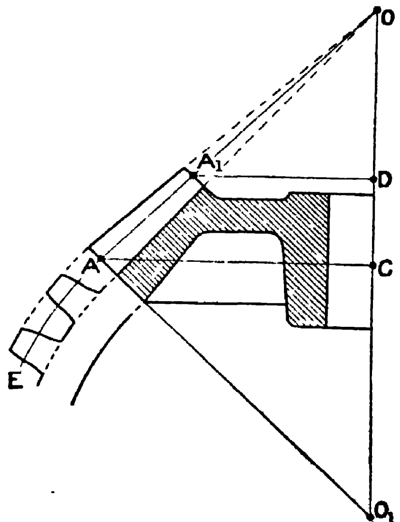


FIG. 597.

OO_1 is the axis of the wheel. $ACDA_1$ is one half of the pitch surface, O being the vertex of the pitch cone. OAO_1 is a right angle. O_1AC is one half of the cone, already referred to as enveloping the sphere whose centre is at O, and whose radius is OA. AE is an arc of a circle struck from O_1 as centre. This arc is the development of part of the base of the cone O_1AC. Then AE is considered as part of the pitch circle of a spur wheel of radius O_1A, and the teeth are constructed on this as for a spur wheel. A thin templet, made to the shape of the teeth on AE, may be used to mark off the shape of the teeth on the edge of the bevel wheel blank.

The theory of involute teeth for bevel wheels may be developed in a similar manner to that of cycloidal teeth. In a spur wheel with involute teeth a plane is taken touching a base cylinder, and a line in this plane parallel to the axis of the cylinder describes the surface of a tooth as the plane rolls on the cylinder. In a bevel wheel the base cylinder of the spur wheel becomes a base cone whose vertex is at the vertex of the pitch cone of the wheel, and as a plane rolls on the base cone a line in the plane, and passing through the vertex of the cone, describes the surface of a tooth on the wheel. Tredgold's method is also applicable to involute teeth.

When the diameter of a bevel wheel is mentioned without qualification, the larger diameter of the pitch surface is understood.

329. Stepped and Helical Teeth.—The smaller the pitch of the teeth of two wheels in gear the smoother is the motion, but the teeth are weaker the smaller the pitch. To combine the smoothness of the motion due to fine pitched teeth with the strength due to coarse pitched teeth, Dr. Hooke invented *stepped teeth*. These teeth are shown in Fig. 598. Imagine a toothed wheel having teeth of a pitch p to be divided into n discs of equal thickness by planes at right angles to the axis of the wheel, and let each disc be placed so that the teeth on it are 1-nth of the pitch p in advance of the teeth on the disc in front of it. These discs would now form a wheel with stepped teeth, which would have the strength of teeth of pitch p, and which would work as smoothly as teeth of pitch p/n.

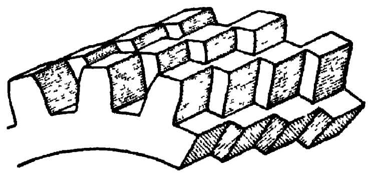


FIG. 598.

If the number of steps on a stepped tooth be made infinite, its surface becomes a screw or helical surface, and the teeth formed in this way are called *helical teeth*. Simple helical teeth on a spur wheel have the appearance shown in Fig. 599. The outline of the section of helical teeth by a plane at right angles to the axis of the wheel is designed as for ordinary teeth, and their outline in the direction of the width of the wheel is drawn by the rule for drawing helices or screw curves. It is obvious that two wheels gearing together and having helical teeth must have their teeth of "opposite hand," that is, one must be right-handed and the other left-handed. It is also evident that the inclinations of the helices must be the same.

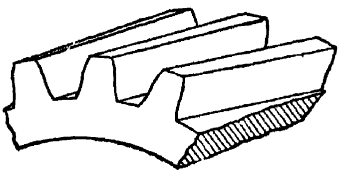


FIG. 599.

The objection to the teeth shown in Fig. 599 is that when at work there is a side pressure which tends to push the wheels out of gear. To overcome this difficulty, the *double helical teeth* shown in Fig. 600 were introduced, and are now largely used. To ensure the proper bearing of the teeth on one another, the shaft of one of a pair of wheels having double helical teeth should have a slight amount of end play.

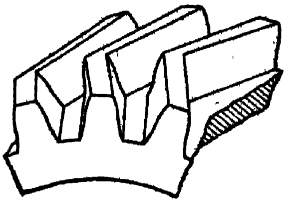


FIG. 600.

Exercises XXIII.

1. A toothed wheel has 95 teeth, whose diametral pitch is $\frac{1}{4}$ inch. Find the diameter of the pitch circle and the circular pitch.

2. Taking the ordinary proportions for teeth, height above pitch line $= 0.3p$, and depth below pitch line $= 0.4p$, where p is the circular pitch, express these in terms of the diametral pitch p'.

3. A wheel A, having 28 teeth, gears with a wheel B, having 35 teeth. How many teeth on A will come in contact with a particular tooth on B? Also, how many revolutions will A make before the same pair of teeth are again in contact? Further, what will the answers be (1) when A has 28 teeth and B 36 teeth, (2) when A has 29 teeth and B 35 teeth?

In the following exercises, 4 to 15, there are given in each two wheels or a wheel and rack in gear. Draw, full size, a side elevation of a portion of the pair in gear, in the neighbourhood of the pitch point, sufficient to show four teeth on each completely. Show clearly in each case the path of approach and the path of recess, the arc of approach and the arc of recess, also the maximum obliquities of action during approach and during recess.

4. Two spur wheels in external contact. Diameters of pitch circles, 10 inches and 16 inches. Numbers of teeth, 15 and 24. Cycloidal teeth. Rolling circle, 5 inches diameter for all curves.

5. Spur wheel and rack. Diameter of pitch circle of wheel, 15 inches. Number of teeth on wheel, 20. Cycloidal teeth. Rolling circle, 5 inches diameter for all curves.

6. Two spur wheels in internal contact. Diameters of pitch circles, 10 inches and 20 inches. Numbers of teeth, 20 and 40. Cycloidal teeth. Rolling circle, 5 inches diameter for all curves.

7. Same as Exercise 4, but with involute teeth.

8. Same as Exercise 5, but with involute teeth.

9. Same as Exercise 6, but with involute teeth.

10. Two wheels in external contact. Diameters of pitch circles, 10 inches and 15 inches. The smaller wheel to have 16 pins $\frac{3}{4}$ inch diameter.

11. Two wheels in internal contact. Diameters of pitch circles, 10 inches and 30 inches. The smaller wheel to have 16 pins $\frac{3}{4}$ inch diameter.

12. Two wheels in internal contact. Diameters of pitch circles, 10 inches and 30 inches. The larger wheel to have 48 pins $\frac{3}{4}$ inch diameter.

13. Two wheels in internal contact. Diameters of pitch circles, 6 inches and 12 inches. The smaller wheel to have 6 pins $\frac{1}{2}$ inch diameter.

14. Wheel and rack. Diameter of pitch circle of wheel, 10 inches. Wheel has 20 teeth. Rack has pins $\frac{1}{2}$ inch diameter.

15. Wheel and rack. Diameter of pitch circle of wheel, 10 inches. Wheel has 20 pins $\frac{1}{2}$ inch diameter.

16. Design for a spur wheel with cycloidal teeth. Diameter of pitch circle, 6 feet. Speed, 70 revolutions per minute. Power transmitted, 350 horse-power. For strength of teeth use the rule $P = 200np^2$, where P is the driving force at pitch circle, n the ratio of breadth of teeth to pitch, and p the pitch. Take $n = 2.75$. Diameter of shaft, $7\frac{1}{2}$ inches, enlarged to $8\frac{1}{2}$ inches inside the nave of the wheel.

17. Design for a bevel wheel with cycloidal teeth. Base angle of pitch cone, 30°. Mean diameter, 5 feet. Speed, 80 revolutions per minute. Power transmitted, 400 horse-power. For strength of teeth use the rule $P = 200np^2$, where $P =$ driving force at mean pitch circle, $n =$ ratio of breadth of teeth to pitch at mean pitch circle, and $p =$ pitch at mean pitch circle. Take $n = 3$. Wheel to have four arms of T section. Diameter of shaft, $7\frac{1}{2}$ inches. Diameter of wheel seat on shaft, $8\frac{1}{2}$ inches.

CHAPTER XXIV

WHEEL TRAINS

330. Wheel Trains.—Two or more wheels in gear form a *wheel train* or *train of wheels.* In Fig. 601 the wheels A and L are shown geared

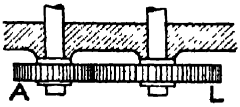


FIG. 601.

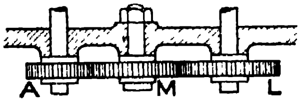


FIG. 602.

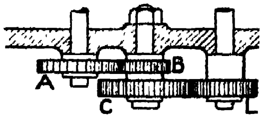


FIG. 603.

directly together, and they will evidently rotate in opposite directions. In Fig. 602 the wheels A and L are shown connected by an intermediate or idle wheel M ; here A and L rotate in the same direction.

In each of the Figs. 603, 604, and 605 the wheels A and L are shown connected by a double or compound wheel BC, the parts B and C being rigidly connected, so that they rotate together as one wheel. In Fig. 603 all the wheels are spur wheels with external teeth. In Fig. 604 L is an annular or internal toothed wheel, and in Fig. 605 all the wheels are bevel wheels. In each of these three examples there is the same number of wheels and the same number of axes, but it should be noticed that while in the arrangement shown in Fig. 603 A and L rotate in the *same* direction, in the arrangements shown in Figs. 604 and 605 A and L rotate in *opposite* directions.

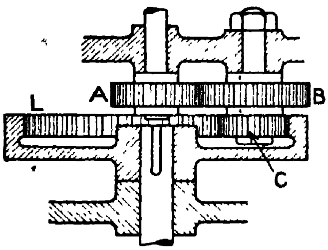


FIG. 604.

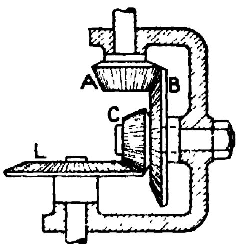


FIG. 605.

Let d_1, d_2, d_3, and d_4 denote the diameters, and n_1, n_2, n_3, and n_4 denote the numbers of teeth in the wheels A, B, C, and L respectively, and let N_1, N_2, and N_4 denote the speeds of the wheels A, B, and L respectively in revolutions in a given time, say per minute, then, from the fact that when two wheels gear together the linear velocities of their pitch circles must be the same, it follows that

$$\frac{N_2}{N_1} = \frac{d_1}{d_2} = \frac{n_1}{n_2}, \quad \text{and} \quad \frac{N_4}{N_1} = \frac{d_1}{d_2} \times \frac{d_3}{d_4} = \frac{n_1}{n_2} \times \frac{n_3}{n_4}.$$

The *velocity ratio* or the *value of a train of wheels* is the ratio of the speed of the last wheel to the speed of the first wheel of the train. The

velocity ratio is *positive* or *negative*, according as the first and last wheels rotate in the *same* or in *opposite* directions.

When the axes of two shafts are parallel and near to one another, but not overlapping, motion may be transmitted from the one shaft to the other by spur wheels, as shown in Fig. 606. The broad intermediate wheel M is here called a *Marlborough wheel*.

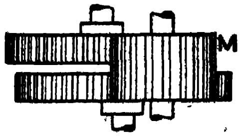


FIG. 606.

331. Change Speed Gears.—There are many cases in practice where it is necessary to drive one shaft from another in a positive manner at different speeds at different times. In a screw-cutting lathe, for example, the leading screw is driven from the lathe spindle by a train of wheels, and the value of the train to be used is the ratio of the pitch of the screw to be cut to the pitch of the leading screw. Hence to cut screws of different pitch different trains of wheels must be used. In the older lathes a set of separate change wheels are used, and different combinations of them have to be mounted to suit the work to be done. Modern lathes and other machine tools are, however, generally fitted with change gears which can be operated by the movement of one or more levers, all the wheels being permanently mounted. Not only is this done for the screw cutting and feed motions, but the main driving of the machine is now largely done by what are called "all-gear drives," that is, the use of stepped pulleys is dispensed with, there being only one driving pulley, and all the changes of speed are obtained by putting different toothed wheels into gear by the simple movements of one or more levers. One important advantage following this substitution of one belt pulley for a stepped pulley is that the belt can be run at the speed most suitable to it, and the power of the machine is not diminished at slower speeds through having to reduce the speed of the belt by shifting it to the larger steps of the stepped pulley.

The application of a sliding cotter key to a change speed gear is shown in Fig. 607. The driving shaft A carries three wheels C, D, and E, which gear with the three wheels F, G, and H respectively, which are firmly keyed to the driven shaft B. The shaft A is hollow for part of its length, and contains a rod R, into which is fitted the cotter key K, which passes through the slots L in the hollow part of A. Each of the wheels C, D, and E has six keyways, and each is counter bored as shown. In the position shown the cotter key is in two of the keyways in D, and the shaft B is being driven through the wheels D and G. If the rod R be moved to the right a distance equal to the width of the cotter key the latter will be in the space formed by the counter bore in E, and the counter bore in the right-hand side of D and all the wheels will be at rest. If the rod R be moved a step further to

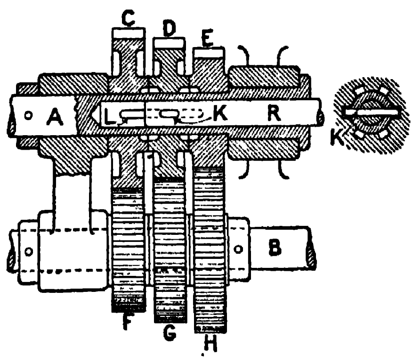


FIG. 607.

the right the cotter key will engage with the wheel E, and the shaft B will be driven through the wheels E and H. The rod R is operated by a lever not shown, which can be locked into definite positions corresponding to the several positions of the cotter key when in gear or out of gear. This form of gear is very suitable for light work, such as is required in feed motions of machine tools.

A form of the sliding wheel change speed gear is shown in Fig. 608. There are three wheels, C, D, and E, rigidly connected together and carried by the driving shaft A. A feather key permits of the wheels C, D, and E being moved longitudinally on A, while at the same time they must rotate with A. The driven shaft B carries three wheels, F, G, and H, which are rigidly fixed to it. By sliding CDE into different positions, B may be put out of gear, or it may be driven through C and F, or through D and G, or through E and H.

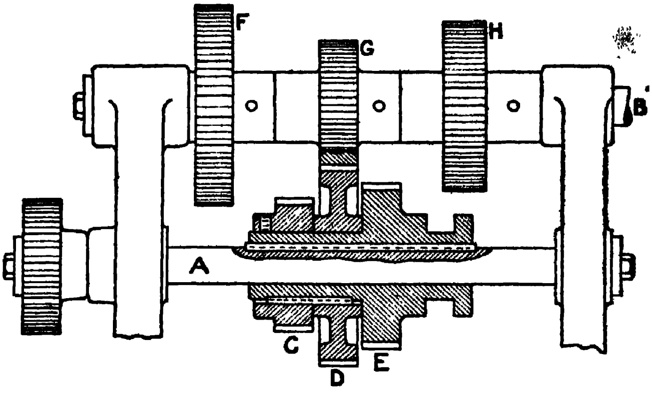


FIG. 608.

Another type of change speed gear is shown in Fig. 609. A is a shaft driven at constant speed by a belt on the pulley B. The pinion C is keyed to A, and is geared permanently to the wheel E through the intermediate wheel F. The wheel E is carried by the shaft H, to which it is connected by a pawl and ratchet wheel, and if H is not driven through the other part of the gear, to be presently described, H is driven through C, F, and E. The shaft A also carries a pinion K, which it drives through a feather key. A wheel L carried by the sliding tumbler MN permanently gears with K, and by lifting and sliding MN the wheel L may be made to engage with any one of the wheels O, P, Q, R, or S, which are all firmly keyed to the shaft H, or L may be placed clear of these wheels. When the shaft H is driven through the tumbler gear it rotates faster than when driven through C, F, and E, but in the same direction, this being possible on account of the ratchet connection of E to H. By using a separate ratchet drive for the slowest speed the shock

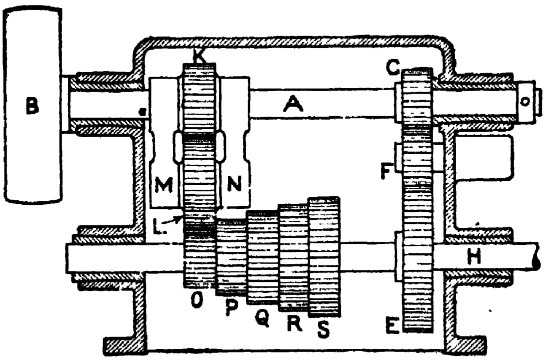


FIG. 609.

due to throwing the tumbler gear in is diminished. The particular example illustrated in Fig. 609 is from a radial drilling machine by the Bickford Drill and Tool Co. of Cincinnati. The gear shown is placed on the base of the machine, and the shaft H drives a vertical shaft in the pillar through bevel wheels.

332. Epicyclic Wheel Trains.—In an ordinary wheel train the axes of the wheels are fixed, while in an epicyclic train at least one axis revolves about another axis which is fixed.

The first point about an epicyclic train to be thoroughly understood is that if a wheel B (Fig. 610) be attached rigidly to an arm EF, and the arm is made to rotate once about an axis at E, the wheel B will turn once on its own axis in the same direction in which the arm rotates. This is made clear by an inspection of Fig. 610, where the arm and wheel are shown in four different positions during one revolution. The arrow on the wheel is supposed to be fixed to it.

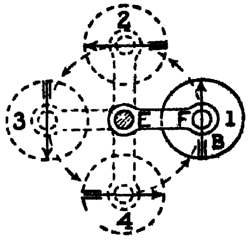


FIG. 610.

Examples of epicyclic trains are shown in Figs. 611 to 616. In each of these examples A is the first, and L the last wheel of the train, and

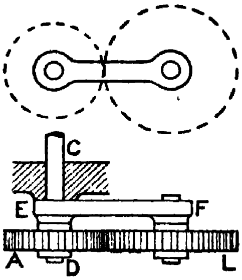


FIG. 611.

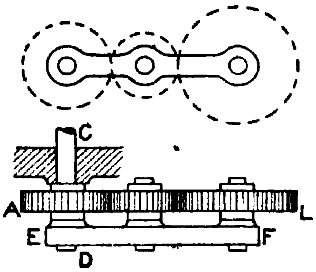


FIG. 612.

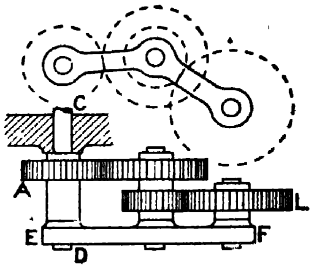


FIG. 613.

EF is an arm carrying certain of the wheels and rotating about an axis CD. The arm may be straight, as in Figs. 611 and 612, or bent, as in Fig. 613. In Fig. 615 the arm takes the form of a spur wheel. In Figs. 614 and 615 the axes of the first and last wheels of the train coincide, and these trains are called *reverted wheel trains.*

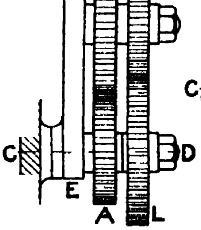


FIG. 614.

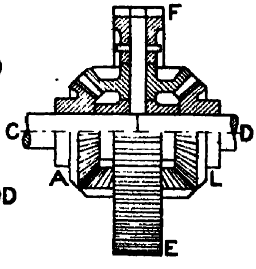


FIG. 615.

The gear shown in Fig. 615 is the well-known *differential gear* used on the driving axles of motor cars to permit of the driving wheels rotating at different speeds in going round a curve. The driving axle CD is divided, the part C carrying one driving wheel and D the other. The wheels A and L are fixed to C and D respectively, and the wheel EF is driven by the engine.

A simple method which may be adopted in solving problems on epicyclic trains will now be illustrated on a fairly complex example. Fig. 616 shows an epicyclic reverted train known as *Humpage's gear.* A is a fixed wheel, that is, a wheel which is not allowed to rotate. L is fixed to the shaft H, and D is fixed to the shaft K. B and C are fixed or cast together, but turn freely on an arm EF, which can rotate about the common axis of the shafts H and K. The wheels B and C and the arm EF are duplicated, as shown, for the sake of balance and pure torque. Let the numbers of teeth in the different wheels be as follows: A, 48; B, 40; C, 25; D, 12; and L, 40. First suppose the whole system to be turned once round in the direction S. The wheels A, L, and D have therefore made one revolution in the direction S. If now A is turned back through one revolution in the direction T, the arm EF being at rest, the various wheels will then occupy the positions which they would have occupied had A been fixed while the arm EF turned once in the direction S. In turning A back through one revolution in the direction T the wheel L will evidently turn in the direction T through $\frac{48}{40} \times \frac{25}{40} = \frac{3}{4}$ of a revolution. But L previously made one revolution in the direction S, therefore the actual motion of L is $1 - \frac{3}{4} = \frac{1}{4}$ of a revolution in the direction S. Again, in turning A back through one revolution in the direction T, the arm EF being at rest, the wheel D will make $\frac{48}{12} = 4$ revolutions in the direction S. But D previously made one revolution in the direction S, therefore the actual motion of D is $4 + 1 = 5$ revolutions in the direction S. Hence the speed of L is to the speed of D as $\frac{1}{4} : 5$, or as 1 is to 20.

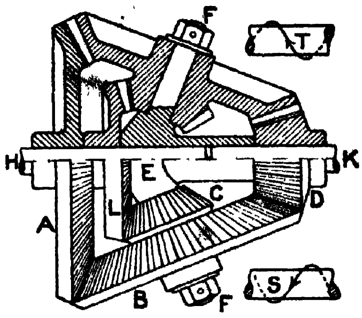


FIG. 616.

The working of the above problem may be tabulated as follows:—

Arm EF.	Wheel A.	Wheel D.	Wheel L.	
$+1$	$+1$	$+1$	$+1$	Revolutions.
0	-1	$+\frac{48}{12} = +4$	$-\frac{48}{40} \times \frac{25}{40} = -\frac{3}{4}$	,,
$+1$	0	$+5$	$+\frac{1}{4}$	,,
or $+4$	0	$+20$	$+1$	,,

Problems on epicyclic trains become quite simple when worked by the above method.

Problems on epicyclic gears may however be solved by aid of a formula constructed as follows:—

Let a = number of revolutions of the arm EF in a given time.
m = number of revolutions of the wheel A in the same time.
n = number of revolutions of the wheel L in the same time.

The speed of A in relation to the arm is $m-a$, and the speed of L in relation to the arm is $n-a$, hence the value of the train or its velocity ratio is $\frac{n-a}{m-a}=e$. In using this formula it is most important that the proper signs be given to the values of a, m, n, and e. For example, if the arm makes 20 revolutions in a direction taken as positive, $a=+20$, and if A makes 30 revolutions in the opposite or negative direction, then $m=-30$ and $m-a=-30-20=-50$. Again, the value of e is positive or negative according as A and L rotate in the same or in opposite directions respectively.

As an example on the use of the formula $e=\frac{n-a}{m-a}$, take the gear shown in Fig. 611, and let the wheels A and L be equal. Here $e=-1$. Let L be prevented from rotating about its axis, then $n=0$, and by formula $-1=\frac{0-a}{m-a}$ or $m=2a$, that is, the wheel A rotates twice as fast as the arm in the same direction. This is the well-known *sun and planet motion* used by Watt as a substitute for the ordinary crank in the steam-engine. A was fixed to the fly-wheel shaft, and L was bolted to the connecting-rod.

In using the formula $e=\frac{n-a}{m-a}$ to solve the problem on Humpage's gear, already worked out, two applications have to be made. First consider the train made up of A, B, C, and L (Fig. 616). Here $e=\frac{48}{40}\times\frac{25}{40}=\frac{3}{4}$, m the speed of A $=0$, and n is the speed of L. Hence $\frac{3}{4}=\frac{n-a}{0-a}$ and $n=\frac{a}{4}$. Next consider the train made up of A, B, and D. Here $e=-\frac{48}{12}=-4$, m the speed of A $=0$, and n is the speed of D. Hence $-4=\frac{n-a}{0-a}$, and $n=5a$. Therefore the speed of L is to the speed of D as $\frac{a}{4}:5a$, or as $1:20$, as before.

Exercises XXIV.

1. The axes of two spur wheels in gear are 37 inches apart. One wheel rotates four times as fast as the other. Find the diameters of the pitch circles of the wheels.

2. It is required to connect two shafts, whose axes are to be as nearly as possible 40 inches apart, by spur wheels so that the velocity ratio may be exactly 9 : 2. Find the number of teeth in each of the two wheels and the distance between the axes of the shafts, to the nearest hundredth of an inch, if the pitch of the teeth is $2\frac{1}{4}$ inches.

3. The crank of a direct double-acting steam-engine is 15 inches long. A spur wheel 9 feet in diameter on the crank shaft drives a pinion 2 feet in

diameter. If the piston travels 500 feet in one minute, what is the speed of the pinion in revolutions per minute? If the diameter of the piston is 17 inches, and the mean effective pressure on it is 25 lbs. per square inch, what is the average force on the teeth of the pinion?

4. The gearing of the fast headstock of a lathe is shown in Fig. 617. The spur wheel B is permanently keyed to the spindle A. The stepped pulley C may be connected to B by a bolt, not shown. When C is not connected to B it can rotate freely on A. D is a pinion fixed to C. E is a wheel and F a pinion, both fixed to the back spindle HK. Let the spindle HK be moved in the direction of the arrow, so that the wheel E comes into gear with the pinion D, and the pinion F with the wheel B, and let C be disconnected from B. If D and F have each 17 teeth, and B and E have each 68 teeth, find the number of revolutions of C for one revolution of A.

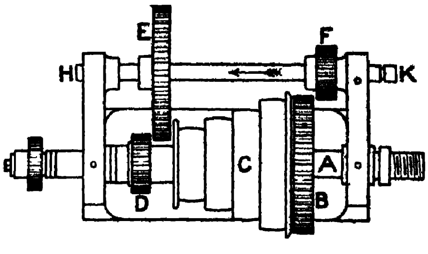


FIG. 617.

5. The leading screw of a lathe has four threads per inch, and is geared to the lathe spindle as follows. On the lathe spindle there is a wheel of 20 teeth, which gears with one of 100 teeth. Attached to the wheel of 100 teeth there is one of 40 teeth, which gears with a wheel of 120 teeth on the leading screw. Find the number of threads per inch in the screw to be cut.

6. In a planing machine the table is driven by a rack and pinion. For the cutting stroke the pulley shaft is connected with the rack through the following gearing. A pinion A (Fig. 618) rigidly connected to one of the pulleys has 24 teeth. This gears with a wheel B, which has 64 teeth. On the same axis as B is a pinion C of 18 teeth, which gears with a wheel D of 72 teeth, and on the axis carrying the wheel D is the pinion E, which has 15 teeth and drives the rack. The pitch of the teeth of the rack is $1\frac{1}{4}$ inches. On the quick return stroke D is driven direct by a pinion F having 18 teeth, and rigidly connected to another pulley on the pulley shaft. The stroke of the table is 6 feet. Find: (a) The number of revolutions per minute of the pulleys, if the cutting speed is not to exceed 25 feet per minute. (b) The time taken for one complete reciprocation of the table. (c) The average force exerted by the tool during one cutting stroke, if the horse-power passing to the planing machine through the belt during the cutting stroke is 3, and if the efficiency of the mechanism is 37 per cent. [B.E.]

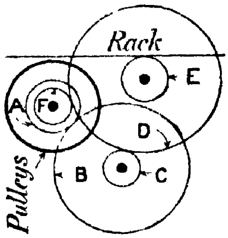


FIG. 618.

7. In the lifting-crab, shown in Fig. 619, the crank handle and the pinion A are fixed to the shaft HK. The wheel B and pinion C are fixed to the shaft LM. The wheel D and pinion E are fixed together, but are loose on the shaft HK. The wheel F and the barrel G are fixed together, but are loose on the shaft LM. The diameters of the wheels and pinions are as follows: A, $4\frac{1}{2}$ inches; B, 16 inches; C, 7 inches; D, $13\frac{1}{2}$ inches; E, $4\frac{3}{4}$ inches; F, $15\frac{3}{4}$ inches. The radius of the crank is $15\frac{1}{4}$ inches, and the effective diameter of the barrel is 10 inches. Neglecting friction, find the weight W, in tons, when an effort of 50 lbs. is applied at the crank handle.

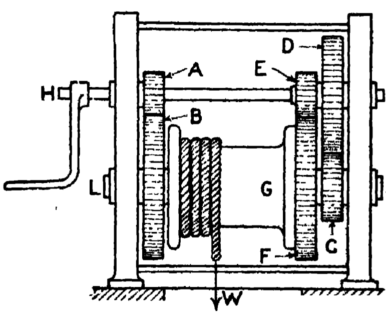


FIG. 619.

8. An epicyclic gear consists of three wheels, as shown in Fig. 612, p. 391. A is a dead wheel having 50 teeth. The arm EF makes +2499 revolutions in a certain time. Find the number of revolutions made by L in the same time when the number of teeth on L is (1) 50, (2) 51, and (3) 49.

9. In the epicyclic train, shown in Fig. 620, the wheel A is fixed. The rotating arm a, which rotates about the axis of A, carries a wheel B, which gears with A, and also a second wheel C, which gears with B. To the wheel C is rigidly fixed an arm b. If the speed of the arm a is n revolutions per minute clockwise, what is the speed of the wheel C about its axis? Find for one revolution of the arm a the path of a point on the arm b, whose distance from the axis of C is equal to the distance between the axes of A and C. Find also the path of a point on the arm b, whose distance from the axis of C is one half that between the axes of A and C. [B.E.]

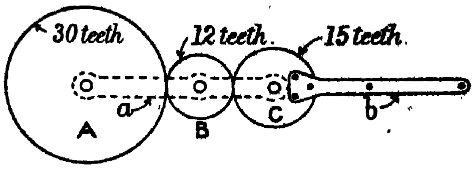


FIG. 620.

10. A and L (Fig. 621) are two wheels of nearly the same diameter. A has 49 teeth, and L has 50 teeth. A and L gear with a broad wheel M, which turns on a stud or pin attached to the arm EF, which turns about the axis of A and L. A being a fixed wheel, find the number of revolutions made by the arm while the wheel L turns once.

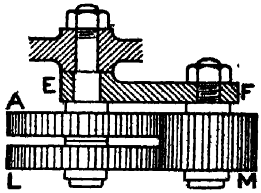


FIG. 621.

11. Referring to the reverted epicyclic wheel train shown in Fig. 614, p. 391, the wheels A, M, N, and L have 25, 35, 20, and 40 teeth respectively, and M and N rotate together. If the arm EF makes + 14 revolutions per minute about the axis CD, find in revolutions per minute (1) the speed of L when A is fixed, and (2) the speed of A when L is fixed.

12. In a "Crypto" front driving gear for a bicycle there is a spur wheel A (Fig. 622), having 14 teeth, and fixed to the fork. There is an annular wheel L having 38 teeth, and fixed to the hub of the front wheel of the bicycle. In one with the crank axle C is a disc carrying four pins, upon which are mounted four pinions M, each having 12 teeth, and each gearing with A and L. If the front wheel of the bicycle is 46 inches in diameter, what would be the diameter of a driving wheel, driven directly by the cranks, which would carry the bicycle the same distance per revolution of crank axle? In other words, what is the above bicycle "geared to"?

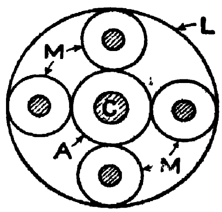


FIG. 622.

13. If a bicycle, having a driving wheel 44 inches in diameter, is geared to 64 inches by means of a Crypto gear in which the wheel A (Fig. 622) has 20 teeth, how many teeth must the wheel L have?

14. An epicyclic train of wheels is constructed as follows. A fixed annular wheel A, and a smaller concentric rotating wheel B, are connected by a compound wheel A_1B_1, the portion A_1 gearing with the wheel A, and B_1 with B. The compound wheel revolves on a stud, which is carried round on an arm which revolves about the axis of A and B. A has 130 teeth, B 20, and B_1 80, the pitch of the teeth of A and A_1 being twice the pitch of the teeth of B and B_1. How many revolutions will B make for one turn of the arm? [Inst.C.E.]

15. A reverted epicyclic train is shown in Fig. 623. A is a fixed annular wheel. BC is a double intermediate wheel mounted on an eccentric E, which is keyed to the shaft H. B gears with A, and C with L, another annular wheel, which is loose on the shaft H. Find the number of revolutions made by the shaft H for + 1 revolution of the wheel L when the numbers of teeth on the wheels A, B, C, and L are 60, 55, 59, and 64 respectively. Show that the friction of this gear would be large.

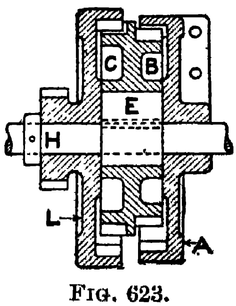


FIG. 623.

16. A pulley block for lifting a heavy weight is constructed as follows

(see Fig. 624). Secured to the block, so as not to revolve, is an annular wheel A of 20 teeth. A second wheel B, of nearly the same diameter, but having 1 tooth more than A, revolves loosely on a spindle concentric with A, and is bolted to a recessed pulley B′, having a diameter of 7 inches, round which is led the chain by which the weight is lifted. A spur wheel C, deep enough to engage with A and B, is mounted, so as to turn freely at the extremity of a short arm keyed to the spindle. To the spindle is keyed a recessed pulley A′, 10 inches diameter, round which is led an endless chain for hauling. Determine the velocity ratio of haul to lift. [U.L.]

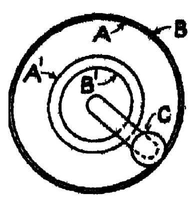


FIG. 624.

17. An epicyclic gear consists of a wheel A with 84 internal teeth, a pinion B, and a spur wheel C of 40 teeth concentric with A, B gearing with C and A. The arm which carries the axis of B rotates at 20 revolutions per minute. If A is fixed, find the speed of C, and if C is fixed, find the speed of A. If a force of 100 lbs. is applied perpendicularly to the arm at a distance of 4 feet from the centre, find the pressure between the teeth of B and C. Take the pitch circle of C as 15 inches in diameter. [U.L.]

18. Referring to the "differential motion" (Fig. 615, p. 391), in which the wheels A and L are equal, if the speeds of EF and A are + 50 and + 30 revolutions per minute respectively, what is the speed of L in revolutions per minute?

19. An arrangement of gearing involving an epicyclic train is shown in Fig. 625. BC is a shaft rotating at the constant speed of + 120 revolutions per minute. The cone pulley MN and the bevel wheel A are keyed to the shaft BC. The bevel wheel L and the wheel T are rigidly connected together, but are loose on the shaft BC. The wheel EF is loose on the shaft BC, and carries the two bevel wheels which gear with A and L, as in the ordinary differential motion shown in Fig. 615, p. 391. The cone pulley PQ and the wheel R are keyed to the shaft HK. The wheel R is geared to EF through the idle wheel S. The shaft HK is driven from the shaft BC by an open belt on the cone pulleys, as shown. The diameters of the cone pulleys at N and P are three-fifths of the diameters at M and Q, and their diameters at the middle are equal. The diameter of the wheel R is half that of EF. Find the speed of the wheel T, in revolutions per minute, when the belt is (1) at the middle of the cone pulleys, (2) at MP, and (3) at NQ.

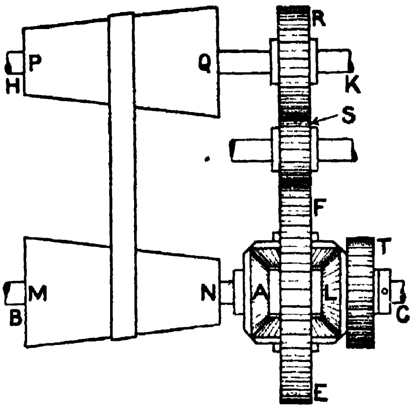


FIG. 625.

20. In the epicyclic bevel gear, shown in the sketch (Fig. 626), the wheels A and B have each 40 teeth, and the wheel C has 20 teeth; the shafts D and E are in one solid piece and rotate together at the rate of 60 revolutions per minute about the axis of E; each wheel is free to rotate on its own spindle, and the wheel A rotates 30 times per minute in a direction opposite to the rotation of the shaft E. Find the speed and direction of rotation of the wheel C. [B.E.]

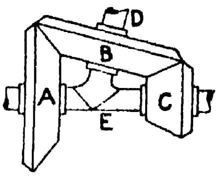


FIG. 626.

21. In an example of Humpage's gear, shown in Fig. 616, p. 392, the numbers of the teeth on the different wheels are as follows: A, 60; B, 48; C, 24; D, 16; and L, 48. If the speed of D is + 266 revolutions per minute, find, in revolutions per minute, (1) the speed of L when A is fixed, and (2) the speed of A when L is fixed.

CHAPTER XXV

MISCELLANEOUS MECHANISMS

333. Cams.—A *cam* is generally a rotating piece which gives a reciprocating or oscillating motion to another piece called the *follower*, the contact between the two being line contact. A cam may however have a reciprocating or oscillating motion as well as the follower.

334. Motion of the Cam Follower.—In general the cam follower has either rectilinear motion or angular motion about a fixed axis. Figs. 627, 629, and 631 show cam followers having rectilinear motion, AB being the

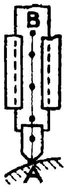


FIG. 627.

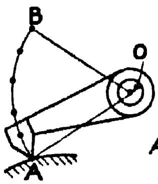


FIG. 628.

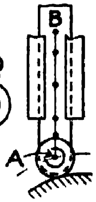


FIG. 629.

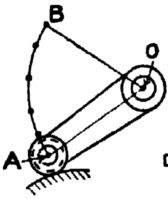


FIG. 630.

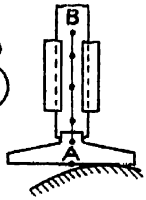


FIG. 631.

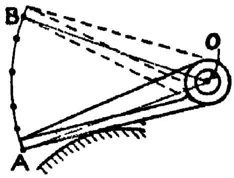


FIG. 632.

length of the travel of the follower. Figs. 628, 630, and 632 show cam followers having angular motion about a fixed axis O, the amount of the movement being the angle AOB.

The velocity of the cam is generally uniform, but the velocity of the follower is usually variable, so that equal movements of the cam are not accompanied by equal movements of the follower, and in designing a cam the first step is to assume equal movements of either the cam or the follower, and then, from the given conditions, to find the corresponding movements of the follower or cam.

Whether the motion or displacement of the cam be rectilinear or angular, it may be represented by a straight line. Let AC (Fig. 633) represent the displacement of the cam during the time that the follower travels from A to B and back again to A. Divide AC into any convenient number of equal parts, say twelve. These parts will represent equal intervals of displacement of the cam, and if the cam is moving with uniform velocity these parts will also represent equal intervals of time. Let it be given that during the 1st and 6th intervals the follower is to remain at rest, that during the 2nd, 3rd, 4th, and 5th intervals the follower is to

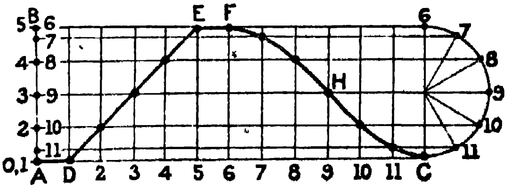


FIG. 633.

move over equal distances, starting from A at the beginning of the 2nd interval and reaching B at the end of the 5th interval, and during the remaining intervals the follower is to have harmonic motion, returning from B to A. It is required to find the positions of the follower corresponding to the positions of the cam at the end of each of the equal intervals of displacement of the cam. The construction is clearly shown in the figure. DE is a straight line. FHC is a sine or harmonic curve constructed in the usual way, as shown. The ordinates of ADEFHC give the positions of the follower corresponding to the abscissæ which give the displacement of the cam. Considering AC as a time base, the diagram ADEFHC is a space-time diagram for the follower.

Fig. 634 shows in a similar manner the case where the follower is to rise half the distance AB with uniform positive acceleration, and to complete its travel to B with uniform retardation or uniform negative acceleration. The curve AED is made up of two parabolas, AE and ED. The parabola AE has AB for its axis and A for its vertex, and the parabola ED has CD for its axis and D for its vertex. The usual and most convenient construction for drawing the parabolas in this case is shown in the figure.

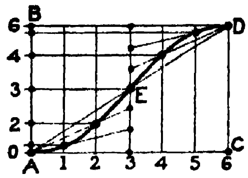


Fig. 634.

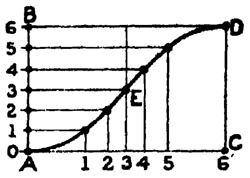


Fig. 635.

In Figs. 633 and 634 the movements of the follower have been found for equal movements of the cam, but from the same diagrams the movements of the cam for equal movements of the follower can be found as shown in Fig. 635, which is the case illustrated in Fig. 634.

When the follower has angular motion, as in Figs. 628, 630, and 632, the length of the straight line AB in Figs. 633, 634, and 635 must be equal to the length of the arc AB in Figs. 628, 630, and 632, and the subdivisions of the arc AB in Figs. 628, 630, and 632 must be equal, each to each, to the subdivisions of the straight line AB in Figs. 633 and 634, that is, the arc AB must be divided similarly to the line AB on the space-time diagram.

335. Plane Sliding Cams.—The flat plate AC (Fig. 636) has a reciprocating horizontal motion in its own plane, and its upper edge works in contact with the lower end of the follower AB. The follower is guided

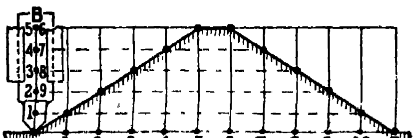


Fig. 636.

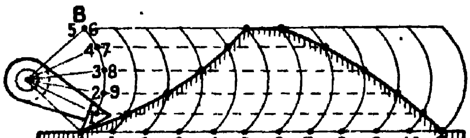


Fig. 637.

in a vertical direction, and rises and falls as the cam plate reciprocates. The force to lift the follower comes from the cam plate, but in this example the force which brings the follower down is independent of the cam, and may be the action of a weight or spring. The cam, however,

restrains the downward motion of the follower, regulating the velocity of fall. In Fig. 636 the acting edge of the cam is made up of straight lines, and it is obvious that the follower will move through equal distances, in rising and falling, for equal movements of the cam. Also, there will be periods of rest for the follower at the bottom and top of its travel.

Fig. 637 shows how the same kind of cam is designed to give the same kind of motion to the follower, except that the motion of the follower is angular instead of rectilinear, that is to say, in both examples the follower moves through equal distances, in rising and falling, for equal movements of the cam, and there are the same periods of rest.

In Figs. 636 and 637 the outline of the cam is obtained by assuming that the cam is fixed and that the follower moves towards the right through equal distances A to 1, 1 to 2, 2 to 3, etc., and at the same time rises through the distances A to 1, 1 to 2, 2 to 3, etc., shown by the divisions on AB.

In Figs. 636 and 637 the lower end of the follower is wedge-shaped, the edge of the wedge being in contact with the cam. Greater durability is obtained by replacing the wedge end by a pin, or by a pin and roller,

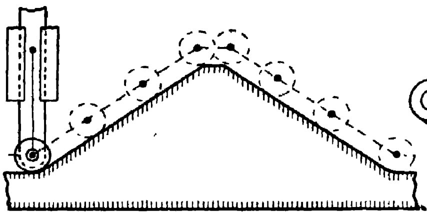


FIG. 638.

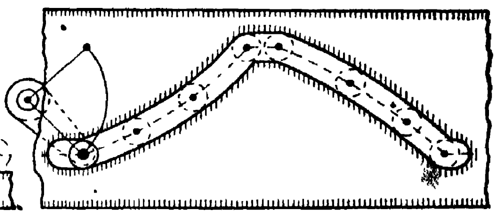


FIG. 639.

the axis of the pin taking the place of the edge of the wedge. With the pin and roller there is less friction than with the pin alone. In designing a cam to work against a pin or roller, the acting surface of the cam is first determined as for contact with a wedge; this acting surface is called the *pitch surface* of the cam, and the trace of the pitch surface on a surface normal to it is called a *pitch line*. The axis of the pin or roller is then supposed to travel so as to generate the pitch surface, and the proper acting surface of the cam is the envelope of the moving pin or roller, as shown in elevation in Figs. 638 and 639. In Fig. 639 the complete envelope is used, and becomes a slot in the cam plate; such a cam will move the follower positively in both directions. In Fig. 638 only one side of the envelope is used, and this cam requires that the follower be pushed against the cam during the downward stroke. A common defect due to the use of a roller is referred to in Art. 339.

336. Plane Rotating Cams.—The method to be adopted in designing plane rotating cams is similar to that already described for plane sliding cams. Figs. 640, 641, and 642 show plane rotating cams for working on wedge-ended followers. The followers for the cams shown in Figs. 640 and 641 have rectilinear motion, and would have the form shown in Fig. 627. In Fig. 640, AB, the path of the end of the follower in contact with the cam, when produced, passes through C, the axis of rotation of the cam, while in Fig. 641, AB produced does not pass through C. The

follower for the cam shown in Fig. 642 has angular motion about the axis O, and would have the form shown in Fig. 628.

AB, the path of the end of the follower in contact with the cam, is first divided into parts, which are the displacements of the follower for

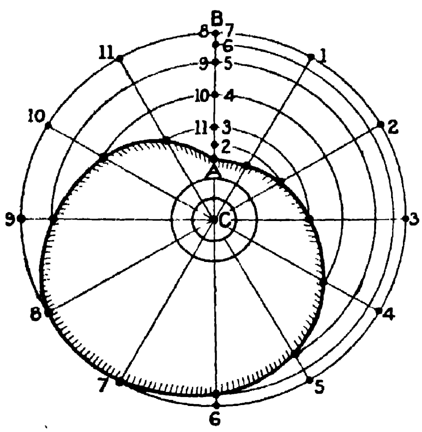


FIG. 640.

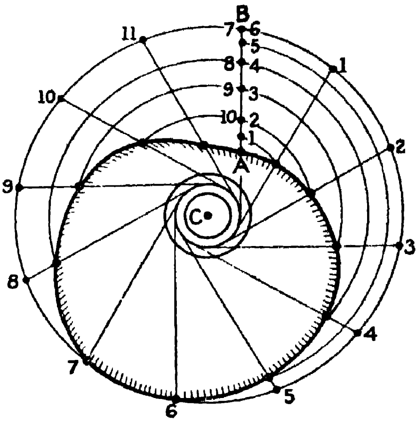


FIG. 641.

equal angular displacements of the cam. The parts of AB are determined, as explained in Art. 334, to suit the particular kind of motion which the follower is required to have. The cam is now supposed to remain at rest, while the path AB of the follower is revolved about the axis C of the cam into as many equidistant positions as there are points of division on AB. It should be noted that two or more points of division on AB may coincide. The next step is to swing round, from the centre C, the various points of division on AB to intersect the corresponding positions into which AB has been placed round the fixed cam, as is clearly shown in Figs. 640, 641, and 642. A fair curve drawn through the points determined in this way is the pitch line of the cam. If the follower is provided with a pin, or a pin and roller, the outline of the cam is determined from the pitch line exactly as described in the latter part of the preceding Article.

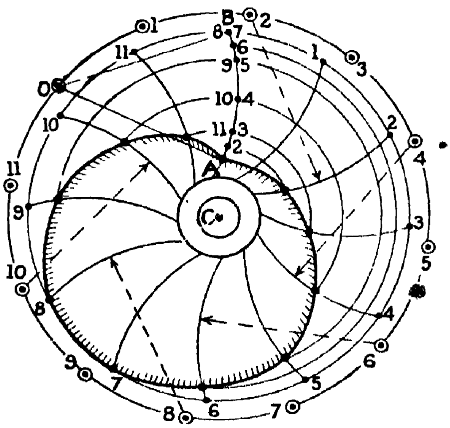


FIG. 642.

Figs. 643 and 644 show how a plane rotating cam is designed to work against followers of the form shown in Figs. 631 and 632 respectively. As in the three cases just considered, the cam is supposed to remain at rest, while the follower is made to revolve about C, the axis of the cam, into as many equidistant positions as there are points of

division on AB, and in addition the follower has given to it its corresponding radial (Fig. 643) or angular (Fig. 644) motions. The contour

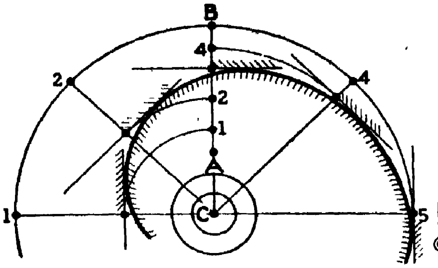


FIG. 643.

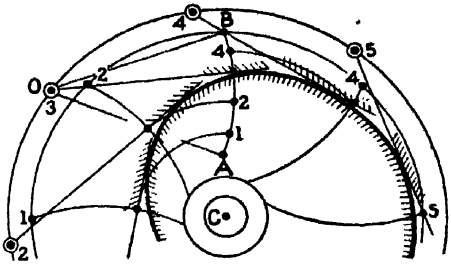


FIG. 644.

of the cam is obtained by drawing a fair curve to touch the various positions of the follower, as shown. The contour is therefore the envelope of the follower, as the latter reciprocates or swings, and at the same time revolves about the axis of the cam, the cam being at rest.

It may be pointed out here that if the plane rotating cam is made a circular eccentric cylinder working against a flat-footed or slotted follower,

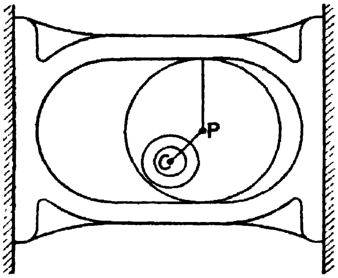


FIG 645.

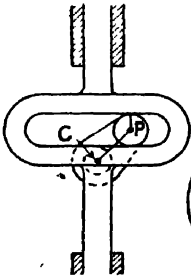


FIG. 646

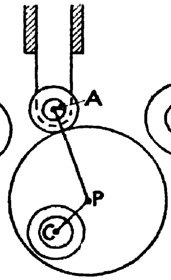


FIG. 647.

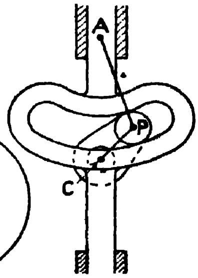


FIG. 648.

as shown in Figs. 645 and 646, the mechanism becomes the equivalent of a crank and infinite connecting-rod. Also the circular eccentric cylinder (Fig. 647) working against the end of a bar, with or without a roller, which can reciprocate in the direction of its length, is the equivalent of a crank CP and connecting-rod AP. If the slot in the follower (Fig. 646) be curved to a radius AP (Fig. 648), the mechanism becomes the equivalent of a crank CP and connecting-rod AP.

337. Cylindrical Cams.—A cylindrical cam may be used to give reciprocating motion to a follower in a direction parallel to the axis of the cam. This form of cam may be looked upon as the plane sliding cam bent round to the form of a cylinder, or the plane sliding cam may be considered as the development of the cylindrical cam.

Fig. 649 shows one half of one form of cylindrical cam, and an approximate method of designing it. The roller shown is conical, and its axis intersects the axis of the cam. To construct a cylindrical cam practically, the acting surface should be cut by a milled roller or cutter having the form of the roller or pin which is to work on it, the axis of the milled cutter being made to move over the pitch surface of the cam

as it cuts out the acting surface. The developments of the edges of the correct cam surface will not be exactly the same as the shaded lines on the developments to the right and left of Fig. 649, but when the roller

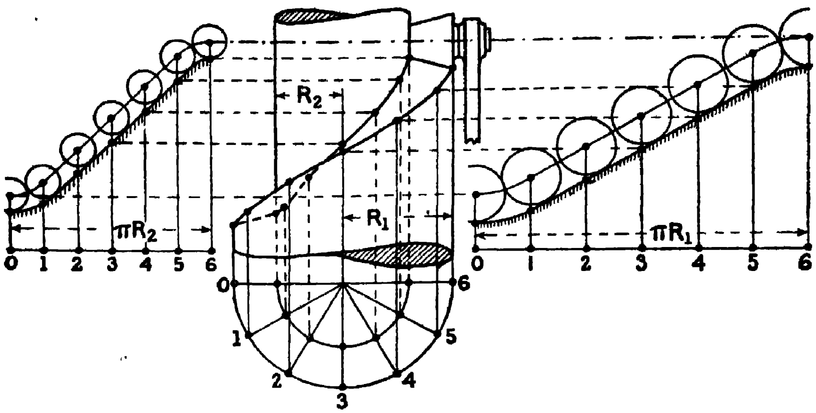


FIG. 649.

is small compared with the diameter of the cam, the differences may generally be neglected.

The cam may be made to drive the follower positively in both directions by having two acting surfaces on opposite sides of the pin or roller. These acting surfaces will then form the opposite sides of a groove on the cylinder.

338. Form of Roller for Cylindrical Cam.—It is obvious that a cylindrical roller will not work correctly on a cylindrical cam, that is, it

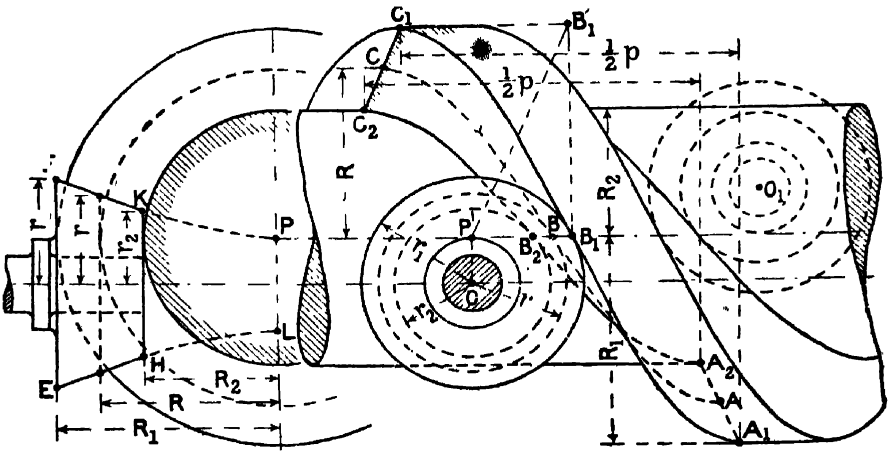


FIG. 650.

will not roll without slipping, since the path upon which the roller has to travel is longer for the outer than for the inner end of the roller. It is also obvious that on that part of the cam which is in contact with the roller when the follower is at rest a conical roller will work correctly if the

vertex of the cone is on the axis of the cam, and the acting surface of the cam is also conical with its vertex at the vertex of the roller cone. For other cases the true form of the roller is not conical, and only when the acting surface of the cam is a screw surface of constant pitch is it possible to give the roller a form which will work correctly on all parts of the cam.

A cylindrical cam in which the acting surface is a screw surface of pitch p is shown in Fig. 650. R_1 and R_2 are the external and internal radii of the screw surface respectively, and R is any other radius. r_1 and r_2 are the radii of the outer and inner ends of the roller respectively, and r is the radius of the roller corresponding to the radius R of the screw surface.

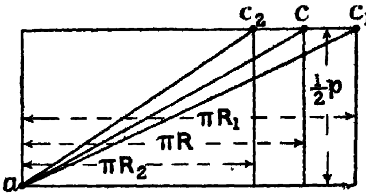


FIG. 651.

Consider the half of the cam on one side of a plane containing the axis of the cam. $A_1B_1C_1$, $A_2B_2C_2$, and ABC are the helices which are the intersections of the screw surface with the surfaces of cylinders of radii R_1, R_2, and R respectively. ac_1, ac_2, and ac, the developments of these helices, are shown to a reduced scale in Fig. 651. Let the cam make half a revolution, then for pure rolling it is evident that

$$\frac{r_1}{ac_1}=\frac{r_2}{ac_2}=\frac{r}{ac}. \text{ But } ac_1=\sqrt{\pi^2R_1^2+\frac{p^2}{4}}, \quad ac_2=\sqrt{\pi^2R_2^2+\frac{p^2}{4}},$$

$$\text{and } ac=\sqrt{\pi^2R^2+\frac{p^2}{4}}. \text{ Hence } r_2=\frac{r_1\sqrt{4\pi^2R_2^2+p^2}}{\sqrt{4\pi^2R_1^2+p^2}},$$

$$\text{and } r=\frac{r_1\sqrt{4\pi^2R^2+p^2}}{\sqrt{4\pi^2R_1^2+p^2}}. \text{ When } R=0, \quad r=\frac{r_1p}{\sqrt{4\pi^2R_1^2+p^2}}$$

and the helix coincides with the axis of the cam, hence the axis of the roller must be at a distance from the axis of the cam equal to $\frac{r_1p}{\sqrt{4\pi^2R_1^2+p^2}}$. Squaring both sides of the equation, $r=\frac{r_1\sqrt{4\pi^2R^2+p^2}}{\sqrt{4\pi^2R_1^2+p^2}}$, and rearranging the terms $\left(\frac{4\pi^2R_1^2+p^2}{p^2r_1^2}\right)r^2-\left(\frac{4\pi^2}{p^2r_1^2}\right)R^2=1$, and this is of the form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, which is the equation to an hyperbola.

The outline of the roller from $R=0$ to $R=R_1$ is EHLPKF, and between $R=R_2$ and $R=R_1$ the form is EHKF. The true form of the roller is an hyperboloid of revolution, but for ordinary cases the part EHKF is practically conical. LP is the throat of the hyperboloid. A plane section of the hyperboloid by a plane parallel to its axis and touching the surface at the throat will be two straight lines, and if this plane also contains the axis of the cam, one of the straight lines will be the line of contact between the roller and the screw surface of the cam. In the side elevation to the right in Fig. 650, $P'B_1$ is the projection on the axis of the cam of the line of contact between the roller and the screw surface of the cam, and $B_1'P'B_1$ is the true inclination

of this line to the axis of the cam. A line C_1CC_2 parallel to $B_1'P'$ determines the section of the screw surface by a plane containing the axis of the cam.

The roller may be turned in an ordinary lathe if the point of the cutting tool is set at a distance below the lathe centres equal to OP′, while the top slide rest is set at an angle to the axis of the lathe equal to the angle $B_1'P'B_1$. In practice the distance OP′ and the size of the roller compared with the outside diameter of the cam will generally be much smaller than shown in Fig. 650.

If the follower is to be driven positively by the cam in both directions, a second roller, shown by the dotted circles at O_1, will be necessary.

It should be pointed out that in the roller designed as just described there is an end thrust which is taken by a collar on the pin carrying the roller, and the friction and wear of the roller on this collar may more than neutralise the saving of friction and wear due to pure rolling between the roller and the cam.

339. Interference in Cams.—In designing a cam to fulfil certain conditions, it may happen that the formation of one part may cut away a part already formed. For example, in Fig. 652, let ABC be a part of the pitch line of a cam to work against a roller. As the axis of the roller moves along AB the envelope DLE is the corresponding part of the outline of the cam, and as the axis of the roller moves along BC the envelope FLH is the corresponding part of the outline of the cam. It will be seen that the parts DLE and FLH interfere with one another, and the possible outline for the cam is DLH. The axis of the roller will therefore move along the path AKC instead of along ABC, the dotted part at K being an arc of a circle whose centre is L. The amount of interference in this case will evidently be greater the more acute the angle between AB and BC at B is, and also the larger the roller is. Interference may also occur in other cases, as, for instance, when the part of the follower which works against the cam is a flat plate. Fig. 653 shows such a case, the required outline of cam being the envelope of the lines A, B, C, etc. It will be seen that the fair curve which touches the lines A, B, D, etc., will not touch the line C. In a case like this all that can be done is to make a compromise by drawing a curve to more nearly approach C and cut the adjacent lines at acute angles, as shown by the dotted curve.

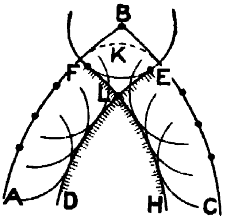


Fig. 652.

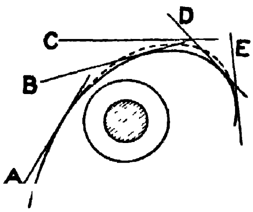


Fig. 653.

340. Velocity Ratio of Follower and Cam.—In Figs. 654, 655, and 656 EPF is part of the pitch line of a cam, RPT is the tangent, and CPD is the normal to EPF at P. In the position shown the follower is in contact with the cam at P. The point P on the cam has a velocity v_1 = PA in the direction PA, and the point P on the follower has a velocity v_2 = PB in the direction PB. If the velocities v_1 and v_2 be resolved along and perpendicular to the normal CPD, the components along the normal

must each be equal to $v = PC$. α and β are the inclinations of RPT to PA and PB respectively.

For a sliding cam (Fig. 654), in which the direction of the motion of

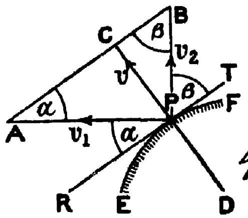


FIG. 654.

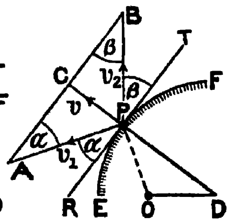


FIG. 655.

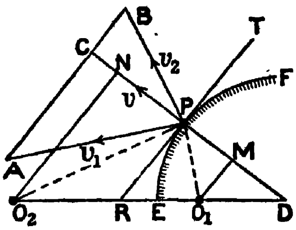


FIG. 656.

the cam is perpendicular to the direction of the motion of the follower,

$$\frac{v_2}{v_1} = \tan \alpha = \cot \beta.$$

For a cylindrical cam, of which Fig. 654 is the development, if R_1 is the radius of the cylinder and ω its angular velocity, then

$$v_1 = R_1\omega, \quad \text{and} \quad \frac{v_2}{\omega} = R_1 \tan \alpha = R_1 \cot \beta.$$

For a plane cam rotating about O, and a follower having rectilinear motion (Fig. 655), $\frac{v_2}{v_1} = \frac{\sin \alpha}{\sin \beta} = \frac{OD}{OP}$. If ω is the angular velocity of the cam, then $\omega = \frac{v_1}{OP}$, and therefore $\frac{v_2}{\omega} = OD$, where OD is perpendicular to the line of stroke of the follower.

For a plane cam rotating about O_1 and a follower swinging about O_2 (Fig. 656), if ω_1 is the angular velocity of the cam and ω_2 is the angular velocity of the follower, then in the position shown,

$$\omega_1 = \frac{v}{O_1M} \text{ and } \omega_2 = \frac{v}{O_2N},$$

where O_1M and O_2N are perpendicular to CPD. Join O_2O_1, and produce it if necessary to meet CPD at D, then $\frac{\omega_2}{\omega_1} = \frac{v}{O_2N} \div \frac{v}{O_1M} = \frac{O_1M}{O_2N} = \frac{O_1D}{O_2D}$.

341. Hooke's Joint or Universal Coupling.—By means of a *Hooke's joint* a motion of continuous rotation may be transmitted from one shaft to another when the axes of the shafts intersect, but are not in the same line. This joint is frequently used when the axes of the shafts are nominally in the same line, but through a lack of rigidity in the frame carrying the bearings of the shafts the axes may get slightly out of line several times during a revolution. The Hooke's joint forms a flexible and yet positive coupling for the shafts. The theory of this coupling will now be considered.

Referring to the lower part of Fig. 657, c and d are two shafts whose axes are assumed to be horizontal and to intersect at o, the acute angle between them being θ. The ends of the shafts are forked, and the forks carry between them a cross aa_1bb_1, the arms of which are at right angles to one another, and the axes of these arms intersect at o. The arms of

the cross are jointed to the forks, so that they may turn freely about their axes. As the shafts rotate the axis of aa_1 describes a circle whose plane is perpendicular to the axis of the shaft c, and the axis of bb_1 describes a circle whose plane is perpendicular to the axis of the shaft d, and as the axes of the shafts are assumed to be horizontal, the planes of these circles are vertical. Referring now to the upper part of Fig. 657, the circles described by the axes of the cross are shown projected on a plane perpendicular to the axis of the shaft c. The circle described by the axis of aa_1 projects into an equal circle AXA_1Y, and the circle described by the axis of bb_1 projects into an ellipse YB_1B, the semi-minor axis of which is equal to $r\cos\theta$, where r is the radius of the circles described by the axes of the cross.

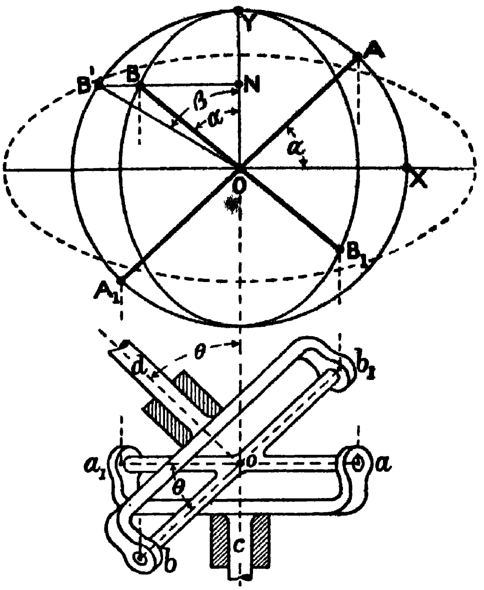


FIG. 657

If OA be the projection of the axis of the arm oa carried by the shaft c, then OB, the projection of the axis of the arm ob carried by the shaft d, must be perpendicular to OA, since these axes are perpendicular to one another, and they are projected on a plane containing one of them. Also, when OA has turned from the horizontal position OX through an angle α, OB will have turned through an equal angle α from the vertical position OY. But the actual angle β through which the arm ob has turned is not the angle BOY but the angle B′OY, the point B′ being on the circle AXA_1Y and in a line B′BN perpendicular to OY.

The connection between α and β has now to be found.

$$BN = B'N\cos\theta, \text{ hence } \frac{BN}{ON} = \frac{B'N}{ON}\cos\theta, \text{ therefore } \tan\alpha = \tan\beta\cos\theta.$$

The angular velocity ω_b of ob at any instant is evidently not necessarily the same as ω_a, the angular velocity of oa at the same instant, and the ratio of these two angular velocities will be the ratio of the indefinitely small increase of β to the corresponding increase in α. Differentiating the equation $\tan\alpha = \tan\beta\cos\theta$, the result is $\dfrac{d\beta}{d\alpha} = \dfrac{\sec^2\alpha}{\cos\theta\sec^2\beta} = \dfrac{\omega_b}{\omega_a}$.

Eliminating β, this reduces to $\dfrac{\omega_b}{\omega_a} = \dfrac{\cos\theta}{1 - \sin^2\theta\cos^2\alpha}$.

$\dfrac{\omega_b}{\omega_a}$ has a maximum value $= \dfrac{1}{\cos\theta}$ when $\cos\alpha = 1$ or -1, that is, when $\alpha = 0°$ or $180°$.

$\dfrac{\omega_b}{\omega_a}$ has a minimum value $= \cos\theta$ when $\cos\alpha = 0$, that is, when $\alpha = 90°$ or $270°$.

Hence the ratio of the fluctuation of speed to the mean speed is $\frac{1}{\cos\theta} - \cos\theta = \sin\theta\tan\theta$, assuming ω_a to be constant.

$$\omega_b = \omega_a \text{ when } \cos\theta = 1 - \sin^2\theta\cos^2\alpha, \text{ that is, when } \cos\alpha = \pm\frac{\sqrt{1-\cos\theta}}{\sin\theta}.$$

If ω_a be assumed constant, and be represented graphically by the radius of the circle AXA_1Y, and if ω_b be calculated for various values of α and the results be measured off from O on the projections, such as OA, of the axis of the arm *oa*, the polar curve shown dotted is obtained, which exhibits graphically the variations in the angular velocity of *ob* for all positions of the arm *oa*. It will be seen that the angular velocities of *oa* and *ob* are equal four times in each revolution. In Fig. 657, θ is 45°.

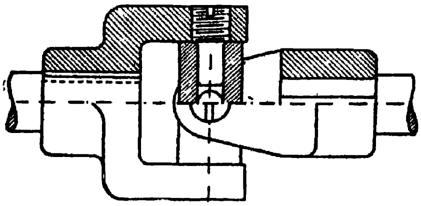


FIG. 658.

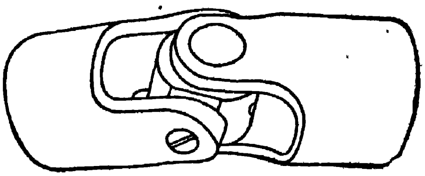


FIG. 659.

The actual form of Hooke's joint varies greatly in practice. One design is shown in Fig. 658, and a more compact form, known as Bocorselski's universal joint, is shown in Fig. 659.

By using a double Hooke's joint, as shown in Fig. 660, the shafts A and C will have the same angular velocity at every instant, provided that their axes are in the same plane and make equal angles with the axis of the intermediate shaft B. This follows at once from the formula already proved. Thus if the shafts A, B, and C turn

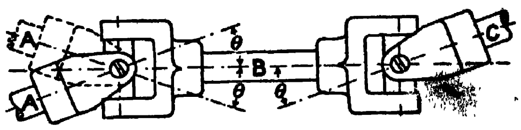


FIG. 660.

through angles α, β, and γ respectively from the position shown in the same time, then $\tan\alpha = \tan\beta\cos\theta = \tan\gamma$, therefore $\alpha = \gamma$. It is however very important to observe that for the above to be true *the axes of the joints in the forks on the intermediate shaft B must be in the same plane* as shown in Fig. 660. Judging from the number of examples to be met with in practice on motor cars and machine tools, in which the forks on the intermediate shaft are arranged wrongly, it would seem that the theory of Hooke's joint is not properly understood by many who have to construct it. Assuming that ω_a, the angular velocity of the shaft A, is constant, it is easy to show that if the axes of the joints in the forks on the intermediate shaft B are arranged at right angles to one another, as is commonly but erroneously done, the fluctuation of speed of the shaft C is from $\frac{\omega_a}{\cos^2\theta}$ to $\omega_a\cos^2\theta$, whereas if C were coupled direct to A with a single Hooke's joint the fluctuation of speed would only be from $\frac{\omega_a}{\cos\theta}$ to $\omega_a\cos\theta$.

342. Oldham's Coupling.—When the axes of two shafts are parallel, and the distance between them is small and variable, the shafts may be coupled, so that one will drive the other at the same speed by means of *Oldham's coupling*, which is shown in Fig. 661. KL and MN are the axes of the shafts. A and B are flanges secured or forged to the shafts E and F respectively. C is an intermediate piece. In one with

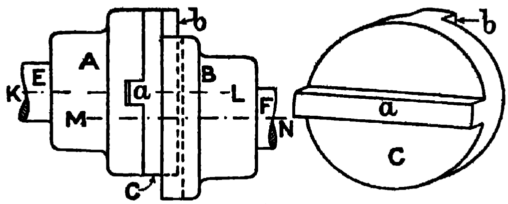


FIG. 661.

C on its opposite faces are prismatic pieces a and b at right angles to one another. These pieces fit into grooves formed in A and B, as shown. It is evident that whatever angle A turns through C must turn through the same angle, and whatever angle C turns through B must turn through the same angle; hence A, C, and B must, at every instant, have the same angular velocity.

343. Ratchets.—The principal parts of a ratchet mechanism are, a wheel or sector or rack having teeth, and a *ratchet, click, or pawl*, which engages with the teeth. In general the ratchet mechanism is used either to give intermittent motion in one direction, or to permit of motion in one direction and prevent it in the opposite direction.

In Fig. 662 A is a ratchet wheel, and B a pawl carried on a pin attached to a lever C. The lever has an oscillating motion, in this case

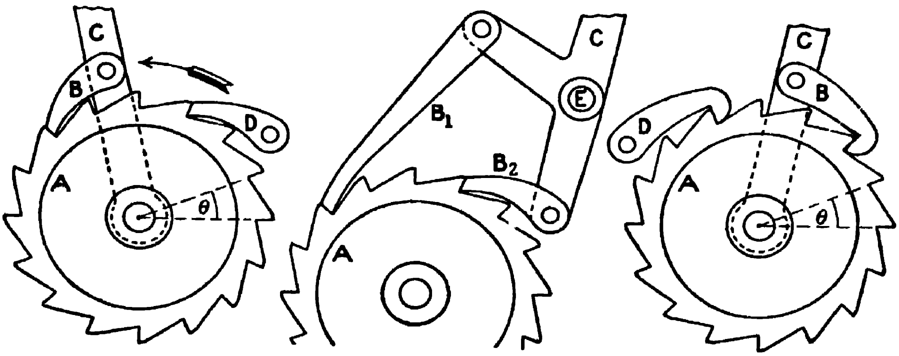


FIG. 662. FIG. 663. FIG. 664.

about the axis of A. When the lever is moving in the direction of the arrow, the pawl B engages with a tooth on the wheel, and the lever and wheel move as one piece. When the motion of the lever is reversed the pawl B rides over the teeth of the wheel, which remains at rest, either because of some resistance, such as friction, or because of the action of the pawl or *catch* or *detent* D, which is mounted on a fixed pin.

In the arrangement shown in Fig. 663 an almost continuous rotation of the wheel A in the direction of the arrow is obtained by the use of two pawls B_1 and B_2 mounted on pins attached to arms on the lever C, which oscillates on a fixed pin E. In Figs. 662 and 663 the teeth of the wheel exert a thrust on the pawls when the latter are in action, but the

pawls may be arranged to be in tension when in action, as shown in Fig. 664.

If friction is neglected, the reaction of a tooth on the pawl acting on it will be perpendicular to the face of the tooth, and in order that the pawl may not slip out of gear the line of action of the reaction must evidently pass between the axis of the wheel and the axis of the pin which carries the pawl when the pawl acts with a thrust, as in Figs. 662 and 663; but when the pawl acts with a pull, as in Fig. 664, then the above-mentioned line of reaction must lie beyond the axis of the pawl pin away from the axis of the wheel.

The limiting position of the line of the reaction of the tooth on the pawl, when friction is considered, is shown in Fig. 665 for a pushing

FIG. 665.

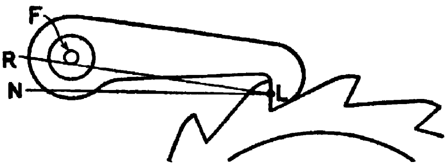


FIG. 666.

ratchet, and in Fig. 666 for a pulling ratchet. LN is the normal to the face of the tooth. When slipping is about to take place between the tooth and the pawl, LR, the line of action of the reaction of the tooth on the pawl, will make with LN an angle NLR equal to ϕ, the friction angle, and when this force is just about to rotate the pawl on its pin, LR will touch the friction circle F, as shown.

In order that there may be no lost motion of the lever C in Figs. 662 and 664, the angle through which it swings must be an exact multiple of the angle θ subtended by one tooth of the wheel at its centre and the amount of possible lost motion or back lash will be slightly less than θ when the angle of swing of the lever is not an exact multiple of θ. The possible back lash is therefore smaller the smaller the pitch of the teeth of the wheel. Reducing the pitch of the teeth reduces their strength, and in order to reduce the possible back lash without reducing the pitch of the teeth, two or more pawls are generally used in the manner shown in Fig. 667, which represents the ratchet mechanism of the Williams universal ratchet drill. In this example the ratchet wheel has twelve teeth, and there are five pawls, but only one pawl at a time can gear with the wheel. The maximum possible back lash is in this case just under one-sixtieth of a revolution, or just under 6°.

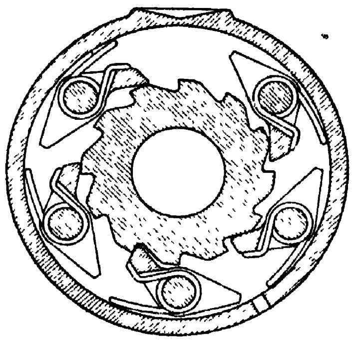


FIG. 667.

Reversible ratchets are used when it is desired to drive the ratchet wheel in either direction. Fig. 668 shows a form of reversible ratchet

used on a screw-jack. A common example of the reversible ratchet is to be found in the feed motions of planing and shaping machines.

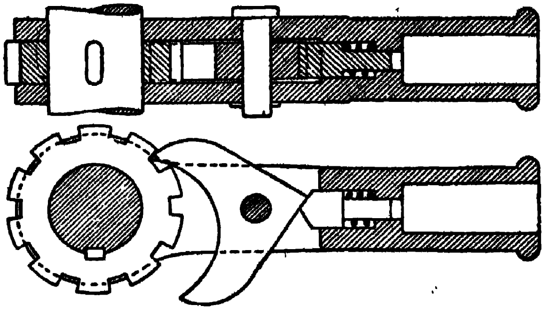


FIG. 668.

In the various forms of ratchet which have been illustrated, the ratchet is kept in contact with the wheel either by the weight of the ratchet or by the action of a spring, and when the pawl is moving backwards over the teeth it drops from one tooth on to the next with a clicking noise. To avoid this clicking several forms of *silent ratchet* have been designed. In the form shown in Fig. 669, the pawl B has attached to it an arm C, at the lower end of which there is a recess containing a plug D pressed outwards by a spring against a facing E on the ratchet wheel A. When the relative motion between the wheel and the pawl would cause the latter to rise and fall on the teeth of the wheel, the friction between E and D causes the arm C and pawl B to swing round until B comes in contact with the stop S, and the pawl remains clear of the teeth on A.

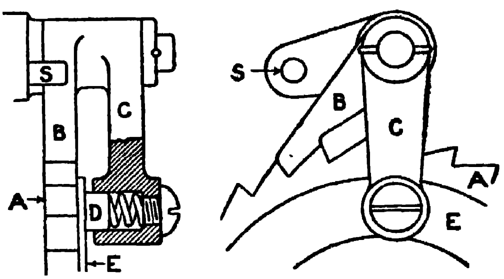


FIG. 669.

When the relative motion between the wheel and the pawl is in the opposite direction, the friction between E and D causes C and B to swing back until B engages with a tooth on A. Another method of operating the arm C is shown in Fig 670. An extension of the rim of the ratchet wheel has a groove cut in it into which is sprung a ring H, between the ends of which there is a gap to receive the lower end of the arm C. When C is pressing against one end of the ring H, the pawl is pressed against the stop, and is out of gear, and when C is in contact with the other end of H, the pawl is in gear with a tooth on the wheel. The friction between the ring H and the bottom of the groove into which it fits is sufficient to operate C when there is relative motion between the wheel and the pawl.

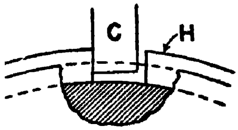


FIG. 670.

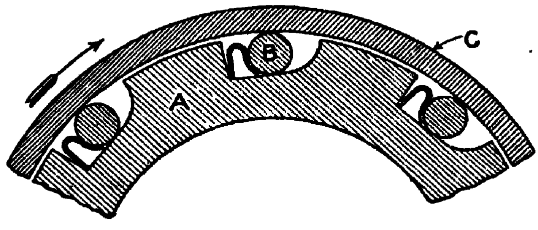


FIG. 671.

Friction ratchets have been very successful, their application to

"free-wheel" bicycles being well known. In Fig. 671, A is a ratchet wheel of special form connected to the hub of the driving wheel of the bicycle. C is a ring encircling A and forming part of the sprocket wheel. Recesses formed on A contain hard steel rollers B, which act as ratchets. The recesses containing the rollers are of variable depth, and the rollers are pressed lightly forward towards the shallower ends of the recesses by light springs behind them. When C is driven forwards in the direction of the arrow, the rollers are rolled up the slight inclines on A, and A, B, and C are locked together; but if C should stop, A will continue to move forward, or if A should tend to overrun C, it may do so because the rollers are then rolled back on the inclines and A is free from C. A slight modification of this gear is shown in Fig. 672, where the springs are spiral and are partly concealed within guide blocks behind the rollers.

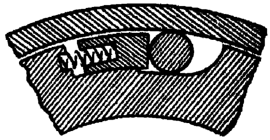


FIG. 672.

A form of friction ratchet, known as the *autoloc*, has been applied in a number of ways. One application of this mechanism is shown in Fig. 673. A is a fixed case or cup which is mounted on a pin or stud, which also carries the separate levers B and E. Between the case A and the boss H on B there is an annular space which contains two lugs FF formed on the lever E. The lever B lies in the right-hand space between the lugs FF. In the left-hand space between the lugs FF there are two balls CC separated by a spiral spring D. The balls CC have bearings in a shallow groove in A and also on the boss H of B, but the surface of H in contact with the balls is not concentric with A, but is shaped so that the annular space containing the spring D and the balls CC gets shallower in both directions from the centre line XX. The object of the contrivance is to move the lever B through any angle, and then lock it in the new position automatically. In the position shown, the lever B is locked by the balls CC, which are wedged between A and H by the action of the spring D. If the lever E be moved, say upwards, the lower lug on E unlocks the lower ball, and the upper lug moves the lever B downwards, and as soon as the force actuating E is removed, the spring again wedges the balls between A and H, so that no force applied to B will move it.

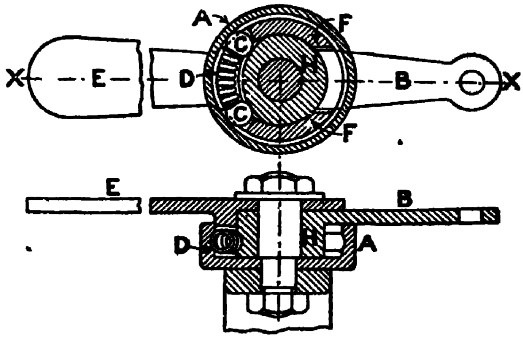


FIG. 673.

Exercises XXV.

1. A plane reciprocating cam has uniform motion and a stroke of 5 inches. The follower reciprocates at right angles to the line of stroke of the cam and in the plane of the cam. For the first $\frac{1}{2}$ inch of the forward stroke of the cam the follower is at rest at the bottom of its stroke. For the next 2 inches of the cam stroke the follower rises $1\frac{1}{2}$ inches with uniform acceleration. For the

next 2 inches of the cam stroke the follower rises 1½ inches with uniform retardation, and then remains at rest until the cam has completed its forward stroke. The follower is provided with a roller 1¼ inches in diameter, which works on the cam. Draw the outline of the cam.

2. Same as Exercise 1, except that the cam has simple harmonic motion, instead of uniform motion.

3. A straight lever oscillates in the plane of a sliding cam, about an axis at one end, though angles of 20° on opposite sides of a line parallel to the line of stroke of the cam. The lever has simple harmonic motion, and one complete oscillation of the lever is performed during two strokes of the cam. The stroke of the cam is 5 inches. The cam works against a roller 1 inch in diameter, whose axis is at the free end of the lever and 6 inches from the axis about which the lever swings. Assuming that the cam has uniform motion, draw its contour.

4. Same as Exercise 3, except that the cam has simple harmonic motion, instead of uniform motion.

5. Draw the profile of a cam to do the following work:—It has to lift a bar vertically with uniform velocity, the length of the travel of the bar being 6 inches; it then has to allow the bar to descend again with uniform velocity, but at one half the speed of the ascent. The two movements occupy one revolution of the uniformly rotating cam. The diameter of the roller working on the cam is ½ inch, and the least thickness of metal round the cam centre must be 2 inches. The line of stroke of the moving bar passes through the cam centre. [B.E.]

6. Set out the form of a plane cam, rotating with uniform velocity, to give a bar reciprocating motion of the following character. During each stroke the bar is to have simple harmonic motion. The out stroke is to be performed while the cam makes one-half of a revolution, and the in stroke while the cam makes one-third of a revolution. There are to be equal periods of rest at each end of the stroke. Stroke of bar, 3 inches. Line of stroke, ¼ inch to one side of axis of cam. Diameter of roller which works on cam, 1 inch. Minimum distance between axis of cam and axis of roller, 2 inches. If the cam makes 30 revolutions per minute, what is the maximum speed of the bar, in feet per minute, (*a*) during the out stroke, (*b*) during the in stroke?

7. O is the axis about which an arm OA swings OA = 3·5 inches. A is the axis of a roller, 0·5 inch in diameter, carried by the arm, and this roller works against a cam which rotates with uniform velocity, and whose axis C is 4 inches from O. The greatest and least distances of A from C are 3·5 inches and 1·25 inches respectively. Design the cam so that the arm shall have uniform angular velocity when swinging, and periods of rest at each end of the swing corresponding to one-twelfth of a revolution of the cam.

8. A cam mechanism is shown in Fig. 674. The cam C rotates uniformly about O, and actuates the slider S by means of the bent lever LL. The slider has an intermittent motion as follows: (*a*) A period of rest while the cam turns through 150°. (*b*) The upward half of a simple harmonic motion from A to B while the cam turns through the next 120°. (*c*) The downward half of another simple harmonic motion while the cam turns through 90°. Set out the true shape of the cam profile, working to the given dimensions and not copying the diagram. [B.E.]

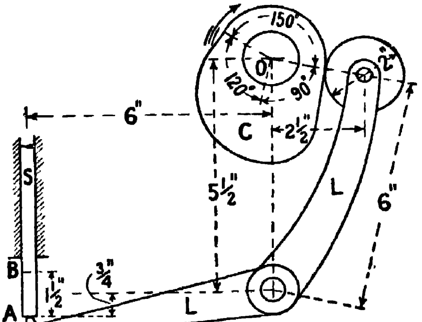


FIG. 674.

9. A vertical bar with a flat horizontal foot (see Fig. 631, p. 397) is driven upwards with simple harmonic motion, and lowered with uniform acceleration, by a cam mounted on a horizontal shaft, and having uniform angular velocity. The up stroke of the bar is performed while the cam turns through an angle of 180°, and the down stroke while the cam turns through an angle of 90°. The bar is at rest at the bottom

of its stroke while the cam turns through an angle of 90°. In its lowest position the sole of the foot of the bar is 3 inches above the axis of the cam, and the stroke of the bar is 5 inches. Draw the outline of the cam.

10. The arm EF (Fig. 675) swings about the axis O through an angle of 30°. In its lowest position the under surface of EF is inclined at 20° to the horizontal. The swinging motion of EF is controlled by a cam rotating about the axis C, the cam being always in contact with the under surface of EF. The cam has uniform angular velocity, and the arm has simple harmonic motion during its upward and downward swings. The time of each swing is one-third of a revolution of the cam, and the arm has equal periods of rest in its top and bottom positions. Design the cam.

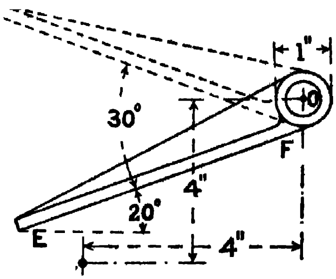
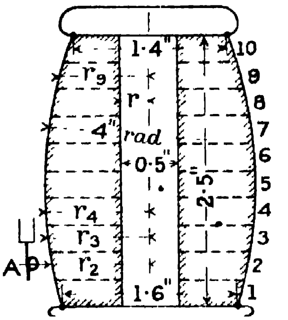


FIG. 675.

11. Design a cam to give reciprocating motion to a frame carrying a bobbin to be filled with yarn to the barrel shape shown in Fig. 676. The yarn is fed on to the bobbin at a fixed level A. There are two cases to be considered, namely, (*a*) the bobbin revolves at constant speed, and (*b*) the yarn is delivered at a constant rate. [*Hints.*—Divide the bobbin into zones 1, 2, 3, etc., of equal height. Let r_1, r_2, r_3, etc., be the mean external radii of these zones respectively. The angle through which the cam turns while the zones 1, 2, 3, etc., are passing the level A are proportional to $(r_1 - r)$, $(r_2 - r)$, $(r_3 - r)$, etc., respectively in case (*a*), and to $(r_1^2 - r^2)$, $(r_2^2 - r^2)$, $(r_3^2 - r^2)$, etc., respectively in case (*b*)].



12. In the case shown in Fig. 643, p. 401, where a cam works against a flat-footed follower, show that, if the displacement of the follower is proportional to the displacement of the cam, the curve of the latter is the involute of a circle.

13. Referring to the Hooke's joint shown in Fig. 657, p. 406, $\theta = 30°$, and the shaft *c* has a uniform speed of 200 revolutions per minute. Construct the angular velocity curve and also the polar angular acceleration curve[1] for the shaft *d*.

14. In a single Hooke's joint θ is the acute angle between the axes of the shafts. One of the shafts has a uniform speed. Express the fluctuation of the speed of the other shaft as a percentage of the speed of the first for values of θ from 0° to 50°, and plot the results.

15. In a double Hooke's joint (Fig. 660, p. 407) the axes of the joints in the forks of the intermediate shaft B are wrongly placed, being at right angles to one another instead of parallel. The shaft A has a uniform speed. Express the fluctuation of the speed of the shaft C as a percentage of the speed of A for a number of values of θ from 0° to 50°, and plot the results.

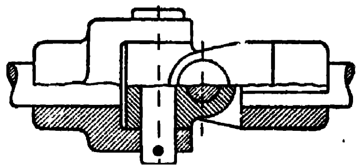


FIG. 677.

16. Hooke's joint is frequently made with the axes of the cross not intersecting, as shown in Fig. 677. Examine this arrangement, and discuss its defects.

[1] Angular acceleration $= \dfrac{d\omega_b}{dt} = \dfrac{d\omega_b}{da} \cdot \dfrac{da}{dt} = \dfrac{\omega_a^2 \cos\theta \sin^2\theta \sin 2a}{(1 - \sin^2\theta \cos^2 a)^2}$.

CHAPTER XXVI

BALANCING

344. Centrifugal Force of Revolving Mass—Equivalent Mass at Different Radius.—If a mass A (Fig. 678) of weight W be attached to a straight arm OA at a distance r from an axis O about which the arm revolves with an angular velocity ω, the centrifugal force of A is $\frac{W\omega^2 r}{g}$ (Art. 31, p. 19). If the mass A be removed and another mass B of weight W_1 be attached to the arm at a distance r_1 from the same axis O about which it revolves with the same angular velocity ω, then the centrifugal force of B is $\frac{W_1\omega^2 r_1}{g}$. If the centrifugal force of B is equal to that of A, then

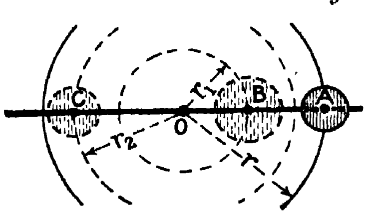


FIG. 678.

$\frac{W\omega^2 r}{g} = \frac{W_1\omega^2 r_1}{g}$, that is, $W_1 r_1 = Wr$, and when this relation holds, the mass B at the radius r_1 is said to be equivalent to the mass A at the radius r.

345. Balancing one Revolving Mass by Another.—Referring to Fig. 678, the centrifugal force of the mass A revolving about the axis O causes a tension in the arm OA, and this force will be transmitted to the bearings of the axle carrying the arm. As the arm revolves the direction of the forces on the bearings due to the centrifugal force of A will be continually changing, with the result that serious vibrations may be set up in the framing carrying the bearings, and through the framing the vibrations will extend to the foundations. If, however, a mass C of weight W_2 be placed on the arm produced beyond the axis O, and at a distance r_2 from O, so that A and C are on opposite sides of O, and if $W_2 r_2 = Wr$, then the centrifugal force of C will cause a pull at O equal and opposite to the pull caused by the centrifugal force of A. The revolving masses will then balance one another, and there will be no straining actions on the bearings of the axle or the frame carrying them due to the centrifugal forces.

346. Balancing any Number of Revolving Masses by means of one Mass, all the Masses being in the same Plane of Revolution.—Let A, B, C, etc. (Fig. 679), be a number of masses of weights W_1, W_2, W_3, etc., respectively at distances r_1, r_2, r_3, etc., respectively from an axis O about which they revolve in the same plane with angular velocity ω; it is required to balance these by one mass in the plane of the others.

Let W denote the weight of a mass X at a radius r which will balance the given masses. The centrifugal forces of the given masses

A, B, C, etc., are $W_1\omega^2r_1/g$, $W_2\omega^2r_2/g$, $W_3\omega^2r_3/g$, etc., respectively, and the centrifugal force of the mass X is $W\omega^2r/g$. Since ω^2/g is common to all the forces, it will be sufficient to consider the forces as represented in magnitude by W_1r_1, W_2r_2, W_3r_3, etc., and Wr. The problem evidently reduces to the simple one of finding a force Wr which will balance the forces W_1r_1, W_2r_2, W_3r_3, etc., all the forces acting in the same plane and at the same point. This is easily done by drawing the polygon of forces shown to the right in Fig. 679. The closing line x gives the direction and magnitude of the force Wr. The radius r may be chosen, and then $W = x/r$.

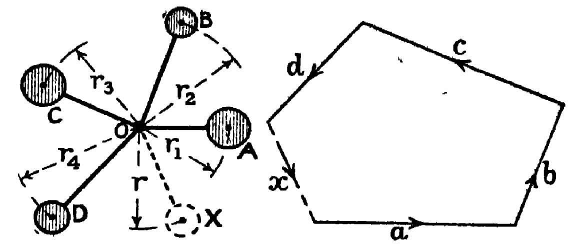


FIG. 679.

347. Variation in the Pressure on the Road of an Unbalanced Rolling Wheel.—If the want of balance of a wheel carrying a load W (including the weight of the wheel) rolling on a road is equivalent to a weight w at a distance r from its axis, and if the angular velocity of the wheel is ω, there will be at every instant a radial force F equal to $w\omega^2r/g$ acting from the centre of the wheel, and the vertical component of this force will cause a variation in the pressure of the wheel on the road. The greatest pressure on the road will be $W + F$ when the centrifugal force is acting vertically downwards, and the least pressure will be $W - F$ when the centrifugal force is acting vertically upwards. When the line of action of the centrifugal force F makes an angle θ with its position when acting vertically downwards (Fig. 680) the vertical component of F is $F \cos\theta$, and the pressure on the road is then $W + F\cos\theta$.

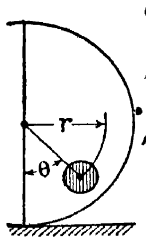


FIG. 680.

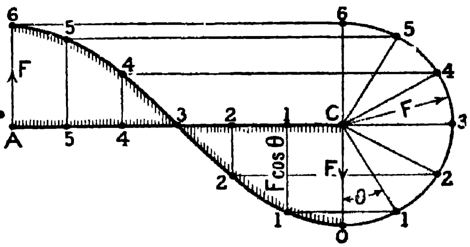


FIG. 681.

Fig. 681 shows the obvious construction for drawing the curve whose ordinates represent the term $F\cos\theta$ on a base CA, representing the distance travelled by the centre of the wheel during half a revolution.

If F is greater than W, then once during each revolution the wheel will rise off the road and return with a blow.

If D is the diameter of the wheel in feet, V the speed of the centre of the wheel in miles per hour, then the number of revolutions made by the wheel in one second is $\dfrac{5280V}{\pi D \times 60 \times 60} = \dfrac{44V}{30\pi D}$, and ω, the angular velocity of the wheel in radians per second, is $\dfrac{2\pi \times 44V}{30\pi D} = \dfrac{88V}{30D}$.

348. Balancing one Revolving Mass by two Others, all the Masses being in Separate Planes of Revolution.—Let A be the one mass, and B and C the other two masses (Figs. 682 and 683). Let the weights of the masses be W_1, W_2, and W_3 respectively, and let the distances of their centres of gravity from the axis of revolution be r_1, r_2, and r_3 respectively. Also let the distances of the planes of revolution of A and C from the plane of revolution of B be a and c respectively. The centres of gravity of the three masses must obviously lie in a plane containing the axis of revolution, and in Figs. 682 and 683 the plane of the paper has been taken as this plane.

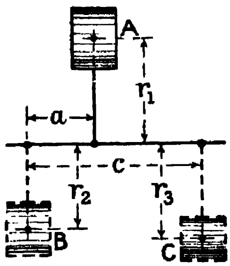


FIG. 682.

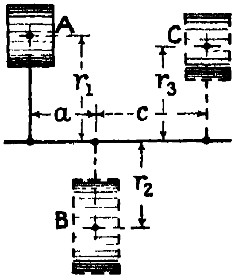


FIG. 683.

The centrifugal forces of the three masses A, B, and C are proportional to W_1r_1, W_2r_2, and W_3r_3 respectively, and the problem evidently reduces to the balancing of three parallel forces in the same plane. The conditions of equilibrium are, (1) for the case shown in Fig. 682,

$$W_1r_1 = W_2r_2 + W_3r_3, \text{ and } W_1r_1a = W_3r_3c;$$

and (2) for the case shown in Fig. 683,

$$W_1r_1 + W_3r_3 = W_2r_2, \text{ and } W_1r_1a = W_3r_3c.$$

For each case there are therefore two equations from which two unknown quantities may be determined.

If the planes of revolution of the masses B and C (Figs. 682 and 683) be the central transverse planes of the bearings of a revolving shaft carrying the given mass A, then when the given revolving mass is unbalanced by other revolving masses the centrifugal forces of the masses B and C determined as above will be the forces exerted by the bearings on the shaft due to the centrifugal force of the mass A.

349. Balancing two or more Revolving Masses by two Others, the Masses being in Separate Planes of Revolution.—Let A and B (Fig. 685) be two given masses which have to be balanced by masses P

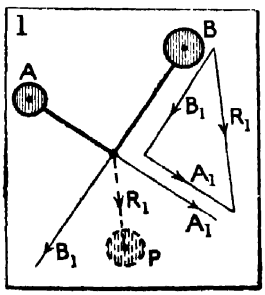


FIG. 684.

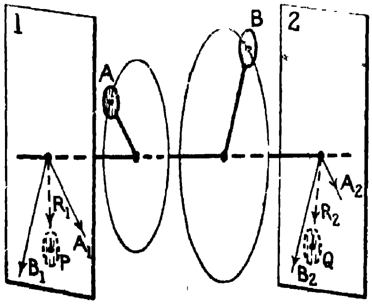


FIG. 685.

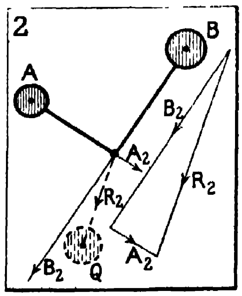


FIG. 686.

and Q in the planes of revolution 1 and 2 respectively. Figs. 684 and 686 are face views of the planes 1 and 2 respectively, with the masses A

and B, and the arms to which they are attached projected on to them. By the preceding Article the forces A_1 and A_2 acting in the planes 1 and 2 which will balance the centrifugal force of the mass A are determined. Also by the same Article the forces B_1 and B_2 acting in the planes 1 and 2 which will balance the centrifugal force of the mass B are determined. By the triangle of forces (Figs. 684 and 686) R_1, the resultant of A_1 and B_1, and R_2, the resultant of A_2 and B_2, are determined, and R_1 and R_2 are the centrifugal forces of the masses P and Q respectively. Hence if the radii at which the masses P and Q act are fixed, the weights of P and Q can be found.

If there are more than two given masses to be balanced the procedure is the same, but instead of the triangle of forces, the polygon of forces will be used to find R_1 and R_2.

In solving this problem it is desirable to tabulate the working, as shown in the form below :—

Mass.	Weight of Mass. W	Radius. r	Centrifugal Force when $\omega^2=g$. Wr	Distance of Mass from Plane 2. x	Balancing Forces when $\omega^2=g$. In Plane 1. $F_1=\frac{Wrx}{l}$	In Plane 2. $F_2=Wr-F_1$.
A						
B						
C						
etc.	. . . . .	. . . . .	. . . . .	. . . . .	. . . .	. . . .
P	$w_1=R_1/r_1$	r_1	$w_1r_1=R_1$	l	R_1	——
Q	$w_2=R_2/r_2$	r_2	$w_2r_2=R_2$	0	——	R_2

Care must be taken to give the proper signs to the forces in the last two columns. When the mass in the first column lies between planes 1 and 2, then F_1 and F_2 have both the same sign; but when either plane is between the mass and the other plane, F_1 and F_2 have opposite signs.

R_1 and R_2 are obtained from the polygon of forces in planes 1 and 2 respectively.

If the planes 1 and 2 (Fig. 685) are the central transverse planes of the bearings of the shaft carrying the given masses, then, when these masses are unbalanced by other revolving masses, the centrifugal forces of the masses P and Q, determined as above, will be the forces exerted by the bearings on the shaft due to the centrifugal forces of the given masses.

Exercises XXVIa.

1 Three masses, A, B, and C, revolve in the same plane about an axis which cuts the plane of revolution at O. The centres of gravity of A, B, and C are 15 inches, 18 inches, and 20 inches respectively from O, and the angle AOB is 90°. The weights of A and B are 80 lbs. and 50 lbs. respectively. Find the weight of C and the angle BOC in order that the revolving masses may balance one another.

2. Two masses, of 10 lbs. and 20 lbs. respectively, are attached to a balanced disc at an angular distance apart of 90°, and at radii 2 feet and 3 feet respec-

tively. Find the resultant force on the axis when the disc is making 200 turns per minute, and determine the angular position and magnitude of a mass placed at 2·5 feet radius which will make the force on the axis zero at all speeds. [Inst.C.E.]

3. A, B, and C are the centres of gravity of three masses revolving in the same plane about a centre O in that plane. OA = 18 inches, OB = 25 inches, OC = 15 inches, angle AOB = 90°, angle BOC = 120°. The weight of the third mass is 50 lbs. Find the weights of the first and second masses, so that the three masses may balance.

4. A locomotive wheel 6 feet in diameter is out of balance to the extent of 200 lbs. at a radius of 1 foot. The load on the wheel, including its own weight, is 7 tons. What are the maximum and minimum pressures of the wheel on the rail, in tons, when the speed of the locomotive is 60 miles per hour? Draw for a complete revolution of the wheel a diagram to show the variation of the pressure of the wheel on the rail. At what speed, in miles per hour, would the locomotive have to run to make the minimum pressure on the rail zero?

5. A wheel weighing 2100 lbs. has its centre of gravity 0·[illegible] inch from its axis. The wheel is mounted on a shaft which runs in two bearings 5 feet apart on opposite sides of the wheel, one bearing being 2 feet from the plane of revolution of the wheel. What are the forces on the bearings due to the centrifugal force of the unbalanced wheel when the latter is making 200 revolutions per minute? What weight placed at a radius of 3 feet 6 inches in the plane of revolution of the wheel will balance it?

6. The crank shaft of a gas-engine carries two fly-wheels A and B, the planes of revolution of which are 3 feet 6 inches apart. The plane of revolution of the crank is between the wheels, and 1 foot 7 inches from the plane of revolution of A. The crank arms and crank pin are equivalent to a weight of 108 lbs. at a radius of 10 inches in the plane of revolution of the crank. What weights placed at a radius of 2 feet, one on each wheel, will balance the crank?

7. The centrifugal force of an overhung crank is equal to that of a weight of 644 lbs. at a radius of 1 foot. The crank shaft is supported on two bearings 5 feet apart, the one nearest to the crank being at a distance of 1 foot 6 inches from the plane of revolution of the crank. Find the forces on the bearings due to the centrifugal force of the crank when the speed of the shaft is 150 revolutions per minute. What weights placed at 2 feet 6 inches radius, one in each of two planes 2 feet 6 inches apart, between the bearings, and equally distant from them, will balance the crank?

8. The following particulars relate to an ordinary inside cylinder locomotive. There are two cranks at right angles, the left-hand crank leading. Distance between centre lines of cylinders, 25 inches. Stroke of pistons, 24 inches. Distance between planes of revolution of balance masses in wheels, 60 inches. The revolving parts which have to be balanced are equivalent to 700 lbs. at the centre of each crank pin. Find the weights of the masses and their angular positions in relation to the cranks to balance the revolving parts, the centres of gravity of the balance masses being at a radius of 32 inches.

9. A shaft, 10 feet span between the bearings, carries two weights of 10 lbs. and 20 lbs. acting at the extremities of arms $1\frac{1}{2}$ feet and 2 feet long respectively, the planes in which the weights rotate being 4 feet and 8 feet respectively from the left-hand bearing, and the angle between the arms 60°. If the speed of rotation be 100 revolutions per minute, find the displacing forces on the two bearings of the machine. Moreover, if the weights are balanced by two additional rotating weights, each acting at a radius of 1 foot, and placed in planes 1 foot from each bearing respectively, estimate the magnitude of the two balance weights and the angles at which they must be set relative to the two arms. [Inst.C.E.]

10. A, B, C, and D are the planes of revolution, taken in order, of four masses connected to a shaft. The weights of the masses are 10, 16, 12, and 20 lbs. respectively, and the distances of their centres of gravity from the axis of the shaft are 2, 1·5, 1, and 1·25 inches respectively. The angular positions of the radii from the axis to the centres of gravity of the masses with respect to a reference radius OX are 0°, 90°, 150°, and 240° respectively. The distances of the planes B, C, and D from the plane A are 10, 18, and 32 inches respectively. The given masses are to be balanced by masses at 1 inch radius, one in a plane

P midway between A and B, and one in a plane Q midway between C and D. Find the weights of the balancing masses and the angular positions of the radii from the axis to their centres of gravity with respect to the reference radius OX.

11. Four masses, A, B, C, and D, weighing 70, 90, 120, and x lbs. respectively, revolve about an axis, each at a radius of 1 foot, in planes which are at equal intervals apart. Determine x and the angular positions of B, C, and D in relation to that of A in order that the masses may balance one another.

350. Disturbing Forces due to Acceleration of Reciprocating Parts.—Consider the steam-engine mechanism shown in Fig. 687, where the connecting-rod is of the slotted bar type, which is the equivalent of the connecting-rod of infinite length. The resultant of the steam pressures on the cylinder ends is a force P, which tends to move the engine frame to the left. The resultant of the steam pressures on the piston is a force Q, equal to P, which tends to move the piston to the right.

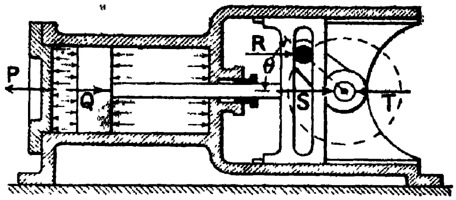


FIG. 687.

If the piston is at rest, or if it is moving with uniform velocity, there will be a thrust R on the crank pin equal to Q and to P. If forces S and T, each equal to R, be applied to the crank shaft in the plane of revolution of the crank pin, as shown, then R and T will form an effort couple whose moment is the turning moment on the shaft, and which will be balanced by the resistance couple. The remaining force S will be the thrust of the shaft on its bearings carried by the frame. Hence the frame is pushed to the right by a force S, and at the same time it is pushed to the left by a force P, which is equal to S; there is therefore no tendency for the frame to move on its foundations.

If, however, the piston is increasing or decreasing in speed, the force R, and therefore the force S, will be less or greater than P, and there will therefore be a resultant force on the frame tending to move it to the left or right. The force tending to move the frame on its foundations is the difference between the forces P and S, and this is evidently the force necessary to accelerate the piston, and this force can be determined without any reference to the steam pressures within the cylinder. For the mechanism shown in Fig. 687, if the crank shaft is rotating with uniform velocity, the piston and the parts reciprocating with it have harmonic motion. When the piston is at a distance x from the middle of its stroke, the force in lbs. required to give the reciprocating parts the necessary acceleration is $\frac{WV^2x}{gr^2} = \frac{W\omega^2x}{g} = \frac{W\omega^2r\cos\theta}{g}$ (Art. 259, p. 298), where W is the weight of the reciprocating parts in lbs., V the velocity of the crank pin in feet per second, ω the angular velocity of the crank in radians per second, r the radius of the crank or half the stroke of the piston in feet, and θ the inclination of the crank to the line of stroke (Fig. 687).

During the first half of a stroke the accelerating force on the piston varies uniformly from $W\omega^2r/g$ to zero, and acts in the direction of the motion of the piston. During the second half of the stroke the accelerating force varies from zero to $W\omega^2r/g$, and acts in the opposite direction, that is, the

accelerating force becomes negative. The force tending to displace the frame of the engine is at every instant equal and opposite to the accelerating force on the piston. It follows that if the engine frame is not bolted down it will oscillate backwards and forwards, making one complete oscillation for each revolution of the crank shaft. Bolting down the engine frame will not eliminate the oscillations, but will reduce their amplitude to an amount depending on the rigidity of the connections and the mass of the foundations.

It is important to understand that the disturbing forces on the frame, as determined above, are independent of the way in which the reciprocating masses are driven, whether by the action of fluid pressure on the piston, as in a steam-engine, or by a torque on the crank shaft, as in a pump or air compressor.

351. Effect of Transferring Reciprocating Mass to Crank Pin.—Suppose that the reciprocating parts which are connected to the crank pin by a slotted bar connecting-rod are removed, and that a mass C of equal weight W is placed at the crank pin (Fig. 688). The centrifugal force of this revolving mass at the crank pin is $W\omega^2 r/g$. Let this force be represented by OA, and let OX be the line of stroke of the reciprocating parts (now removed). Draw AB perpendicular to OX. The force OA is equivalent to two forces OB and BA acting at O, OB being equal to $\frac{W\omega^2 r \cos\theta}{g}$, and BA equal to $\frac{W\omega^2 r \sin\theta}{g}$. It was shown in the preceding Article that the effect of the acceleration of the reciprocating parts was to cause a thrust on the crank shaft in the line of stroke, the magnitude of this thrust being $\frac{W\omega^2 r \cos\theta}{g}$. Hence a mass equal to that of the reciprocating parts, but placed at the crank pin, has the same disturbing effect on the frame *in the line of stroke* as the reciprocating parts themselves have.

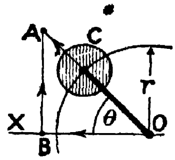


FIG. 688.

352. Changing the Direction of the Disturbing Force.—If two masses DD (Fig. 689) be placed on the crank arms produced so as to balance the mass C referred to in the preceding Article, then at every instant the component of the centrifugal force of DD in the line of stroke will balance the component of the centrifugal force of C in that line. Hence if the mass C be removed from the crank pin, and the reciprocating parts be again connected to it, the thrust on the frame *in the line of stroke*, due to the acceleration of the reciprocating parts, will be entirely balanced. But the centrifugal force of DD has a component at right angles to the line of stroke, the magnitude of which is $-\frac{W\omega^2 r \sin\theta}{g}$, and this is unbalanced.

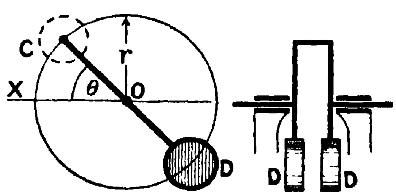


FIG. 689.

The effect of the balance weights DD is therefore to balance the disturbing force in the line of stroke, and to introduce another disturb-

ing force at right angles to the first. The second disturbing force evidently goes through the same variations in magnitude as the first. The first varies from $W\omega^2 r/g$ at the beginning of the stroke to zero at the middle of the stroke, and from zero at the middle it varies to $-W\omega^2 r/g$ at the end of the stroke. The second varies from zero at the beginning of the stroke to $W\omega^2 r/g$ at the middle of the stroke, and diminishes again to zero at the end of the stroke.

If the mass of DD at crank radius, and opposite to the crank pin, as in Fig. 689, be less than the mass of the reciprocating parts, then the disturbing force in the line of stroke will be only partly balanced, and the disturbing forces will now be, first, a force in the line of stroke equal to $\dfrac{(W-w)\omega^2 r\cos\theta}{g}$, and second, a force at right angles to the line of stroke equal to $\dfrac{w\omega^2 r\sin\theta}{g}$, where w is the combined weight of the masses DD.

The obvious construction for finding R, the resultant disturbing force for any position of the crank, is shown in Fig. 690, where $r_1 = (W-w)\omega^2 r/g$, and $r_2 = w\omega^2 r/g$.

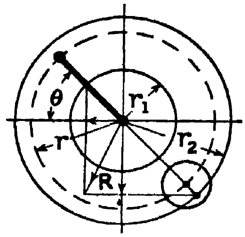


Fig. 690.

In some cases the kind of balancing just described, where the disturbing force in the line of stroke is entirely or partially balanced, may be advantageous, as in locomotives, where a horizontal disturbing force is generally more injurious than a vertical one, but in many other cases there would be no advantage whatever.

353. Distribution of Weight of Connecting-rod.—The part of the connecting-rod in the neighbourhood of the crank pin has almost pure rotary motion with the crank pin, and the part in the neighbourhood of the cross-head has almost pure reciprocating motion with the piston. If A is the centre of the cross-head end of the connecting-rod, B the centre of the crank pin end, and C the centre of gravity of the

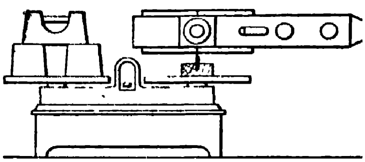


Fig. 691.

whole rod, and if W is the total weight of the rod, then the usual practice is to reckon $\dfrac{AC}{AB}\cdot W$ as the part of the weight of the rod which is to be credited as a revolving weight at the crank pin centre, and the remaining part $\dfrac{BC}{AB}\cdot W$ is to be credited as reciprocating with the piston.

The position of the centre of gravity of an actual connecting-rod may be determined by balancing it in a horizontal position on a knife edge. The values of the weights to be reckoned as revolving and reciprocating respectively may, however, be obtained by direct weighing, as shown in

Fig. 691. The connecting-rod is supported on two knife edges placed at right angles to its axis and passing through the centres of the faces of the end brasses. One of the knife edges is placed on the platform of a weighing machine, and the other rests on a support on the ground. When the crank or large end of the rod is on the platform, the weight indicated is the weight to be credited as revolving, and when the cross-head or small end is on the platform, the weight indicated is the weight to be credited as reciprocating.

In many cases in practice the weight of the part of the connecting-rod to be credited as revolving is about two-thirds of the total weight of the rod.

354. Balancing of Locomotives.—As regards the revolving parts of a locomotive there is no difficulty in balancing them entirely, but the difficulties in the way of completely balancing the reciprocating parts are so great, that in practice only an approximate solution is attempted. If the horizontal disturbing forces due to the acceleration of the reciprocating parts be completely balanced in the manner explained in Art. 352, the vertical disturbing forces introduced may be so great as to cause serious damage to the permanent way, to the bridges, and to the wheel tyres. As the result of the extensive experience of locomotive engineers, the practice now generally adopted is to balance all the revolving parts completely and two-thirds of the reciprocating parts. That is to say, as regards the reciprocating parts, the horizontal disturbing forces due to the acceleration of two-thirds of the reciprocating parts are balanced by revolving masses, but it must be remembered that these revolving masses introduced cause vertical disturbing forces equal to those which they balance horizontally.

The balance weights are, in practically all cases, placed between the spokes of the wheels near the rims.

In what follows, the pistons are assumed to have harmonic motion.

355. Inside Cylinder Uncoupled Locomotives.—These engines have two cylinders placed between the frames, and the driving axle is cranked, the two cranks being at right angles to one another. The important revolving weights which have to be balanced are, the crank arms, the crank pins, and the parts of the connecting-rods, determined as explained in Art. 353. All these should be reduced to equivalent weights at the crank pin centres.

To the equivalent revolving weight at each crank pin centre has to be added two-thirds of the weight of the reciprocating parts for one cylinder. The problem is then to find the weights to be placed in the wheels, at a given radius depending on the diameter of the wheels, in order to balance the weights assumed to be at the crank pin centres. This problem is a simple case of the one considered in Art. 349, but on account of its importance the solution for this case will be given here.

Since all the revolving masses have the same angular velocity, the centrifugal forces may be represented by the products of the weights of these masses and the radii of the respective circles described by their centres of gravity.

Referring to Fig. 692, 1 and 2 are the circles described by the centres of gravity of the required balance weights. A is the left-hand and B

the right-hand crank. P and Q are the centrifugal forces of the revolving masses and two-thirds of the reciprocating masses at crank radius, as already explained. P and Q are in general equal.

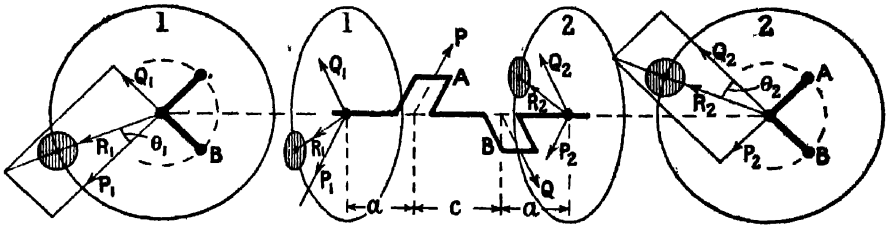


FIG. 692.

Forces P_1 and P_2, in the planes of the circles **1** and **2** respectively, which will balance P, are determined from the equations

$$P_1(2a+c) = P(a+c), \text{ and } P_2(2a+c) = Pa.$$

Forces Q_1 and Q_2, in the planes of the circles **1** and **2** respectively, which will balance Q, are determined from the equations

$$Q_1(2a+c) = Qa, \text{ and } Q_2(2a+c) = Q(a+c).$$

R_1, the resultant of P_1 and Q_1, and R_2, the resultant of P_2 and Q_2, can now be determined. $R_1 = \sqrt{(P_1^2+Q_1^2)}$, and $R_2 = \sqrt{(P_2^2+Q_2^2)}$. The angles θ_1 and θ_2 are determined from $\tan\theta_1 = Q_1/P_1$ and $\tan\theta_2 = P_2/Q_2$.

$$\text{If } P = Q, \text{ then } P_1 = Q_2, \ P_2 = Q_1, \ R_1 = R_2, \text{ and } \theta_1 = \theta_2.$$

Let w = weight at each crank pin, including equivalent revolving weight and two-thirds of the reciprocating weight for one cylinder. W = weight of each balance weight. r = radius of cranks. R = radius of circles described by the centres of gravity of balance weights.

$$\text{Then } P = wr, \quad P_1 = \frac{wr(a+c)}{2a+c}, \quad Q_1 = \frac{wra}{2a+c}, \text{ and}$$

$$R_1 = WR = \frac{wr}{2a+c}\sqrt{(a+c)^2+a^2}, \text{ also } \tan\theta_1 = \frac{a}{a+c}$$

356. Outside Cylinder Uncoupled Locomotives.—These engines have two cylinders placed outside the frames, and the two cranks, which are at right angles to one another, are placed at the ends of the driving axle, which is straight. Generally the crank arms are formed in the driving wheels.

Referring to Fig. 693, **1** and **2** are the circles described by the centres of gravity of the required balance weights which are placed in the driving wheels. P and Q are the centrifugal forces of the weights which are considered as revolving at the centres of the crank pins A and B respectively. The weight considered as revolving at each crank pin includes two-thirds of the weight of the reciprocating parts for one cylinder, two-thirds of the weight of that part of the connecting-rod which is reckoned as reciprocating, the weight of that part of the connecting-rod which is reckoned as revolving, and the weight of the part of the crank pin which

projects from the crank arm. S and T are the centrifugal forces of the crank arms and the parts of the crank pins which they contain.

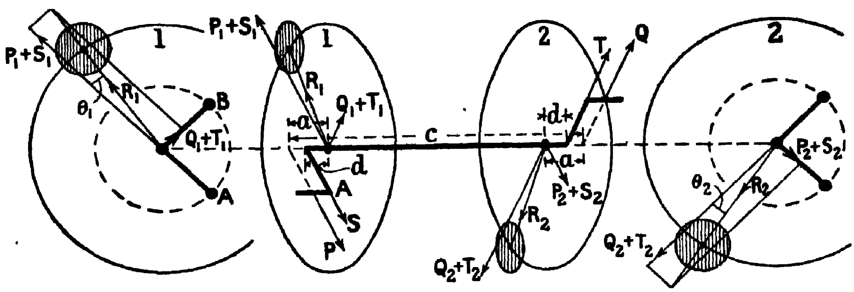


FIG. 693.

Forces P_1 and P_2, in the planes of the circles **1** and **2** respectively, which will balance P, are determined from the equations

$$P_1(c-2a)=P(c-a), \text{ and } P_2(c-2a)=Pa.$$

Forces Q_1 and Q_2, in the planes of the circles **1** and **2** respectively, which will balance Q, are determined from the equations

$$Q_1(c-2a)=Qa, \text{ and } Q_2(c-2a)=Q(c-a).$$

Forces S_1 and S_2, in the planes of the circles **1** and **2** respectively, which will balance S, are determined from the equations

$$S_1(c-2a)=S(c-2a+d), \text{ and } S_2(c-2a)=Sd.$$

Forces T_1 and T_2, in the planes of the circles **1** and **2** respectively, which will balance T, are determined from the equations

$$T_1(c-2a)=Td, \text{ and } T_2(c-2a)=T(c-2a+d).$$

P_1 and S_1, P_2 and S_2, Q_1 and T_1, Q_2 and T_2, act respectively in the same straight lines and in the same directions, as shown.

$$R_1=\sqrt{(P_1+S_1)^2+(Q_1+T_1)^2}, \text{ and } R_2=\sqrt{(P_2+S_2)^2+(Q_2+T_2)^2},$$

$$\tan\theta_1=\frac{Q_1+T_1}{P_1+S_1}, \text{ and } \tan\theta_2=\frac{P_2+S_2}{Q_2+T_2}.$$

If $P=Q$, and $S=T$, then $P_1=Q_2$, $P_2=Q_1$, $S_1=T_2$, $S_2=T_1$, $R_1=R_2$, and $\theta_1=\theta_2$.

Using the same notation as for inside cylinder engines, with in addition w_c = the weight of one crank arm and the part of the crank pin which it contains, reduced to crank radius, then $P=wr$, $P_1=\dfrac{wr(c-a)}{c-2a}$, $Q_1=\dfrac{wra}{c-2a}$,

$$S=w_cr, \quad S_1=\frac{w_cr(c-2a+d)}{c-2a}, \quad T_1=\frac{w_crd}{c-2a},$$

and $$R_1=WR=\frac{r}{c-2a}\sqrt{\{w(c-a)+w_c(c-2a+d)\}^2+(wa+w_cd)^2},$$

also, $$\tan\theta_1=\frac{wa+w_cd}{w(c-a)+w_c(c-2a+d)}.$$

357. Coupled Locomotives.—In order to increase the adhesion or resistance to slipping, and thus enable a larger tractive force to be utilised, two or more pairs of wheels of a locomotive may be coupled together. The coupling together of two wheels is effected by a coupling-rod connecting the crank pins on two equal cranks, formed one in each wheel. The cranks may, however, be separate from the wheels and be fixed to the axles. It is obvious that coupled wheels must be of the same diameter.

So far as the coupling-rods affect the balancing of the engine they behave exactly as revolving weights at their crank pins, the amount of weight at any one crank pin being equal to the portion of the weight of the rod supported by that pin. Hence the coupling-rods with their cranks and crank pins may be completely balanced in the manner explained in Art. 349 by weights in the wheels. In the case of driving wheels, the balance weights for the coupling-rod cranks, with their pins and the weights of the coupling-rods carried by them, may be combined with the balance weights determined, as in the two preceding Articles, and resultant balance weights found. The resultant balance weights on the driving wheels of a coupled engine may, however, be obtained directly by simply including in the planes 1 and 2 (Figs. 692 and 693) the forces which will balance the centrifugal forces of the coupling-rod cranks, crank pins, and the weights of the coupling-rods which they carry.

Fig. 694 shows the arrangement of the various cranks in an inside cylinder engine with two pairs of wheels coupled. It will be observed that the cranks connected by the coupling-rod CD are at right angles to

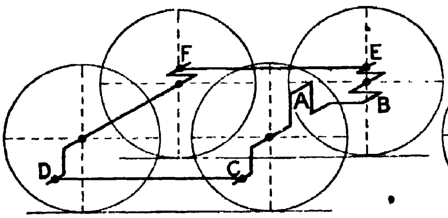


FIG. 694.

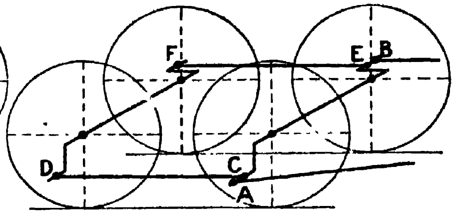


FIG. 695.

the cranks connected by the coupling-rod EF, also the crank pins C and E are opposite to the main crank pins A and B respectively.

In the case of an outside cylinder coupled engine (Fig. 695), the driving cranks serve also as coupling-rod cranks, each crank pin on these cranks being long enough to carry a connecting-rod end and a coupling-rod end. Here also the cranks on one side of the engine are at right angles to those on the other side.

The coupling-rod cranks on an inside cylinder engine may be, and generally are, shorter than the driving cranks, but on outside cylinder engines all the cranks are of the same radius.

358. Balancing Reciprocating Parts in Coupled Locomotives.—The reciprocating parts to be balanced in a coupled engine may be balanced in the driving wheels, or this balancing may be distributed amongst all the coupled wheels. All that is necessary is to consider each axle as a driving axle, with imaginary cranks parallel to the respective cranks on the real driving axle, the imaginary crank pins on a particular

axle carrying revolving masses equal to the portions of the reciprocating *masses to be balanced in the wheels of that axle. The revolving masses of the imaginary cranks are of course neglected.*

Another way of proceeding is to find the separate masses necessary in the driving wheels to balance the reciprocating masses which are to be balanced and then divide these up into parts, which are placed in similar positions in the various wheels. For example, if A_1 and B_1 (Fig. 696) are the masses in the left-hand and right-hand driving wheels respectively which will balance, say, two-thirds of the reciprocating masses, and if equal portions of A_1 and B_1 be transferred to A_2 and B_2 in the left-hand and right-hand coupled trailing wheels, the radii from A_2 and B_2 being equal and parallel to the radii from A_1 and B_1 respectively, then this new distribution of balance weights will have the same effect *horizontally* in balancing the reciprocating parts, but vertically there will now be a less variation in the pressure on the rails per wheel. The balance weights thus found in the wheels to balance the reciprocating parts are then combined with the balance weights for the revolving masses, and the resultant balance weights determined.

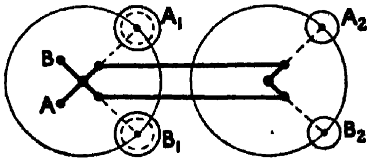


FIG. 696.

359. Complete Balancing of Reciprocating Parts having Harmonic Motion.—In Art. 351 it was shown that the disturbing forces due to the acceleration of the reciprocating parts are the same as those produced in the line of stroke by a revolving mass equal to that of the reciprocating parts concentrated at the crank pin, but this revolving mass produces equal disturbing forces at right angles to the line of stroke.

Now suppose a number of sets of reciprocating masses to be connected to the same number of cranks on a shaft. Next suppose that these reciprocating masses are removed, and masses equal to them are concentrated at their respective crank pins. The disturbing forces in the various lines of stroke will now be the same as before, but if the various imaginary revolving masses at the crank pins be of such magnitudes, and if their relative positions be such that they balance one another, then it is obvious that not only will the disturbing forces in the lines of stroke balance one another, but the disturbing forces in the directions at right angles to the lines of stroke will also balance one another.

The problem of the complete balancing of reciprocating masses therefore reduces to that of balancing a number of revolving masses, a problem which was discussed in Art. 349. It must, however, be remembered that the revolving masses now being considered are imaginary, and that *reciprocating masses can only be completely balanced by other reciprocating masses.*

A few cases will now be considered in illustration of the foregoing. First, take the case of a single-cylinder engine (Fig. 697), the piston being connected to a crank pin A revolving in a circle of radius r. Let B and C be two other crank pins revolving in circles of radii r_1 and r_2 respectively, the planes of revolution of these pins being at distances b and c respectively from the plane of revolution of the crank pin A. Let w be the weight of the reciprocating masses connected to the crank pin A. Then, if the crank pins B and C be in the same plane with the

crank pin A and the axis of the shaft, and if B and C are on the opposite side of the axis to A, revolving masses of weights w_1 and w_2 placed at B and C respectively will balance a revolving mass of weight w at A if $w_1r_1(b+c)=wrc$, and $w_2r_2(b+c)=wrb$.

Hence if reciprocating masses D and E of weights w_1 and w_2 respectively be connected to the crank pins B and C, as shown, the reciprocating masses D and E will completely balance the reciprocating masses connected to the crank pin A. Reciprocating masses, such as D and E, introduced to balance other reciprocating masses are called *bob-weights*, and they serve no other purpose than that of balancing. Instead of the masses D and E, which are useless except for balancing purposes, two additional cylinders might be introduced, and the reciprocating parts belonging to these cylinders would take the place of the bob-weights, and the reciprocating parts of this three-cylinder engine would then completely balance one another.

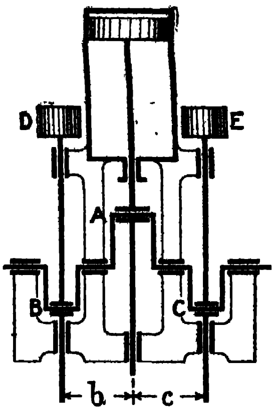


FIG. 697.

If the stroke of the bob-weights is small, they may be driven by eccentrics instead of ordinary cranks. The weights w_1 and w_2 of the bob-weights must of course include the weights of the reciprocating parts connected to them.

As a second illustration, consider a three-cylinder engine with three cranks at definite angles apart, say, 120° each. A, B, and C (Fig. 698) are the three crank pins, and it is required to balance the reciprocating parts by bob-weights driven by cranks or eccentrics on the crank shaft in the planes 1 and 2.

Imagine masses equal to the reciprocating masses to be concentrated at their respective crank pins. The centrifugal force of the mass at A is

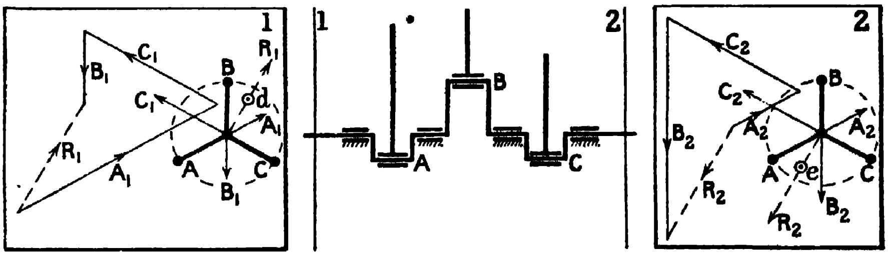


FIG. 698.

balanced by forces A_1 and A_2 in the planes 1 and 2 respectively, and these forces are determined as in Art. 348, p. 416. In like manner forces B_1 and B_2, C_1 and C_2, which will balance the centrifugal forces of the masses at B and C, are determined. R_1, the resultant of A_1, B_1, and C_1, also R_2, the resultant of A_2, B_2, and C_2, are determined by the polygons of forces shown. A crank pin d at radius r_1 in plane 1, and a crank pin e at radius r_2 in plane 2, will be the drivers for the required

bob-weights. If w_1 and w_2 are the weights of the bob-weights, including the weights of the reciprocating parts connected to them, then $w_1r_1=R_1$, and $w_2r_2=R_2$, the various centrifugal forces being represented by the products of weight and radius.

A four-cylinder engine with four cranks is one that lends itself to complete balancing of the reciprocating parts without the addition of balance weights. The quantities to be considered are: first, the ratios of the weights of the four sets of reciprocating parts to that of one of them (3 quantities); second, the ratios of the distances between the centre lines of the cylinders to one of the distances (3 quantities); third, the angles between the cranks (3 quantities); in all 9 quantities, and if any 5 of these be given, the other 4 can be found.

In this Article reciprocating masses only have been referred to. Revolving masses, including any cranks or eccentrics introduced to drive bob-weights, must be balanced separately by other revolving masses.

Exercises XXVIb.

1. The reciprocating parts of a single cylinder horizontal steam-engine weigh 200 lbs., and the remaining parts of the engine weigh 6400 lbs. The stroke of the piston is 16 inches, and the crank shaft makes 300 revolutions per minute. Assuming that the engine is not bolted down, but is free to oscillate, find the amplitude of the oscillations, and the magnitude of the displacing force at the end of each oscillation. Assume that the reciprocating parts have harmonic motion.

2. After the engine of the preceding exercise is bolted down, suppose that it is found that a force of 2000 lbs., applied to the crank shaft in the line of stroke, displaces the engine frame 0·001 inch, and that the displacement up to five times this amount is proportional to the displacing force. Assuming that all the yield takes place between the frame and the foundations, what will now be the amplitude of the oscillations of the engine frame?

3. A connecting-rod, 6 feet 2 inches long between centres, was found to balance in a horizontal position on a knife edge placed at 24½ inches from the large end centre. When a weight of 14 lbs. was placed at the small end centre, it was found that the whole balanced in a horizontal position on a knife edge placed at 26¼ inches from the large end centre. Find the weight of the rod and the weights of the parts which should be credited as revolving and reciprocating respectively.

4. Find the balance weights at crank radius, in order to balance all the revolving masses and two-thirds of the reciprocating masses of an inside single locomotive, having given the following data. Cranks at right angles, left-hand crank leading. Distance centre to centre of cylinders, 2 feet. Distance between planes containing mass centres of balance weights, 5 feet. Mass of reciprocating parts per cylinder, reduced to crank radius, 500 lbs. Mass of revolving parts per cylinder, reduced to crank radius, 700 lbs. [Inst.C.E.]

5. The reciprocating parts of an inside cylinder uncoupled locomotive weigh 550 lbs. per cylinder. The revolving parts are equivalent to 650 lbs. per cylinder at crank pin. The stroke of the pistons is 24 inches, and the distance between the centre lines of the cylinders is 25 inches. Find the balance weights which must be placed in the driving wheels at 2 feet 6 inches radius, their planes of revolution being 5 feet apart, in order to balance the whole of the revolving parts and two-thirds of the reciprocating parts. Cranks at right angles, left-hand crank leading.

6. The following particulars relate to an outside cylinder uncoupled locomotive. Stroke of pistons, 26 inches. Length of connecting-rod, 78 inches. Distance of centre of gravity of connecting-rod from centre of large end, 26 inches. Weight of connecting-rod, 450 lbs. Weight of reciprocating parts per cylinder, 400 lbs. Equivalent weight of one crank arm and the portion of the crank pin within it, 130 lbs. at 13 inches radius. Weight of one overhanging

crank pin, 30 lbs. Distance between centre lines of cylinders, 74 inches. Distance between planes containing centres of gravity of balance weights, 59 inches. Distance between planes containing centres of gravity of crank arms, 62 inches. Determine the balance weights to be placed in the driving wheels at 3 feet radius. All the revolving parts and two-thirds of the reciprocating parts are to be balanced.

7. Take the data of Exercise 5 as applying to an inside cylinder engine with two pairs of wheels coupled, together with the following additional data. Radius of wheel cranks, 11 inches. Distance between planes of motion of coupling-rods, 6 feet 3 inches. Distance between planes of rotation of wheel cranks, 5 feet 2 inches. Weight of each coupling-rod, 270 lbs. Weight of each wheel crank, including the weight of the portion of the crank pin within it, reduced to 11 inches radius, 120 lbs. Weight of each overhanging crank pin, 30 lbs. Determine the balance weights in the driving and trailing wheels, at 2 feet 6 inches radius, to balance all the revolving parts and two-thirds of the reciprocating parts, one-third of the reciprocating masses being balanced in the driving wheels, and one-third in the trailing wheels.

8. Determine, from the following data, the balance weights for an outside cylinder locomotive with two pairs of wheels coupled. Stroke of pistons, 26 inches. Distance between centre lines of cylinders, 75 inches. Distance between planes of motion of coupling-rods, 67 inches. Distance between planes of revolution of balance weights, 60 inches. Distance between planes of revolution of cranks, 60 inches. Weight of reciprocating parts per cylinder, 350 lbs. Weight of one connecting-rod, 225 lbs. One-third of weight of connecting-rod to be considered as reciprocating with cross-head and two-thirds as revolving with crank pin. Weight of one coupling-rod, 230 lbs. Weight of one crank pin within coupling-rod, 15 lbs. Weight of one crank pin within connecting-rod, 12 lbs. Weight of one crank with part of crank pin within it, reduced to 13 inches radius, 90 lbs. Centres of gravity of balance weights at 26 inches radius in driving wheels and 27½ inches in trailing wheels. All the revolving and two-thirds of the reciprocating parts to be balanced, the balance for the reciprocating parts to be in the driving wheels only.

9. Find the difference between the maximum and minimum pressures on the rail of a driving wheel of the engine in Exercise 7 when the speed is 60 miles per hour, the diameter of the wheel being 7 feet.

10. Calculate the difference between the maximum and minimum pressures on the rail of a driving wheel of the engine in Exercise 8 when the speed is 50 miles per hour, the diameter of the wheel being 6 feet 1 inch.

11. The piston of a single cylinder direct-acting engine has a stroke of 2 feet. The weight of the reciprocating parts is 300 lbs., and these parts are to be balanced by two bob-weights driven by cranks of 6 inches radius. The lines of stroke of the bob-weights are 5 feet apart, and the line of stroke of the piston is between the lines of stroke of the bob-weights and 2 feet from one of them. Determine the weights of the bob-weights.

12. A, B, and C are the parallel lines of stroke of the pistons of a three-cylinder engine. B is between A and C, and is 25 inches from A and 30 inches from C. Each piston has a stroke of 24 inches The reciprocating parts in the line A weigh 210 lbs. Find the weights of the reciprocating parts in the lines B and C, and show how the cranks must be placed so that the reciprocating parts of the engine may balance one another.

13. The diagram (Fig. 699) shows the crank shaft of a three-cylinder triple expansion engine for a torpedo boat. The cranks make equal angles with one

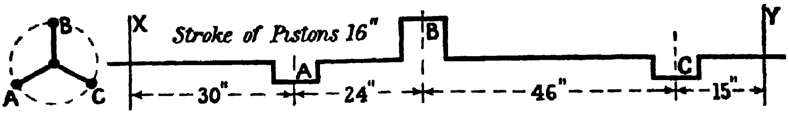


FIG. 699.

another. The reciprocating parts connected to the crank pins A, B, and C weigh 150 lbs., 160 lbs., and 260 lbs. respectively, and they are to be balanced by bob-weights in the planes X and Y. The one bob-weight in the plane X has

a stroke of 4 inches, and the other in plane Y has a stroke of 5 inches. The bob-weights are driven by eccentrics on the crank shaft. Determine the weights of the bob-weights and the angular positions of their eccentrics.

14. A, B, C, and D are the parallel centre lines of the cylinders of a four-cylinder engine, taken in order. The distances between A and B, B and C, and C and D are 5, 7, and 6 feet respectively. The angle between the cranks of the first two cylinders is 150°, and the reciprocating parts connected to them weigh 3500 and 4800 lbs. respectively. Find the weights of the reciprocating parts of the third and fourth cylinders and the angular positions of their cranks in order that the reciprocating parts may balance one another completely. The first crank (A) leads. All the pistons have the same stroke. Assume that the pistons have harmonic motion.

CHAPTER XXVII

HYDROSTATICS

360. Fluids.—Fluids are of two kinds, *Liquids* and *Gases*. A fluid is not capable of resisting change of shape or volume unless it is constrained by the sides of a vessel surrounding it. For instance, if a cylinder (Fig. 700), closed at one end and fitted with a frictionless piston, have the space between the piston and the closed end of the cylinder full of fluid, the piston will support a force P tending to push it further into the cylinder, but if a hole be made in the side of the cylinder (Fig. 701), a force P, *however small*, will be sufficient to push the piston in and cause the escape of the fluid through the hole.

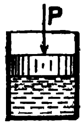


FIG. 700.

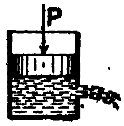


FIG. 701.

If a definite volume of a liquid be placed in a vessel of greater volume the liquid will only occupy a portion of the vessel equal to the original volume of the liquid, but if a quantity of a gas, however small, be introduced into a vessel, however large, it will expand and fill the whole of the vessel.

A liquid when confined, as in Fig. 700, offers a very great resistance to decrease in volume; in fact, for the purposes of the engineer, a liquid can in general be taken as incompressible. Water, for example, loses 0·007 of its original volume for each ton of pressure per square inch applied to it, and the compressibility of mercury is only about one-tenth that of water. A gas, on the other hand, is, within certain wide limits, readily compressible.

A fluid is said to be *more fluid* or more like a *perfect fluid* the less the friction of its particles on one another, or the less the resistance which it offers to a body moving through it. A fluid is said to be more *viscous* the greater the friction of its particles on one another, or the greater the resistance which it offers to a body moving through it. The viscosity of a fluid does not affect its equilibrium, but it does affect its motion.

The branch of mechanics which treats of the equilibrium of fluids and the forces acting on them when at rest is called *hydrostatics*. That part of hydrostatics which deals with gases is called *pneumatics*.

361. Direction of Fluid Pressure on a Surface.—The pressure of a fluid on a surface is perpendicular to that surface at every point (Fig. 702). This is true for viscous as well as for non-viscous fluids *at rest*. A viscous fluid in motion exerts a slight tangential force on a surface over which it is moving, and the pressure of the fluid on the surface will therefore not be perpendicular to the surface in this case.

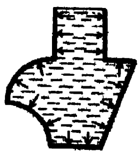


FIG. 702.

362. Transmission of Pressure.—If a fluid at rest have any pressure applied to any part of its surface, that pressure is transmitted equally to all parts of the fluid. For example, if the vessel shown in Fig. 703 be full of water or air, and if it has attached to its sides equal cylinders fitted with frictionless pistons, which are kept at rest by suitable forces, any additional force applied to one of the pistons will require that an equal additional force be applied to each of the other pistons to keep them at rest.

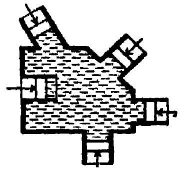


FIG. 703.

It follows from this that if one of the cylinders be enlarged until the piston which fits it has double the area, this piston will require double the force to keep it in equilibrium against the fluid pressure, and generally if a is the area of one piston and q the force pushing it in, and if A is the area of another piston and Q the force pushing it in, then for equilibrium $a/\mathrm{A} = q/\mathrm{Q}$. This is the principle of the hydraulic press (Fig. 704), in which a comparatively small force P acting on a small piston or plunger is able to balance a large force W on a large piston or ram.

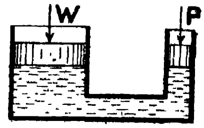


FIG. 704.

363. Pressure at any Point of a Liquid due to its Weight.—Let A (Fig. 705) be a very small horizontal disc of area a immersed in a liquid at a vertical depth h below its free surface. The pressure of the liquid on the top of the disc will not be altered if a cylindrical tube, open at both ends, and having an internal diameter equal to that of the disc A, be placed over it in a vertical position, as shown. This tube above the level of the disc contains a cylindrical column of liquid whose volume is ah and whose weight is ahw, where w is the weight of a unit of volume of the liquid. If the liquid surrounding the tube AB be removed, the pressure on the upper surface of the disc A will not be altered, because the pressure of the surrounding liquid on the tube is horizontal, the sides of the tube being vertical, and therefore self-balancing. The load on the top of the disc is now ahw, and the pressure per unit of area is $ahw/a = hw = p$, which shows that at any point in a liquid the intensity of the pressure due to the weight of the liquid is directly proportional to its depth below the free surface of the liquid.

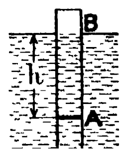


FIG. 705.

If the area a be in square feet, the height h in feet, and w the weight of 1 cubic foot of the liquid, then hw will be the pressure per square foot at the depth h. The depth h is called the *head* of liquid at A, and the head is evidently a measure of the pressure.

364. Total Pressure on a Plane Horizontal Surface Immersed in a Liquid.—From the preceding Article it follows that the total pressure on a plane horizontal surface due to the weight of a liquid in which it is immersed is equal to the weight of a right prism of the liquid, whose base is the given surface, and whose height is the depth of the surface below the free surface of the liquid.

365. Total Pressure on a Plane Inclined Surface Immersed in a Liquid.—Let MN (Fig. 706) be a plane inclined surface immersed in a liquid, and let the surface be divided into a large number of narrow

horizontal strips, of which EF is one. Let y be the depth of the strip EF below the free surface of the liquid, and let a denote the true area of the strip. By Art. 363 the intensity of the pressure at the depth y is wy, and therefore the total pressure on the strip EF is wya, and the total pressure on the whole surface MN will be the sum of all the pressures on the separate strips, and will therefore be equal to Σwya or $w\Sigma ya$. But by a property of the centre of gravity of a surface $\Sigma ya = hA$, where h is the depth of the centre of gravity of the surface below the free surface of the liquid, and A is the true area of the surface. Hence the total pressure on the surface MN is whA. But whA is the weight of a right prism of the liquid, whose base is the given surface, and whose height is the depth of the centre of gravity of the surface below the free surface of the liquid.

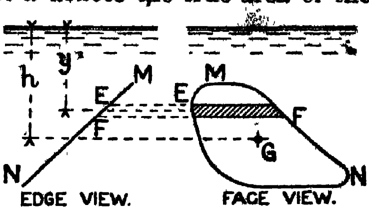


FIG. 706.

366. **Artificial Head.**—In the three preceding Articles the effect of any external pressure on the free surface of the liquid has been neglected. If the free surface of the liquid is exposed to the atmosphere, or to steam, as in a boiler, or if it support a loaded piston in a cylinder, then the pressure on the free surface will be transmitted to the surface immersed in the liquid. Let p_1 be the intensity of the pressure due to the weight of the liquid at a depth h_1 below its free surface. Let p_2 be the intensity of the external pressure on the free surface of the liquid, and let h_2 be the head of liquid, which, by its weight, would cause a pressure of intensity p_2. Also, let p be the intensity of the total pressure at the depth h_1, and lastly, let h be the head of liquid which, by its weight, would cause a pressure of intensity p. Then $p = p_1 + p_2$, and since $h = \frac{p}{w}$, $h_1 = \frac{p_1}{w}$, and $h_2 = \frac{p_2}{w}$, it follows that $h = h_1 + h_2$. The head h_2, which is the head of liquid equivalent to the external pressure, may be called the artificial head.

367. **Resultant of Pressure.**—If a surface be exposed to pressure, either uniform or varying, the single force, acting at a point on the surface, which will produce the same effect on the surface as a whole as the pressure over the surface, is called the resultant of the pressure. The magnitude of this resultant is, for plane surfaces, the same as what has been called the total pressure in preceding Articles.

368. **Centre of Pressure.**—If a surface be exposed to pressure, either uniform or varying, the point on the surface at which the resultant of the pressure acts is called the *centre of pressure*. In what follows, the surfaces exposed to pressure will be assumed to be plane surfaces.

If the pressure is uniform over the surface, the centre of pressure is obviously at the centre of gravity of the surface.

The other important case is where the pressure varies uniformly in one direction, and is uniform in a direction at right angles to this as when an inclined surface is immersed in a liquid.

Let MN (Fig. 707) be a plane surface immersed in a liquid, and let the straight line in which the plane of the surface intersects the free surface of the liquid be taken as the axis about which moments are to be taken. In what follows this axis will be referred to as the axis. Consider a narrow horizontal strip EF which is at a distance x from the axis, and let a be the true area of this strip. The total pressure on the strip EF is $wax \sin \theta$, where w is the weight of a unit of volume of the liquid, and θ is the inclination of the surface MN to the horizontal. The moment of the total pressure on EF about the axis is $wax^2 \sin \theta$, and the sum of all such quantities for the whole area of the given surface is $\Sigma wax^2 \sin \theta$, or $w \sin \theta \Sigma ax^2$, and this must be equal to the moment of the resultant pressure about the axis.

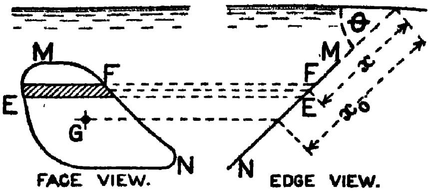


FIG. 707.

The magnitude of the resultant pressure on the whole area is $wAx_0 \sin \theta$, where A is the true area of the given surface, and x_0 the distance of its centre of gravity from the axis. If $\bar{x}$ is the distance of the centre of pressure from the axis, then

$$\bar{x}wAx_0 \sin \theta = w \sin \theta \Sigma ax^2, \text{ therefore } \bar{x} = \frac{\Sigma ax^2}{Ax_0}.$$

But Σax^2 is the moment of inertia of the surface about the axis; calling this I_0, $\bar{x} = \frac{I_0}{Ax_0}$.

If the moment of inertia of the surface about an axis parallel to the above-mentioned axis, and passing through the centre of gravity of the surface, be denoted by I, then (Art. 68) since $I_0 = I + Ax_0^2$,

$$\bar{x} = \frac{I + Ax_0^2}{Ax_0} = \frac{Ak^2 + Ax_0^2}{Ax_0} = \frac{k^2 + x_0^2}{x_0},$$

where k is the radius of gyration of the surface referred to the axis through its centre of gravity.

The foregoing demonstration shows that the position of the centre of pressure is independent of the inclination of the immersed surface if the distances of the various points of the surface from the free surface of the liquid *measured in the plane of the surface* remain unaltered.

The depth of the centre of pressure below the free surface of the liquid is obviously $\bar{x} \sin \theta$.

As to the lateral position of the centre of pressure; if the locus of the middle points of the horizontal lines which can be drawn on the surface is a straight line, this line will obviously contain the centre of pressure, and in practically all cases where the centre of pressure is required in practice, the surface satisfies this condition.

369. Examples of Centre of Pressure.—The following are the cases of most frequent occurrence in practice. In the illustrations, c is the centre of pressure in each case. The distances d, h, and $\bar{x}$ are measured in the plane of the figure or surface.

Fig. 708. A rectangle or parallelogram, with its highest side below

and parallel to the surface of the liquid. c is in the line which bisects the horizontal sides of the rectangle or parallelogram.

$$\bar{x} = \frac{2}{3}\left(\frac{3h^2 + 3hd + d^2}{2h + d}\right). \quad \text{If } h = 0,\ \bar{x} = \tfrac{2}{3}d.$$

Fig. 709. A triangle with its base below and parallel to the surface

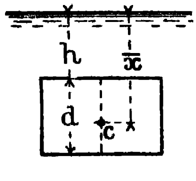


FIG. 708.

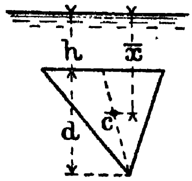


FIG. 709.

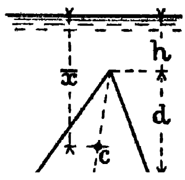


FIG. 710.

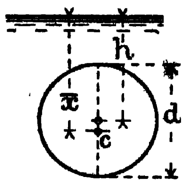


FIG. 711.

of the liquid, the vertex being below the base. c is in the line joining the vertex and the middle point of the base.

$$\bar{x} = \frac{6h^2 + 4hd + d^2}{6h + 2d}. \quad \text{If } h = 0,\ \bar{x} = \tfrac{1}{2}d.$$

Fig. 710. A triangle with its base parallel to the surface of the liquid, the vertex being above the base and below the surface of the liquid. c is in the line joining the vertex with the middle point of the base.

$$\bar{x} = \frac{6h^2 + 8hd + 3d^2}{6h + 4d}. \quad \text{If } h = 0,\ \bar{x} = \tfrac{3}{4}d.$$

Fig. 711. A circle entirely immersed, its centre being at a distance h from the surface of the liquid.

$$\bar{x} = h + \frac{\quad}{16h}. \quad \text{If } h = \frac{d}{2}, \qquad \tfrac{5}{8}d.$$

370. Resultant Pressure on a Body Immersed in a Liquid.—It is evident that the horizontal components of the pressures on the surface of the immersed body balance one another, there being no tendency to move the body horizontally, and the resultant pressure must therefore act vertically.

Suppose the body to be divided into a large number of vertical prisms, of which AB (Fig. 712), having a horizontal sectional area a, is one. Let the upper end of AB be at a depth h_1, and the lower end at a depth h_2 below the free surface of the liquid, and let w be the weight of a unit of volume of the liquid. The downward force exerted by the liquid on the top of AB is wah_1, and the upward force exerted by the liquid on the bottom of AB is wah_2. Hence the resultant upward force on AB is $wa(h_2 - h_1)$, and this is the weight of a volume of liquid equal to the volume of AB. This result will be true for each of the prisms into which the body is divided; hence the resultant pressure of the liquid on the body is an upward force equal to the weight of a volume of the liquid equal to the volume of the body.

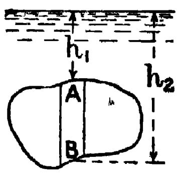


FIG. 712.

Stating the foregoing result in another way: if the body is weighed

while it is immersed in the liquid, its loss of weight is equal to the weight of the liquid which it displaces.

371. Floating Bodies.—A consequence of the result of the preceding Article is that when a body floats in a liquid the weight of the body is equal to the weight of the liquid which it displaces. Another obvious result is that when the floating body is at rest, the straight line which joins the centre of gravity of the body and the centre of gravity of the displaced liquid is vertical. The centre of gravity of the displaced liquid is called the *centre of buoyancy*. The resultant fluid pressure on the body acts vertically upwards in a line through the centre of buoyancy.

Fig. 713 shows a floating body slightly displaced from its position of equilibrium. CG is the line joining the centre of buoyancy and the centre of gravity of the body when the body is in its position of equilibrium. C′ is the new position of the centre of buoyancy. The body is now under the action of two vertical forces, each equal to the weight of the body, one acting downwards through G, and the other upwards through C′. If the vertical line through C′ meets the line CG or that line produced at M, the point M is called the *metacentre* of the floating body. The equilibrium of the floating body is evidently more stable the higher the point M is above G, the centre of gravity of the body, and the equilibrium is unstable when M is below G.

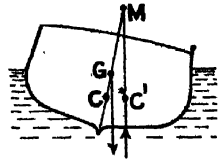


FIG. 713.

372. Weight of Water.—The weight of a cubic foot of water varies with the temperature, as shown in the following table:—

Temp. Cent. . . .	0°	4°	16·67°	40°	60°	80°	100°
„ Fahr. . . .	32°	39·2°	62°	104°	140°	176°	212°
Weight in lbs. of 1 cubic foot	62·34	62·35	62·28	61·87	61·31	60·59	59·76

The above weights are for pure distilled water free from air.

At the temperature 62° Fahr., and the barometer at 30 inches, a cubic foot of distilled water, freed from air, weighs 0·046 lb. more than when nearly saturated with air.

A gallon of water at 62° Fahr. weighs 10 lbs.

Exercises XXVII.

1. Referring to Fig. 704, p 432, if the diameter of the larger piston is 12 inches, and the diameter of the other is $1\frac{1}{2}$ inches, what is the force W when P is 150 lbs.?

2. A vertical tube 3 feet long, and having an internal diameter of 1 inch, is filled with equal volumes of water and mercury. Assuming that the weight of the mercury is 13·56 times the weight of the water, calculate the pressure in lbs. per square inch at the bottom of the tube due to the head of liquid. Also, what is the weight of water in the tube?

3. What pressure, in lbs. per square inch, corresponds to 160 feet head of water, and what head of water, in feet, corresponds to a pressure of 180 lbs. per square inch?

4. Feed water is pumped into a boiler from a tank. Just before starting the feed-pump the levels of the water in the boiler and tank are 38·5 inches and

23·7 inches respectively above the floor level, and when the pump is stopped these levels have changed to 39·4 inches and 17·4 inches respectively above the floor level. A change of 1 inch in the level of the water in the tank corresponds to a change of 19·3 lbs. of water in the contents of the tank. The mean gauge pressure of the steam in the boiler while the pump is at work is 80 lbs. per square inch. Neglecting friction, find the number of ft.-lbs. of work done by the feed-pump.

5. A tank of the form of a right circular cylinder 7 feet in diameter lies with its axis horizontal. Find the total pressure on one end when the tank is full of water.

6. A lock gate is 35 feet wide, and the heights of the water above the bottom of the gate on the two sides are 26 and 13 feet respectively. Find the resultant pressure and the height, measured from the bottom of the gate, at which it acts, the weight of the water per cubic foot being 64 lbs. [Inst.C.E.]

7. The width of a lock is 12 feet, and that of each gate $6\frac{1}{4}$ feet. If the height of the water be 10 feet inside the lock and 4 feet outside, find the resultant fluid pressure on each gate, and also the pressure (assumed to be acting symmetrically) between the two gates. [Inst.C.E.]

8. A box in the form of a cube, of internal dimensions 1 foot, has its base horizontal, and is half-filled with water. One vertical side is kept in its position by four screws only, one at each angular point. Find the tensions in these screws due to the water pressure. [Inst.C.E.]

9. In the vertical side of a tank there is a rectangular opening 2 feet high and 1 foot wide, the shorter sides being horizontal. This opening is covered by a door held by two bolts placed in the middle of the width, one $13\frac{1}{2}$ inches above and the other $13\frac{1}{2}$ inches below the centre of the door. When water stands in the tank at a level of 10 feet above the centre of the door, what are the tensions in the top and bottom bolts? What would be the best positions for the two bolts so that they may be subjected to the same tension, and what would then be the tension in each bolt?

10. A vertical wall, 2 feet thick and 18 feet high, weighing 124 lbs. per cubic foot, supports the pressure of water on one side. How high may the water rise without causing the resultant force on the base of the wall to pass more than 8 inches from the middle of the wall's width? [Inst.C.E.]

11. An opening in a reservoir dam is closed by a plate 3 feet square, which is hinged at the upper horizontal edge; the plate is inclined at an angle of 60° to the horizontal, and its top edge is 12 feet below the surface of the water. If this plate is opened by means of a chain attached to the centre of the lower edge, find the necessary pull in the chain if its line of action makes an angle of 45° with the plate. The weight of the plate is 400 lbs. [U.L.]

12. A dock entrance, whose level floor is 24 feet below the water, has a width of 62 feet at the water level and 50 feet at the floor, the side walls being built with a straight batter. The entrance is closed by a caisson, and on one side of the caisson the floor is dry. Calculate the total horizontal pressure upon the caisson, and the height of its centre of action above the floor. Take weight of water as 64 lbs. per cubic foot. [Inst.C.E.]

13. A vessel of water is weighed on a parcel spring-balance, the reading of which shows that the vessel and water weigh 11 lbs. A 7 lb. iron weight is suspended by a fine wire from the hook of an ordinary spring-balance, and is lowered into the water until it is completely immersed. Under these conditions find (i.) the reading of the spring-balance from which the weight is suspended, (ii.) the reading of the parcel spring-balance on which the vessel stands. Give the reasons for any change in the readings of the balances. (Specific gravity of iron$=7·5$.) [Inst.C.E.]

14. A rectangular wooden barge, without a deck, is 20 feet long, 11 feet wide, and 3 feet deep, outside measurements, and the sides, ends, and bottom have a uniform thickness of 3 inches. Taking the weight of the wood at 50 lbs. per cubic foot, determine the position of the water line when the empty barge floats in water weighing 63 lbs. per cubic foot. What load, in tons, will this barge carry when the water-line is 2 feet from the bottom?

15. If the area of the horizontal section of a ship at the water line is 15,000 square feet, and the sides are vertical where they cut the water, find the extra depth the ship will sink when loaded in fresh water with 750 tons of cargo.

What depth would the ship sink if floating in salt water of specific gravity 1·026 when loading? [Inst.C.E.]

16. A steamer loading 25 tons to the inch in fresh water dock is found after ten days' voyage, burning 52 tons of coal a day, to have risen 25 inches in sea water. Determine the displacement in tons, taking 35 and 36 cubic feet per ton as the specific volumes of salt and fresh water respectively. [U.L.]

17. If a uniform triangular prism floats freely in water with one edge on the surface (Fig. 714), prove that the opposite face must be vertical. [Inst C.E.]

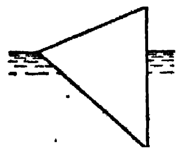


FIG. 714.

18. A cube, edges 5 feet long, floats in water with half its volume immersed, the bottom face being horizontal. The centre of gravity of the cube is 20 inches below its geometrical centre in a vertical line through it. A weight equal to one-fiftieth part of the weight of the cube is placed at the middle point of one of the top edges of the cube. Determine the angle through which the cube will tilt under the additional weight.

CHAPTER XXVIII

GENERAL PRINCIPLES OF HYDRAULICS

373. Energy of Water—Bernoulli's Theorem.—In connection with hydraulics, the total energy in a given quantity of water consists of three parts: (1) The *potential energy*, or the energy due to the height through which it may fall, or the energy due to its position; (2) the *pressure energy*, or the energy due to the pressure which the water exerts on the sides of the containing vessel or pipe; (3) the *kinetic energy*, or the energy due to its motion.

In what follows, the energy of one pound weight of the water will be considered.

(1) The potential energy of 1 lb. of water which is capable of falling through a height of h feet is h foot-pounds.

(2) The pressure energy of 1 lb. of water which exerts a pressure of P lbs. per square foot is P/w, where w is the weight of a cubic foot of the water. For if 1 cubic foot of water be admitted into a cylinder which is fitted with a piston having an area of 1 square foot, then the piston will move through a distance of 1 foot; and if the water exerts all the time a pressure of P lbs. per square foot, the work done by the cubic foot of water will be P foot-pounds. Therefore the work done by 1 lb. of water is P/w foot-pounds. It is important to notice that in proving that the pressure energy of 1 lb. of water is P/w, it is assumed that the full pressure P is kept up during the time that the 1 lb. of water is being used to do the work P/w.

(3) The kinetic energy of 1 lb. of water which is moving with a velocity of v feet per second is $v^2/2g$.

A portion of water may have all three of the above forms of energy, but one, two, or all of them, may be zero. Also, the energy in one form may be so small compared with the energy in another form, that it may be neglected. For example, in the transmission of power by water pressure amounting to, say, 1000 lbs. per square inch, the water will have a velocity seldom exceeding 5 feet per second. Here the pressure energy is $\frac{1000 \times 144}{62 \cdot 3} = 2311$ ft.-lbs., and the kinetic energy when the velocity is 5 feet per second is $\frac{5^2}{2 \times 32 \cdot 2} = 0 \cdot 39$ ft.-lb., a quantity so small compared with 2311 that it may be neglected.

If H is the total energy in one pound weight of liquid, then

$$H = h + \frac{P}{w} + \frac{v^2}{2g}.$$

If all the particles of a 1 lb. mass of water are moving with the same velocity at any instant, and there is no frictional resistance to the motion,

also if the liquid may be considered as incompressible, so that there is no internal work due to change of volume, then, if the mass of liquid considered moves without doing external work or without having external work done upon it, not only is $H = h + \frac{P}{w} + \frac{v^2}{2g}$ at any instant, but H is a constant quantity, although the quantities h, v, and P may vary. This is known as *Bernoulli's theorem.*

374. Flow through a Smooth Pipe of Varying Section.—Fig. 715 shows a pipe of varying cross section conveying water from a tank to a datum level, say the level of the sea. The free surface of the water in the tank is at a height H above datum. Neglecting friction, and assuming that the motion of the water is steady, the total energy of a 1 lb. mass of the water will be the same in every position. The table below shows the amounts of the various forms of energy in a 1 lb. mass of the water in four different positions. Where P_1 and P_2 are the pressures of the water at A_1 and A_2 respectively, w is the

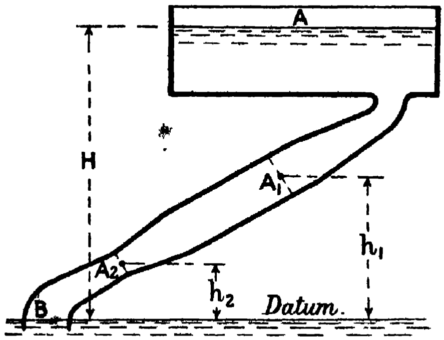


FIG. 715.

Position.	Potential Energy	Pressure Energy.	Kinetic Energy.	Total Energy.
A	H	nil	nil	H
A_1	h_1	$\frac{P_1}{w}$	$\frac{v_1^2}{2g}$	$H = h_1 + \frac{P_1}{w} + \frac{v_1^2}{2g}$
A_2	h_2	$\frac{P_2}{w}$	$\frac{v_2^2}{2g}$	$H = h_2 + \frac{P_2}{w} + \frac{v_2^2}{2g}$
B	nil	nil	$\frac{v^2}{2g}$	$H = \frac{v^2}{2g}$

specific weight of the water, v_1, v_2, and v are the velocities of the water at A_1, A_2, and B respectively.

If a_1, a_2, and a are the areas of the cross sections of the pipe at A_1, A_2, and B respectively, then, if the pipe is running full, the quantity of water passing through each cross section in a given time must be the same, hence $a_1v_1 = a_2v_2 = av$.

375. Venturi Water Meter.—Bernoulli's theorem has an important and interesting application in the Venturi water meter, by means of which the rate of flow of water through a water main may be determined without interposing any moving part in the flowing water. Figs. 716–719 * show the main parts of a Venturi meter. There are two conical pipes AB and CD (Figs. 716 and 717), whose smaller ends are connected by a short pipe BC, forming the *throat* of the meter. This combination is introduced so as to form a part of the water main, the

* Figs. 716–719 have been prepared from working drawings kindly supplied by Mr. George Kent, High Holborn, London.

delivery of which is to be measured. The axis of the water main in the neighbourhood of the meter is horizontal. The water enters the meter at A, and leaves at D. A hollow belt is cast round the pipe AB at EF, and the interior of this belt communicates with the interior of the pipe by four small holes, the positions of which are shown in the cross section, Fig. 718. These holes are bushed with vulcanite to prevent incrustation. A copper tube leads from the annular space at EF to the top of a vertical cast-iron cylinder containing mercury. The throat is lined with a gun-

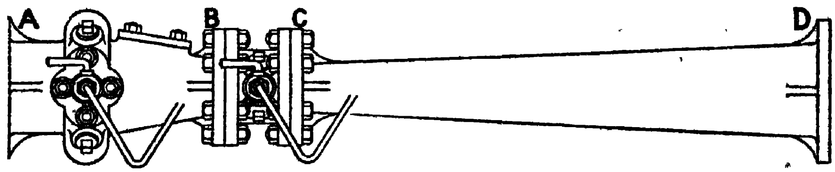


FIG. 716.

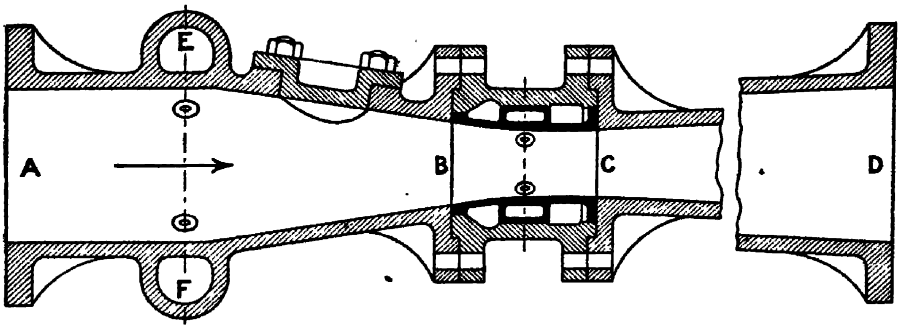


FIG. 717.

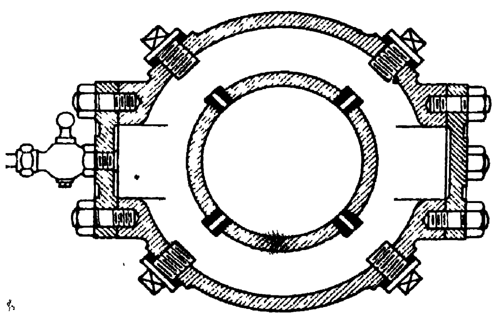


FIG. 718.

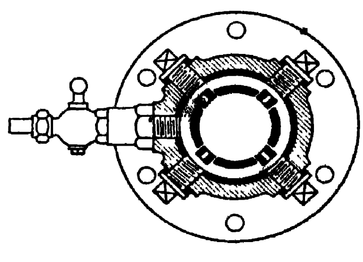


FIG. 719.

metal casting, having an annular space round its centre which communicates with the interior of the throat by four small holes arranged as at EF, and shown in Fig. 719, which is a cross section at the throat. A second copper tube leads from the annular space round the throat to the top of a second cast-iron cylinder containing mercury. The two vertical cylinders containing mercury communicate with one another at their bottom ends, so as to form the equivalent of a U tube mercury gauge,

the levels of the mercury being indicated by cast-iron floats with vertical rods attached to them.

It may be left as an exercise to the student to show that the quantity of water flowing through the main in a given time is equal to $c\sqrt{k}$, where c is a constant for a given meter, and k is the head of mercury equivalent to the difference between the pressures in the main and in the throat of the meter.

The diameter of the throat of the meter is frequently one-third of the diameter of the main.

In order that Bernoulli's theorem may apply without sensible error, it is necessary that the interior of the Venturi tube lying between the annular pressure chambers should be as smooth as possible. In practice, the error in the Venturi meter does not exceed 2 per cent.

It is usual to fit a recording apparatus to the Venturi meter, consisting of a clock-driven drum, upon which a diagram is traced by a pen actuated by one of the cast-iron floats mentioned above. The abscissæ of this diagram represent time, and the ordinates rate of flow, and the area of the diagram between any two ordinates represents the quantity of water delivered in the time represented by the distance between these ordinates. A mechanical integrator operated by a clock and the second float is generally added; this shows on a dial the total quantity of water delivered.*

Venturi meters are suitable for mains of almost any diameter, and have been made for mains as large as 10 feet in diameter. They are, however, not suitable when the velocity of the water is very small.

376. Radiating Current.—Fig. 720 shows two horizontal co-axial discs whose distance apart is a. At the centre of the lower disc there is an opening into a pipe, from which water flows into the space between the discs. Consider the flow across a section of the water between the discs made by the surface of a cylinder of radius r whose axis coincides with the axis of the discs. Let the velocity of the water across this section be v, and let Q be the volume passing per unit of time, then $Q = 2\pi rav$, and $rv = Q/2\pi a$. But for all values of r the quantity Q is constant, therefore $rv =$ a constant, and if r and v be plotted in a plane containing the axis of the discs, the resulting curve is a rectangular hyperbola.

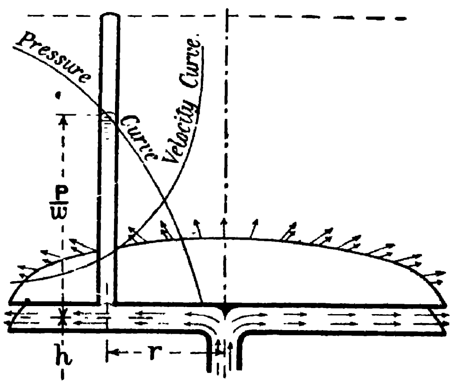


Fig. 720.

Let P be the pressure of the water at radius r, as shown by a pressure gauge, then the pressure head is P/w, and the kinetic energy per unit of weight is $v^2/2g$. Let h be the height of

* For an illustrated description of the recording apparatus of a Venturi meter, see *Engineering*, Feb. 22, 1907.

the horizontal stream between the discs above datum, then by Bernoulli's theorem

$$h+\frac{P}{w}+\frac{v^2}{2g}=H=\text{a constant.}\quad \text{Hence } H-h-\frac{P}{w}=\frac{v^2}{2g}=\frac{Q^2}{8g\pi^2a^2r^2}, \text{ and}$$

$$r^2\left(H-h-\frac{P}{w}\right)=\frac{Q^2}{8g\pi^2a^2}=\text{a constant.}$$

If r and $\frac{P}{w}$ be plotted in a plane containing the axis of the discs, a curve known as *Barlow's curve* is obtained.

The foregoing discussion will obviously also apply to the case where the current is reversed, flowing inwards instead of outwards.

377. Vortices.—A mass of rotating fluid is called a *vortex.* When the motion is produced by the action of forces of weight and fluid pressure only, the vortex is called a *free vortex.* When the law of motion in a vortex is different from that of a free vortex, it is called a *forced vortex.* The simplest form of forced vortex is that in which all the particles have the same angular velocity; this form of forced vortex is considered in Art. 380, under the heading of "whirling liquids."

378. Free Circular Vortex.—If instead of having simple radial motion the water between the discs in Fig. 720 moves in circular currents, and at the same time moves slowly in a radial direction from one circular current to another, assuming freely the velocities proper to the currents which it enters, a *free circular vortex* is produced.

Consider a portion ABCD (Fig. 721) of a ring of the water in a free circular vortex. Let r be the internal radius of this ring, and dr its radial thickness. Let the length and depth of ABCD be such that the area of the vertical face AB is unity. The faces AD and BC are radial, and since the thickness dr of the ring is very small, the area of the face CD may also be taken as unity. Hence the volume of ABCD is dr, and its weight wdr. Let P and $P+dP$ be the fluid pressures, and v and $v+dv$ be the velocities at the inner and outer faces of the ring respectively.

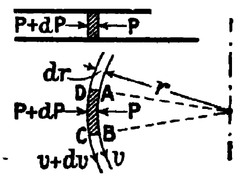


FIG. 721.

The centrifugal force of ABCD is $\frac{wv^2dr}{gr}$, and this must be balanced by the difference between the fluid pressures on the outside and inside, namely, dP. Hence, $dP=\frac{wv^2dr}{gr}$. Again, by Bernoulli's theorem,

$$h+\frac{P+dP}{w}+\frac{(v+dv)^2}{2g}=h+\frac{P}{w}+\frac{v^2}{2g}, \text{ which gives the result } \frac{dP}{w}+\frac{vdv}{g}=0.$$

Substituting for dP its value $\frac{wv^2dr}{gr}$, the result $vdr+rdv=0$ is obtained. Hence $vr=$ constant, and v varies inversely as r as in the radiating current. It follows that the law of variation of pressure will also be the same as in the radiating current.

379. Free Spiral Vortex.—By superposing on the fluid particles the motions of a radiating current and of a free circular vortex, a *free spiral*

vortex is produced. Let A (Fig. 722) represent a fluid particle in a free circular vortex whose axis is O. Let AC at right angles to OA represent the velocity of A in the free circular vortex. Again, let A represent a fluid particle in a current radiating from O. Let AB on OA produced represent the velocity of A in the radiating current. If the two motions be combined, A will have, when in the position considered, a velocity represented by AD, a diagonal of the rectangle BC. Since

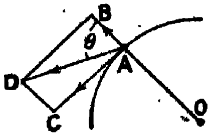


FIG. 722.

$$AC \cdot AO = \text{a constant, and } AB \cdot AO = \text{a constant,}$$

it follows that $AC/AB = \text{a constant} = \tan\theta$. Hence the path of the fluid particle will at every instant make the constant angle θ with the radius drawn from the particle to the axis, and this is a property of the logarithmic or equiangular spiral.

380. Whirling Liquids.—Let a cylinder of radius r (Fig. 723), containing a liquid, revolve about its axis YY_1, which is vertical, with an angular velocity ω. Let P be a point on the free surface of the liquid, and let w be the weight of a very small portion of the liquid at P. Consider the forces acting on the liquid at P in a vertical plane containing P and the axis YY_1. There is the weight w acting vertically downwards, the centrifugal force Q acting horizontally, and the fluid pressure p, which must be perpendicular to the free surface of the liquid, and which must balance the resultant R of Q and w. Let the horizontal through P meet the axis at N, and let the line of action of R meet the axis at G.

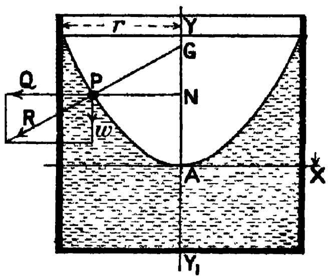


FIG. 723.

$$Q = \frac{w}{g}\omega^2 PN, \text{ and } \frac{GN}{PN} = \frac{w}{Q} = \frac{wg}{w\omega^2 PN} = \frac{g}{\omega^2 PN}. \quad \therefore GN = \frac{g}{\omega^2}.$$

Hence GN, the sub-normal of the free surface at P, is a constant, and therefore the free surface is a paraboloid, and the section of the free surface by a plane containing the axis YY_1 is a parabola.

The equation to the parabola, taking the axes as AX and AY, where A is the vertex, is (Art. 11, p. 10) $x^2 = 4ay$, where a is the focal distance of the vertex, $PN = x$, and $AN = y$. Now in a parabola the sub-normal is constant and equal to the semi-latus rectum. But the semi-latus rectum is the value of x in the equation $x^2 = 4ay$ when $y = a$, therefore $GN = 2a$, but $GN = \frac{g}{w^2}$, therefore $2a = \frac{g}{\omega^2}$, and the equation to the parabola is

$$x^2 = \frac{2g}{\omega^2}y, \text{ or } y = \frac{\omega^2}{2g}x^2.$$

If h is the height of the cup, then $h = \frac{\omega^2}{2g}r^2$.

The volume of a paraboloid is half the volume of the circumscribing cylinder, hence the volume of liquid in the cylinder above the level AX is $\frac{\pi r^2 h}{2}$. If CD (Fig. 724) is the level of the liquid when at rest, then

$\pi r^2 k = \frac{\pi r^2 h}{2}$, and therefore $k = \frac{1}{2}h = \frac{\omega^2 r^2}{4g}$, which shows that the depth of A, the bottom of the cup below the original level of the liquid, is proportional to the square of the angular velocity.

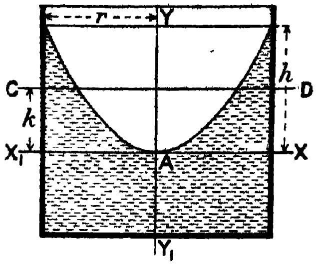


FIG. 724.

When the top of the cup reaches the top of the cylinder, as shown by the dotted parabola in Fig. 725, $h_1 = 2k_1$, where k_1 is the depth of the original level of the liquid below the top of the cylinder.

Suppose now that the top of the cylinder is closed, and that the angular velocity is still further increased. The cup will still be a paraboloid. Let the total depth of the cup be now y_1, and its greatest radius x_1; then, since the volume of the cup of height h_1 must be the same as that of the cup of height y_1, $h_1 r^2 = y_1 x_1^2$, or $x_1^2 = \frac{h_1}{y_1} r^2$, but $y_1 = \frac{\omega_1^2}{2g} x_1^2$, therefore $y_1^2 = \frac{\omega_1^2 h_1 r^2}{2g}$, and $y_1 = \omega_1 r \sqrt{\frac{h_1}{2g}}$, where ω_1 is the angular velocity. This shows that after the cup touches the top of the cylinder its total depth is directly proportional to the angular velocity.

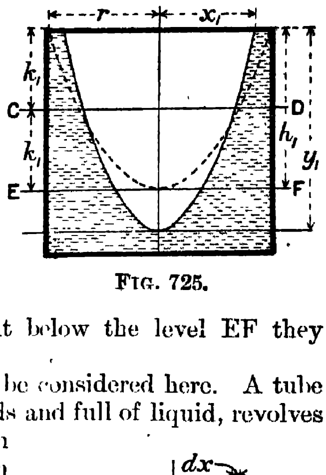


FIG. 725.

The preceding investigation gives the theory of a well-known instrument for indicating the speed of revolution of a shaft at any instant. The cylinder containing the liquid is made of glass, and the spindle upon which it is mounted is geared to the shaft whose speed is required. A graduated scale placed beside the cylinder shows the speed at a glance by indicating the position of the vertex of the paraboloid. From the level CD to the level EF (Fig. 725) the graduations of the scale are unequal, but below the level EF they are equal.

Another problem on whirling liquids may be considered here. A tube AB (Fig. 726), of length $2r$, closed at both ends and full of liquid, revolves with its axis in a horizontal plane about an axis bisecting the axis of the tube with an angular velocity ω. It is required to find the pressure exerted by the liquid on the ends of the tube. Let a be the area of the cross section of the tube, and w the weight of a unit of volume of the liquid. Consider a small portion of the liquid between two planes perpendicular to the axis of a tube and at a distance apart equal to dx, and let the distance of this portion of liquid from the axis of revolution be x.

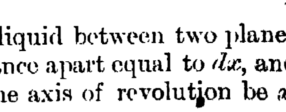


FIG. 726.

The centrifugal force exerted by this position of liquid is $\frac{wadx\omega^2 x}{g}$, and

the total force transmitted to each end of the tube will be

$$\int_0^r \frac{wadx\omega^2 x}{g} = \frac{wa\omega^2}{g}\int_0^r xdx = \frac{wa\omega^2 r^2}{2g},$$

and the intensity of the pressure will be $\frac{w\omega^2 r^2}{2g}$.

381. Torricelli's Theorem.—Fig. 727 shows a tank containing liquid with jets issuing at a depth h below the free surface of the liquid and at a height h_1 above a datum line AB.

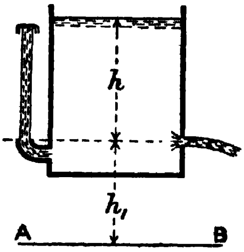


FIG. 727.

Let P_0 be the pressure of the atmosphere, and v the velocity of the liquid as it leaves the orifice and issues into the atmosphere. At the free surface of the liquid in the tank the potential energy is $h+h_1$, the pressure energy is P_0/w, and the kinetic energy is zero, per pound of liquid. The pressure of the liquid in the jet is P_0, and its pressure energy is therefore P_0/w, the potential energy is h_1, and the kinetic energy is $v^2/2g$, per pound. If the loss of energy due to friction be neglected, then

$$h+h_1+\frac{P_0}{w}+0=h_1+\frac{P_0}{w}+\frac{v^2}{2g}, \text{ and therefore } h=\frac{v^2}{2g}.$$

That is, the velocity of the issuing liquid is that which a body would acquire in falling freely from rest under the action of gravity through a height equal to the depth of the orifice below the free surface of the liquid. If the jet be directed vertically upwards, as shown to the left in Fig. 727, the liquid will rise to nearly the level of the free surface of the liquid in the tank. It will not quite reach the level of the free surface of the liquid, on account of the air resistance and the friction of the liquid on the sides of the orifice or nozzle.

If the jet enters into a second tank (Fig. 728) in which the liquid stands at a height h_2 above the jet, then the pressure of the liquid in the jet is P_0+wh_2, and if v is its velocity, the total energy of the jet per pound of liquid is $h_1+\frac{P_0+wh_2}{w}+\frac{v^2}{2g}$, and as before, neglecting friction,

$$h+h_1+\frac{P_0}{w}+0=h_1+\frac{P_0+wh_2}{w}+\frac{v^2}{2g},$$

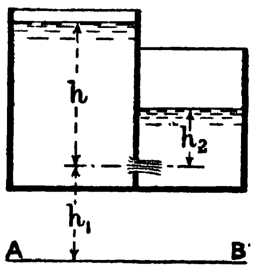


FIG. 728.

and therefore $h-h_2=\frac{v^2}{2g}$, which shows that the velocity is that due to a head equal to the difference of level of the liquid in the two tanks, and is independent of the depth of the jet below the free surface of the liquid in the second tank.

In like manner it follows that if the jet enters a vessel in which there is a partial vacuum, such as a steam-engine condenser, the head due to the pressure in this vessel will be negative and equal to, say, $-h_3$, then $h+h_3=\frac{v^2}{2g}$.

382. Influence of Velocity of Approach.—In the preceding Article the total energy per pound of the water at the free surface in the tank in Fig. 727 was assumed to be $h+h_1+\frac{P_0}{w}$, the water being assumed to be at rest. But since the water is leaving at the orifice, the water in the tank above the orifice must have a downward velocity, called the *velocity of approach*. Let a denote the area of the cross section of the jet, and A the area of the free surface of the water in the tank, then if v is the velocity of the jet, and V the downward velocity at the free surface in the tank, the energy per pound at the free surface is $h+h_1+\frac{P_0}{w}+\frac{V^2}{2g}$, and the energy of the jet per pound is $h_1+\frac{P_0}{w}+\frac{v^2}{2g}$; also $V=\frac{av}{A}$, hence $h_1+\frac{P_0}{w}+\frac{v^2}{2g}=h+h_1+\frac{P_0}{w}+\frac{a^2v^2}{2gA^2}$, from which $\frac{v^2}{2g}=\frac{h}{1-\left(\frac{a}{A}\right)^2}$.

Generally a/A is so small that a^2/A^2 may be neglected.

383. Flow through Sharp-edged Orifices. — When water issues through a sharp-edged orifice (Fig. 729) in the side or bottom of a tank it is found that the jet contracts to a minimum section, called the *contracted section* or *vena contracta*, which is a little distance in front of the orifice. This contraction of the jet is due to the fact that the water particles in approaching the orifice are not moving in parallel lines. For a circular orifice the distance of the contracted section in front of the sharp edge of the orifice is about half the diameter of the orifice. The ratio of the area of the contracted section of the jet to the area of the orifice is called the *coefficient of contraction*. If A is the area of the orifice, a the area of the contracted section, and k the coefficient of contraction, then $a=kA$. The value of k for sharp-edged orifices may be taken as 0·63, but it varies slightly with the shape of the orifice and the head of water. The value of k in different cases may be determined from direct measurements of the jet.

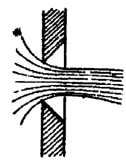


FIG. 729.

On account of friction the velocity of the water at the contracted section of the jet is less than $\sqrt{2gh}$, given by Torricelli's theorem (Art. 381), and the ratio of the actual velocity to the theoretical velocity is called the *coefficient of velocity*. If v is the actual velocity, and c the coefficient of velocity, then $v=c\sqrt{2gh}$. An average value of c is about 0·97. The value of c may be found from observations on the path of the jet (Fig. 730). If the face of the orifice is vertical, then in t seconds a particle of water will travel a horizontal distance $x=vt$ from the contracted section, and it will in the same time fall a distance $y=\frac{1}{2}gt^2$. Hence $\frac{x^2}{y}=\frac{2v^2}{g}=4c^2h$, and $c=\frac{x}{2\sqrt{hy}}$. It will be seen from the equation $x^2=4c^2hy$ that the path of the jet is a parabola whose

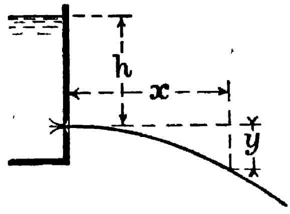


FIG. 730.

vertex is at the contracted section, and whose directrix is horizonta at a distance c^2h above the vertex.

If Q is the actual volume of water flowing through the orifi second, then $Q = av = kAc\sqrt{2gh} = CA\sqrt{2gh}$, where C (which is eq ck) is called the *coefficient of discharge*, and is the ratio of the actu charge to the theoretical discharge. By the theoretical disch meant the discharge neglecting friction and the contraction of th

The coefficients k, c, and C are called the hydraulic coeffici an orifice. The coefficient C is the one which is of most impor practice, and it is the one which is most easily determined by measurements. Taking $k = 0{\cdot}63$, and $c = 0{\cdot}97$, then $C = 0{\cdot}63 \times 0{\cdot}97 = 0{\cdot}61$, which agrees with the mean value of C, as determined directly from numerous experiments with sharp-edged orifices.

384. Miner's Inch.—In selling water in mining districts the water is frequently measured by delivering it through rectangular orifices under a small but constant head. The *miner's inch* is the quantity of water delivered per minute through an orifice 1 inch square, in a vertica plate, under a head which varies in different localities from 6 to inches, measured to the centre of the orifice. With a head of $6\frac{1}{4}$ inch measured to the centre of the orifice, the miner's inch is equivalent about $1\frac{1}{2}$ cubic feet of water per minute.

385. Entire and Partial Suppression of Contraction of Je Fig. 731 shows the form of the jet issuing through a sharp-edged orifi in a plate, the thickness of which is such that its outside face is in the plane of the smallest section of the jet. The space shown in black is the empty space between the surface of the orifice and that of the jet. It is obvious that if the space shown in black be filled up, or if the orifice be shaped to the natural form of the jet within the plate, the smallest diameter of the jet will then be the same as the smallest diameter of the orifice, and the coefficient of contraction will become unity.

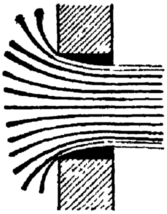


FIG. 731.

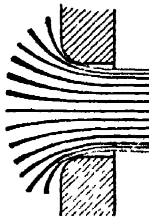


FIG. 732.

Rounding the inside edge of the orifice to a greater or less extent, as shown in Fig. 732, will evidently have the effect of diminishing the contraction of the jet, and therefore of increasing the coefficient of contraction.

An orifice which is to be used for the measurement of the water delivered by it should be sharp-edged and of the form shown in Fig. 729, because the coefficient of contraction for an orifice with a rounded edge is uncertain, varying with the amount of rounding.

386. Loss of Energy or Head.—When water is discharged through an orifice under a head h, it has been shown that the actual velocity at the *vena contracta* is equal to $c\sqrt{2gh}$, where c is the coefficient of velocity. The energy in 1 lb. of water at the *vena contracta* is therefore equal to c^2h. If the water lost no energy in reaching and passing through the orifice, its energy at the *vena contracta* would be h. The *loss of energy* per lb. of water is therefore $h - c^2h = h(1 - c^2)$. This is also the expression for the *loss of head*, that is to say, the head which would produce the actual

…ocity $c\sqrt{2gh}$, if there were no friction, is less than the actual head …he amount $h(1-c^2)$. The ratio $\frac{h(1-c^2)}{c^2h}=\frac{1-c^2}{c^2}$ is called the …cient *of resistance* for the orifice.

…87. Drowned or Submerged Orifices.—It was shown in Art. 381 … 728) that when the water stands above the orifice on both sides, …ective head to be used in calculating the velocity through the … neglecting friction, is the difference between the levels on the two … This is also the head to be taken in calculating the discharge …gh a drowned orifice when friction and the contraction of the jet are considered, but it has been found by experiment that in the case of a drowned sharp-edged orifice the coefficient of discharge is slightly less (about 2 per cent.) than when the discharge is directly into the atmosphere.

388. Time of Flow through an Orifice for a given Change of Water Level in a Vessel.—Let a be the area of the orifice, and k its coefficient of discharge.

The simplest case (Fig. 733) is where the level changes from AB to …D, the water flowing in under a constant head h. Let V denote the …lume of water ABCD. The discharge through the orifice per second is … $\sqrt{2gh}$, and therefore the time in seconds to discharge the quantity V is … $a\sqrt{2gh}$.

A common case is that in which the level is to be raised from AB to CD (Fig. 734), the water passing through an orifice at O, the level EF

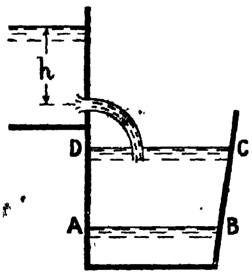


FIG. 733.

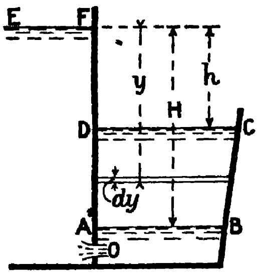


FIG. 734.

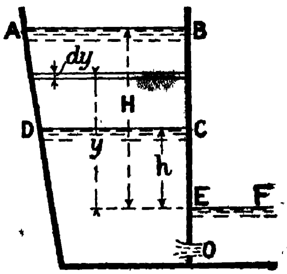


FIG. 735.

of the water on the *inflow* side being at a constant height above O; or the level is to be lowered from AB to CD (Fig. 735), the water passing through an orifice at O, the level EF of the water on the *outflow* side being at a constant height above O. When the free surface of the water in the vessel is at a distance y from the level EF, let its area be Y. In the time dt let y change to $y-dy$, then

$$ka\sqrt{2gy}\cdot dt=\mathrm{Y}dy. \quad \text{Hence } t=\int_h^{\mathrm{H}}\frac{\mathrm{Y}dy}{ka\sqrt{2gy}},$$

where t is the time required to change the level from AB to CD. In practical cases the vessel ABCD is generally either a vertical cylinder or a vertical prism, and Y is a constant $=\mathrm{A}$, then $t=\frac{2\mathrm{A}}{ka\sqrt{2g}}(\sqrt{\mathrm{H}}-\sqrt{h})$.

An important application of the foregoing formula is to the filling or emptying of a canal lock (Fig. 736). The upper and lower reaches of the canal may be assumed to have constant levels during the operation of filling or emptying the lock. When the gates are shut, communication between the lock and the upper or lower reaches of the canal is either through sluices in the gates themselves, or through culverts in the walls of the lock.

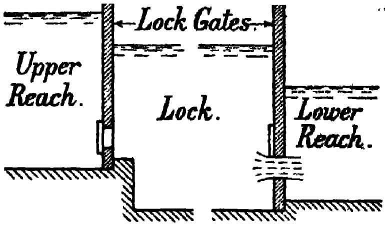


FIG. --

* In the foregoing cases the head on one side of the orifice has been assumed to be constant, but when one vessel of limited capacity discharges into another, the level in the second rises as the level in the first falls. Assuming that the vessels (1) and (2), Fig. 737, have vertical sides, let A and B denote the areas of the free surfaces of the water in (1) and (2) respectively, and let the level in (1) fall from CD to EF by discharging into (2) through the orifice O below the level KL of the water in (2).

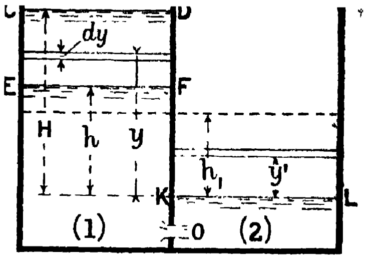


FIG. 737.

When the free surface of the water in (1) is at the height y above KL, the level in (2) will have risen through the height y' such that $A(H-y)=By'$. In the time dt let y change to $y-dy$, then

$$ka\sqrt{2g(y-y')}\cdot dt = A\,dy, \text{ but } y-y' = \frac{(A+B)y-AH}{B},$$

therefore

$$dt = \frac{A\sqrt{B}\cdot dy}{ka\sqrt{2g}\sqrt{(A+B)y-AH}} = \frac{A\sqrt{B}}{ka\sqrt{2g}}\left\{(A+B)y-AH\right\}^{-\frac{1}{2}}dy.$$

Hence
$$t = \frac{A\sqrt{B}}{ka\sqrt{2g}}\int_h^H\left\{(A+B)y-AH\right\}^{-\frac{1}{2}}dy$$

$$= \frac{2A\sqrt{B}}{(A+B)ka\sqrt{2g}}\left\{\sqrt{BH}-\sqrt{(A+B)h-AH}\right\}.$$

When the level has become the same in both vessels, h will become $h_1 = \dfrac{AH}{A+B}$, and then $t = \dfrac{2AB\sqrt{H}}{(A+B)ka\sqrt{2g}}$.

*389. **Large Rectangular Orifices.**—When the orifice in the side of a vessel is small compared with the head of water over it, the head may be assumed as the same for all parts of the orifice; but when the orifice is so large that this assumption involves serious error, the formula for the discharge must be determined by taking into account the variation of head.

Large orifices are generally rectangular. Consider a rectangular

orifice (Fig. 738) of breadth b, the upper and lower edges being at depths h_1 and h_2 below XX, the free surface of the water. Consider a narrow horizontal strip of the orifice of depth dy at a depth y below XX. Neglecting contraction and all losses, the discharge through this strip is $dQ = bdy\sqrt{2gy}$, and the theoretical discharge through the whole orifice is

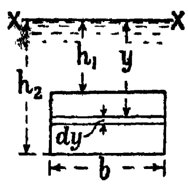


FIG. 738.

$$Q = \int_{h_1}^{h_2} bdy\sqrt{2gy} = b\sqrt{2g}\int_{h_1}^{h_2} y^{\frac{1}{2}}dy = \tfrac{2}{3}b\sqrt{2g}\left(h_2^{\frac{3}{2}} - h_1^{\frac{3}{2}}\right).$$

If it may be assumed that the coefficient of discharge k is the same for all values of b, h_1 and h_2, then the actual discharge is

$$\tfrac{2}{3}kb\sqrt{2g}\left(h_2^{\frac{3}{2}} - h_1^{\frac{3}{2}}\right).$$

Experiments on the flow through large vertical rectangular orifices however show that k depends on the proportions of the orifice, and also on the head of water over it. An approximate average value of k is 0·62.

390. Rectangular Notches or Weirs.—If the head h_1 over the rectangular orifice of the preceding Article is zero, the orifice becomes a *rectangular notch* or *weir* (Fig. 739); and if the formula of the preceding Article still applied, the discharge would be given by the expression $\frac{2}{3}kb\sqrt{2g}\cdot h^{\frac{3}{2}}$, where h takes the place of h_2. This expression may be used in determining the discharge through a rectangular notch if the value of the coefficient k is known with sufficient certainty for the particular notch to which it is applied.

When the vertical edges of the notch project into the stream, as in Fig. 739, the notch or weir is said to have *end contractions*. In Fig.

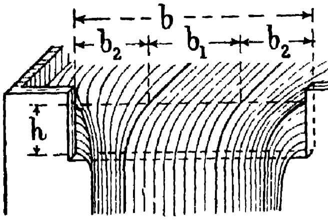


FIG. 739.

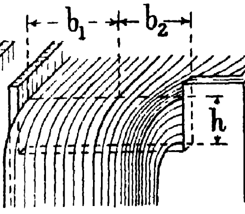


FIG. 740.

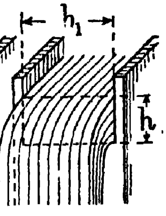


FIG. 741.

739 the weir has two end contractions. In Fig. 740 there is only one end contraction, and in Fig. 741 there are no end contractions.

In a weir with no end contractions, called a *suppressed weir*, the width of the outflowing stream is uniform, and the discharge is directly proportional to the width, and is given by an expression of the form $k_1b_1h^{\frac{3}{2}}$. To prevent any lateral spreading of the stream as it flows over a suppressed weir, the sides should be prolonged, as shown in Fig. 742.

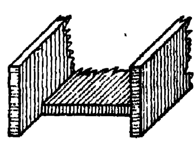


FIG. 742.

The effect of an end contraction is to reduce the effective width of the stream through the weir, but the influence of the end contraction only extends over a limited

width b_2 of the weir, and the discharge over this portion is given by an expression of the form $k_2b_2h^{\frac{3}{2}}$. The width b_2 will depend on the height h, and may be written $b_2 = mh$. Hence for a weir with n end contractions, where n is equal to 2, 1, or 0,

$$Q = (k_1b_1 + nk_2b_2)h^{\frac{3}{2}} = (k_1b - nk_1mh + nk_2mh)h^{\frac{3}{2}}$$
$$= \{k_1b - n(k_1 - k_2)mh\}h^{\frac{3}{2}} = \alpha(b - n\beta h)h^{\frac{3}{2}},$$

where α and β are constants to be determined by experiment.

The above formula $Q = \alpha(b - n\beta h)h^{\frac{3}{2}}$ is known as *Francis's formula*, and although it was first derived empirically from experiments, it will be seen from the foregoing that it has a rational basis. This formula is sometimes called the *Lowell formula*, from the fact that the experiments upon which it was founded were conducted at Lowell, in Massachusetts. The experiments of Francis were made on weirs from 4 to 10 feet long, with heads varying from 0·6 to 1·6 feet, and the mean values of the constants α and β were found to be 3·33 and 0·1 respectively. Francis's formula may therefore be written $Q = 3{\cdot}33(b - 0{\cdot}1nh)h^{\frac{3}{2}}$.

The head h in the discharge formulæ for weirs given in this Article is usually taken as the head measured from the crest of the weir to the still water level just above the weir, as shown in Fig. 743, and not as the depth over the crest. Generally the upper surface of the water drops and curves slightly before reaching the weir. In the experiments of Francis, the head was measured from the crest to the level of the water 6 feet above the weir.

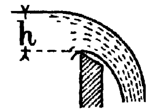


FIG. 743.

The effect of velocity of approach is considered in the next Article.

391. Velocity of Approach in a Stream.—When the water in the stream has velocity before it reaches the weir, this velocity is equivalent to an additional head at the weir. In order that the water in a stream may flow, its upper surface must slope downwards in the direction of motion, and the effective head at any point, neglecting friction, must be measured to the level of still water up-stream. Fig. 744 shows a longitudinal section of the stream in the neighbourhood of the weir, and the horizontal line XX is the still water level up-stream. The height h_0 is the additional head due to the velocity of approach.

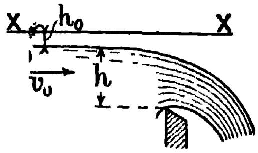


FIG. 744.

Reasoning as in the three preceding Articles, it follows that the ordinary formula $Q = \frac{2}{3}kb\sqrt{2g}\cdot h^{\frac{3}{2}}$ velocity of approach being neglected becomes $Q = \frac{2}{3}kb\sqrt{2g}\{(h + h_0)^{\frac{3}{2}} - h_0^{\frac{3}{2}}\}$ when velocity of approach is considered.

Also Francis's formula becomes $Q = 3{\cdot}33(b - 0{\cdot}1nh)\{(h + h_0)^{\frac{3}{2}} - h_0^{\frac{3}{2}}\}$ when velocity of approach is considered.

Since h_0 is generally small compared with $h + h_0$, the term $h_0^{\frac{3}{2}}$ is often neglected, and the ordinary formula then becomes $Q = \frac{2}{3}kb\sqrt{2g}(h + h_0)^{\frac{3}{2}}$, and Francis's formula becomes $Q = 3{\cdot}3(b - 0{\cdot}1nh)(h + h_0)^{\frac{3}{2}}$.

When it is not convenient to measure h_0 directly, the velocity of approach v_0 may be computed approximately as follows. Let A denote

the area of the cross section of the stream above the weir. Calculate Q', the discharge over the weir, neglecting velocity of approach, then $v_0 = Q'/A$ approximately, and $h_0 = \dfrac{v_0^2}{2g}$.

392. Triangular Notches.—A triangular or V notch has one great advantage over the rectangular notch. In the former the linear dimensions are in a fixed ratio to one another, whatever be the depth of water in the notch, and it follows that the cross sections of the issuing streams will be similar, and the coefficient of contraction therefore constant.

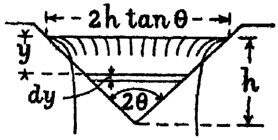


FIG. 745.

Let the edges of the notch (Fig. 745) have equal inclinations to the vertical, and let the angle between them be 2θ. Neglecting in the first instance the contraction of the jet and the effect of friction, consider a strip of the notch at a depth y and of width dy. The length of this strip is $2(h-y)\tan\theta$, its area is $2(h-y)\tan\theta \cdot dy$, and the velocity of the water through it is $\sqrt{2gy}$. Hence

$$dQ = 2(h-y)\tan\theta \cdot dy\sqrt{2gy} = 2\tan\theta\sqrt{2g}(hy^{\frac{1}{2}} - y^{\frac{3}{2}})dy,$$

and

$$Q = 2\tan\theta\sqrt{2g}\int_0^h (hy^{\frac{1}{2}} - y^{\frac{3}{2}})dy = 2\tan\theta\sqrt{2g}(\tfrac{2}{3}h^{\frac{5}{2}} - \tfrac{2}{5}h^{\frac{5}{2}})$$
$$= \tfrac{8}{15}\tan\theta\sqrt{2g}\cdot h^{\frac{5}{2}}.$$

If k is the coefficient of discharge, then the actual discharge is $Q = \frac{8}{15}k\tan\theta\sqrt{2g}\cdot h^{\frac{5}{2}}$.

The late Professor James Thomson found k to be 0·617; taking this value of k, and making $2\theta = 90°$, which is the usual angle of the notch, $Q = 2{\cdot}64h^{\frac{5}{2}} = 2{\cdot}64h^2\sqrt{h}$.

393. Partially Submerged Orifices.—When a rectangular orifice is partially submerged, as shown in Fig. 746, the orifice may be considered as made up of two parts, the upper of depth $h_2 - h_1 - h'$, and breadth b discharging into the atmosphere under a head varying from $h_2 - h'$ to h_1, and the lower of depth h' and breadth b fully submerged and discharging under a constant effective head $h_2 - h'$. Let Q_1 and Q_2 denote the discharges from these upper and lower parts respectively, then, by Articles 389 and 383,

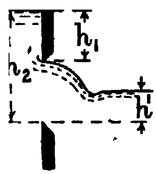


FIG. 746.

$$Q_1 = \tfrac{2}{3}kb\sqrt{2g}\{(h_2 - h')^{\frac{3}{2}} - h_1^{\frac{3}{2}}\}, \text{ and } Q_2 = kbh'\sqrt{2g(h_2 - h')}.$$

The total discharge is $Q_1 + Q_2$.

394. Drowned Weirs.—A weir is said to be *drowned* or *submerged* when the tail water level is above the crest of the weir, as shown in Fig. 747. The formulæ of the preceding Article may be applied to a drowned weir by putting $h_1 = 0$ and changing h_2 to h, then the discharge over the weir is

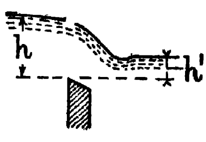


FIG. 747.

$$Q = Q_1 + Q_2 = \tfrac{2}{3}kb\sqrt{2g}(h-h')^{\frac{3}{2}} + kbh'\sqrt{2g(h-h')}$$
$$= \tfrac{2}{3}kb\sqrt{2g(h-h')}(h + \tfrac{1}{2}h').$$

Exercises XXVIIIa.

1. A pipe whose axis is horizontal is full of water in motion. At a section A the velocity of the water is 300 feet per minute, and the pressure is 20 lbs. per square inch. The pipe tapers gradually from 6 inches diameter at A to 4 inches diameter at B. Assuming that there is no loss of energy between A and B, determine the pressure of the water at B. What must be the diameter of the pipe at B if the pressure there is reduced to 4 lbs. per square inch?

2. A horizontal tube is tapered slowly from a diameter of 15 inches to a diameter of 6 inches. Neglecting friction, calculate the difference in the pressures in lbs. per square inch at the two sections when the discharge is 60,000 gallons per hour. [Inst.C.E.]

3. The diameter of a pipe gradually changes from 8 inches at a point A, 40 feet above datum, to 5 inches at a point B, 20 feet above datum. The pressure at A is 30 lbs. per square inch, and the pipe delivers water at the rate of 5 cubic feet per second. Neglecting friction, find the pressure at B.

4. A conical pipe varying in diameter from 4 feet 6 inches at the large end to 2 feet at the small end forms part of a horizontal water main. The pressure head at the large end is found to be 100 feet, and at the small end 96·5 feet. Find the discharge through the pipe. [Inst.C.E.]

5. A Venturi water meter is 15 inches diameter in the main and 6 inches diameter in the throat. The difference between the pressures of the water in the main and in the throat is 9·2 inches of mercury. Find the discharge in gallons per minute. (Specific gravity of mercury, 13·56.)

6. In a particular Venturi water meter the diameter of the main is 3 feet, and the diameter of the throat 1 foot. Q is the number of gallons of water delivered per minute, and h is the effective head, in inches of mercury, in the gauge showing the difference between the pressures in the main and in the throat. Taking the specific gravity of mercury as 13·56, find the numerical value of the constant c in the formula $Q = c\sqrt{h}$ for this meter.

7. Define and describe "forced" and "free" vortices. A glass tube 2 inches diameter, open at the top, containing a liquid, rotates about its axis, which is vertical, at 700 revolutions per minute. What is the depression of the lowest point of the surface below the surface of the liquid when at rest? [U.L.]

8. A glass tube, internal diameter 2 inches, and length 12 inches, has its axis vertical; it is closed at both ends, and contains a liquid which fills three-fourths of the volume of the tube. The tube is made to revolve about its axis. Find the speed of the tube in revolutions per minute (1) when the top of the cup formed by the liquid is at the top of the tube, (2) when the bottom of the cup is at the bottom of the tube. Construct the speed scale, the gradations to show speed increments of 10 revolutions per minute.

9. A glass tube 3 feet long, of uniform cross section, bent into the form of three sides of a square (Fig. 748), and half filled with water, rotates uniformly about the axis of one of the parallel limbs, which is vertical. Find the angular velocity if the other vertical limb is half full of water.

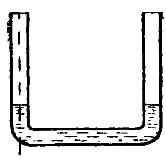


FIG. 748.

10. Neglecting the effect of friction, with what velocity will water flow through an orifice in the shell of a steam boiler at a point 30 inches below the water level when the steam pressure gauge indicates 40 lbs. per square inch?

11. Water under a pressure of 7 lbs. per square inch is fed into a tank containing water to a depth of 15 feet through an orifice in the bottom of the tank. Neglecting friction, find the velocity of flow through the orifice.

12. A jet of water under a head of 3 feet enters a condenser where the absolute pressure is 5 lbs. per square inch. If the pressure of the atmosphere is 14·7 lbs. per square inch, find the velocity of the jet, neglecting friction.

13. A vertical pipe of 3 inches bore contains water which runs out through an orifice at the bottom of the pipe. The diameter of the issuing jet is $\frac{1}{2}$ inch. Neglecting friction, determine the velocity of the jet, in feet per second, when the head of water in the pipe is 10 feet, (i.) neglecting the velocity of approach, (ii.) taking the velocity of approach into account. Construct a curve showing the relation between the velocity of the jet and the head of water over it,

neglecting the velocity of approach, for values of the head between 10 feet and 1 foot.

14. The following results were obtained during an experiment to determine the quantity of water which would be discharged through a small circular orifice in the side of a tank. The diameter of the orifice, which had sharp edges, was 1 inch:—

Number of experiment		1	2	3	4
Duration of experiment	minutes	15	15	15	15
Actual discharge	lbs.	576	660	733	827
Head of water above centre of orifice	inches	1·5	2·0	2·5	3·27
Number of experiment		5	6	7	8
Duration of experiment	minutes	15	15	10	10
Actual discharge	lbs.	915	1011	737	788
Head of water above centre of orifice	inches	4·01	5·0	6·0	7·0

Plot on squared paper a curve to show the relation between the discharge in lbs. per minute and the head of water above the centre of the orifice. From your curve determine the discharge in gallons per hour when the head of water was $5\frac{1}{2}$ inches.

Plot also on squared paper a curve to show the relation between the discharge in lbs. per minute and the square root of the head of water above the centre of the orifice. From your curve, what would you consider the relation to be between the quantity of flow and head?

Determine for each of the experiments in the above table the "coefficient of discharge" for this orifice, and plot a curve to show the relation between "coefficient of discharge" and head of water. [B.E.]

15. Water flows through a sharp-edged circular orifice 0·3 inch in diameter in the side of a tank. The head of water above the centre of the orifice is 4 feet. The jet passes through a ring whose internal diameter is slightly larger than that of the jet, and the centre of this ring is found to be 48 inches horizontally and 13·1 inches vertically from the centre of the *vena contracta*. In 5 minutes the weight of water discharged is 90·2 lbs. Calculate the coefficients of discharge, velocity, and contraction for this orifice

16. If the miner's inch is defined as the flow through an orifice 1 inch square, in a vertical plane, under the head of $6\frac{1}{4}$ inches measured to the centre of the orifice, and if the flow is found to be $1\frac{1}{2}$ cubic feet per minute, what is the coefficient of discharge?

17. A tank 10 feet square and 10 feet deep has a circular orifice 4 inches diameter in the bottom, which may be regarded as a thin plate. Water is admitted to the tank until it is full, and is then shut off. In how many seconds will the tank be empty? [Inst.C.E.]

18. A rectangular chamber 120 feet square contains 15 feet depth of water, which is allowed to flow out through a vertical rectangular orifice 2 feet by 1 foot, the top of which is level with the floor of the reservoir and tail-water. Calculate the time it will take to empty. [Inst.C.E.]

19. Two chambers with vertical sides, each 50 square feet in area, are connected by means of a rectangular sluice, 3 feet by 2 feet, near the bottom. One chamber contains water to a depth of 25 feet, and the other a depth of 10 feet. If the sluice is opened, find how long it will be before the water is at the same level in the two chambers. [Inst.C.E.]

20. Find the answer to the preceding exercise when the chamber in which the depth of the water is 10 feet has an area of 80 square feet instead of 50 square feet, the other particulars being the same.

21. A hemispherical cistern is 20 feet in diameter, and it is full of water. How many minutes will it take to lower the depth of the water 5 feet, if the water escapes through a 3-inch diameter sharp-edged hole in the bottom of the cistern? The coefficient of discharge for the hole is 0·60. [U.L.]

22. A cylindrical tank 2 feet in diameter and 6 feet high is full of water. On opening an orifice 1 inch in diameter in the bottom of the tank it is found

that the water level is lowered 4 feet in 4 minutes. What is the coefficient of discharge?

23. A weir 20 feet long has a head of 15 inches above the crest. Taking the coefficient $k=0{\cdot}6$, calculate the discharge in cubic feet per second.

24. Assuming that the weir of the preceding exercise has two end contractions, calculate the discharge, in cubic feet per second, by Francis's formula.

25. A rectangular weir with one end contraction is required to discharge 500,000 gallons of water per hour with a still-water head of 10 inches. Determine the necessary length of the weir. Use Francis's formula.

26. Find the quantity of water which will flow through a notch 9 feet long, the head of water over the sill being 10 inches, and the area of the approach channel being 30 square feet. [Inst.C.E.]

27. Water flows from a pond over a weir 10 feet long, to a depth of 10 inches; it then flows along a level rectangular channel 8 feet broad, and over a second weir the width of the channel, its crest being 1 foot above the bottom. Find the depth of the water over the 8-foot weir. [Inst.C.E.]

28. What are the advantages and disadvantages attending the use of the **V**-gauge notch, and for what purposes is it specially suitable? The still-water surface level is at a height of 15·5 inches above the bottom of a right-angled **V**-gauge notch. Calculate the discharge in cubic feet per second, taking 0·6 as the coefficient of discharge.

29. A measuring weir is constructed with a 90° angular notch, the edges being bevelled to 45° on the outside to a nearly sharp edge. Give the formula you think best for the discharge over such a weir, and apply it to calculate the discharge in gallons per minute when the water depth above the apex of the angular notch is 9·36 inches, and the water level 5 feet back from the weir is found to be 0·93 inch above that of the weir. [Inst.C.E.]

30. A triangular notch, having an angle of 90 degrees, is used to measure the flow of a stream. Readings at intervals of 1 hour are taken, as shown in the table.

Reading	1	2	3	4	5
Head, in inches	4	5	6	7	6

Draw a curve showing the rate of discharge at any time, and show how you would determine the discharge between the time of the first and last readings. [Inst.C.E.]

31. Water flows over a rectangular notch 3 feet wide to a depth of 6 inches, and afterwards passes through a triangular right-angled notch. Find the depth of water through this notch. The coefficients of discharge for the notches are to be taken as 0·62 and 0·59 respectively.

32. A weir at the edge of a pond is 6 feet wide. The crest of the weir is 6 inches below the water level of the pond, and 3 inches below the tail-water level. Compute the discharge over this weir, in gallons per hour, if the coefficient of discharge $k=0{\cdot}58$.

395. Loss of Energy or Head due to Sudden Enlargement of Pipe.—Let a straight horizontal pipe (Fig. 749) suddenly enlarge in cross section from an area a_1 at A′B′ to an area a. In passing from the smaller to the larger part of the pipe the stream lines will be disturbed, and eddies will be formed as shown, but at some distance forward in the enlarged part of the pipe the motion will again become steady and the stream lines parallel. Where the eddies form there is a churning of the water, and a consequent loss of energy.

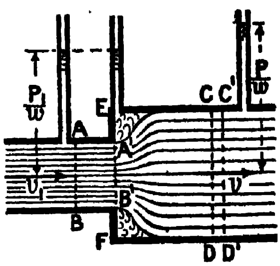


FIG. 749.

Consider a portion of water between the sections AB and CD, the motion being steady at these sections. Let this mass of water move forward to the position

A′B′D′C′ in t seconds. Let the pressure and velocity at AB be P_1 and v_1, and let the pressure and velocity at CD be P and v. Experiment has shown that in the enlarged part of the pipe where it joins the smaller part the pressure is the same as in the smaller part, namely, P_1.

The forces urging the mass of water ABDC forward are, P_1a_1 at AB, and $P_1(a-a_1)$ on the annular surface between A′B′ and EF. The force retarding the forward motion is Pa at CD. Hence the resultant force on ABDC in the direction of motion is $P_1a_1+P_1(a-a_1)-Pa=(P_1-P)a$. The impulse of this force is $(P_1-P)at$, which must be equal to the change in the momentum of the mass of water ABDC in the time t seconds. But since there is no change in the momentum of the mass of water between the sections A′B′ and CD, the change in the momentum of ABDC must be equal to the difference between the momenta of the masses AA′B′B and CC′D′D, that is, the change in the momentum of a mass equal to $v_1a_1t=vat$. Hence

$$(P_1-P)at=\frac{wv_1a_1t}{g}(v-v_1),\text{ therefore }\frac{P_1-P}{w}=\frac{v}{g}(v-v_1),$$

from which it follows that $$\frac{P_1}{w}+\frac{v_1^2}{2g}=\frac{P}{w}+\frac{v^2}{2g}+\frac{(v_1-v)^2}{2g}.$$

But if there had been no loss of energy in passing from the smaller to the larger part of the pipe, Bernoulli's theorem shows that $\frac{P_1}{w}+\frac{v_1^2}{2g}$ would have been equal to $\frac{P}{w}+\frac{v^2}{2g}$, therefore the loss of energy due to the sudden enlargement of the pipe is $\frac{(v_1-v)^2}{2g}$ per lb. of water passing.

In the foregoing discussion no account has been taken of the effect of friction, but in the short length of pipe considered the effect of friction would be very small.

396. Loss of Energy or Head due to Sudden Contraction of Pipe.—In passing from the larger to the smaller part of the pipe (Fig. 750) the stream follows the contour of the larger part almost right up to the smaller part, and then contracts to a cross section of area a_1 at a section AB within the smaller part, a_1 being less than a, the area of the section of the smaller part of the pipe. The only loss up to AB is due to friction, and may here be neglected. After passing AB the stream expands and fills the remainder of the pipe, as shown. In passing from AB to CD there is a loss due to the sudden enlargement, as in the case considered in the preceding Article. Using the formula of the preceding Article, the loss of energy or head between AB and CD is $\frac{(v_1-v)^2}{2g}=\frac{v^2}{2g}\left(\frac{a}{a_1}-1\right)^2=m\frac{v^2}{2g}$. If k is the coefficient of contraction at AB, then $\frac{a}{a_1}=\frac{1}{k}$. The coefficient k varies with the ratio of a to a_0, where a_0 is the area of the section of the larger part of the pipe. If a_0 is very large

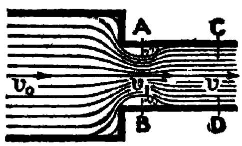


FIG. 750.

Loss = ·45 $\frac{v^2}{2g}$

compared with a, the value of m is about 0·45, which makes $k = 0·6$. If $a_0 = 10a$, the value of m is about 0·36, which makes $k = 0·625$.

397. Loss of Energy or Head due to Obstructions in Pipes.—An obstruction in a pipe, in the form of a diaphragm having a central hole, as shown in Fig. 751, is a case of sudden contraction in a pipe, and the

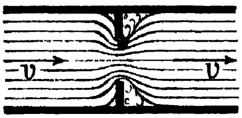


FIG. 751.

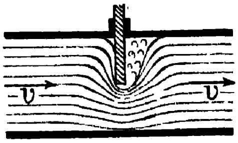


FIG. 752.

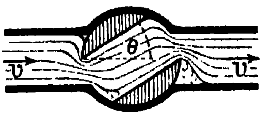


FIG. 753.

expression for the loss of energy or head obtained in the preceding Article, namely, $\frac{v^2}{2g}\left(\frac{a}{a_1} - 1\right)^2 = m\frac{v^2}{2g}$, may be used, where a is the area of the section of the pipe, v the velocity of the water through it, and a_1 the area of the section of the contracted stream as it passes through the hole in the diaphragm. If a_2 is the area of the hole in the diaphragm, the ratio $a_1/a_2 = k$ will depend on the ratio of a_2 to a. Values of m corresponding to various values of a_2/a are given in the following table on the authority of Weisbach :—

a_2/a	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9
m	226	47·8	30·8	7·80	3·75	1·80	0·80	0·29	0·06

The above values of m are also approximately true for a sluice partly open (Fig. 752).

For a cock in a cylindrical pipe (Fig. 753), the loss of energy or head is also given by the expression $m\frac{v^2}{2g}$. Weisbach gives the following values of a_2/a and m for various values of θ, a being the area of the pipe, and a_2 the effective area through the cock when turned through the angle θ :—

θ	5°	10°	20°	30°	40°	45°	50°	60°	65°
a_2/a	0·93	0·85	0·69	0·54	0·39	0·32	0·25	0·14	0·09
m	0·05	0·29	1·56	5·47	17·3	31·2	52·6	206	486

398. Flow through a Cylindrical Mouthpiece.—A short pipe or mouthpiece AF (Fig. 754), having a length of from two to three times its diameter, projects from the side of a tank, as shown. The water on entering the pipe converges, as in sharp-edged orifices, to a jet of sectional area a_1 at AB within the mouthpiece, and then expands until it has a sectional area a equal to that of the mouthpiece.

Let P and v be the pressure and velocity of the jet at EF, and let P_1

and v_1 be the pressure and velocity of the jet at AB. P, the pressure of the water at EF, will be the same as that of the atmosphere.

The coefficient of contraction at AB is

$$\frac{a_1}{a} = k, \text{ and } v_1 = \frac{av}{a_1} = \frac{v}{k}.$$

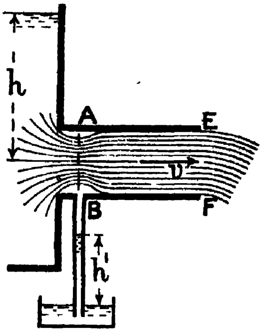


FIG. 754.

Between AB and EF there is a loss of energy or head (Art. 395) equal to

$$\frac{(v_1 - v)^2}{2g} = \frac{v^2}{2g}\left(\frac{1}{k} - 1\right)^2.$$

If c is the coefficient of velocity at AB, then $v_1 = c\sqrt{2gh}$, and the energy at AB is

$$\frac{v_1^2}{2g} = \frac{c^2 2gh}{2g} = c^2 h.$$

The energy at EF is $\frac{v^2}{2g}$. Hence $c^2 h = \frac{v^2}{2g} + \frac{v^2}{2g}\left(\frac{1}{k} - 1\right)^2$. From which it follows that

$$v = \frac{c\sqrt{2gh}}{\sqrt{\left\{1 + \left(\frac{1}{k} - 1\right)^2\right\}}} = C\sqrt{2gh}, \text{ where } C = \frac{c}{\sqrt{\left\{1 + \left(\frac{1}{k} - 1\right)^2\right\}}},$$

is the coefficient of velocity and also the coefficient of discharge at EF, since the jet fills the pipe at EF.

Taking $c = 0{\cdot}97$, and $k = 0{\cdot}63$ for a sharp-edged orifice,

$$C = \frac{0{\cdot}97}{\sqrt{\left\{1 + \left(\frac{1}{0{\cdot}63} - 1\right)^2\right\}}} = 0{\cdot}836.$$

Experiments with mouthpieces having lengths from two to three times the diameter gave $C = 0{\cdot}82$. It should be noted that in the foregoing investigation the effect of friction in the pipe has been neglected, but the pipe being short, this effect will be small.

By Bernoulli's theorem $\frac{P_1}{w} + \frac{v_1^2}{2g} = \frac{P}{w} + \frac{v^2}{2g} + \frac{v^2}{2g}\left(\frac{1}{k} - 1\right)^2$. Hence $\frac{P - P_1}{w} = \frac{v^2}{g}\left(\frac{1}{k} - 1\right)$. Inserting for v its value $C\sqrt{2gh}$, $\frac{P - P_1}{w} = 2C^2 h\left(\frac{1}{k} - 1\right)$.

If a vertical tube be inserted into the mouthpiece at B, and its lower end be placed in a vessel open to the atmosphere and containing water, water will rise in this tube to a height h', given by the equation

$$h' = \frac{P - P_1}{w} = 2C^2 h\left(\frac{1}{k} - 1\right) = \frac{2c^2 h\left(\frac{1}{k} - 1\right)}{1 + \left(\frac{1}{k} - 1\right)^2}.$$

Taking $c = 0{\cdot}97$, and $k = 0{\cdot}63$ as before, the above reduces to $h' = 0{\cdot}82h$. By experiment h' is about $\frac{3}{4}h$, which corresponds to $C = 0{\cdot}82$, and $k = 0{\cdot}64$.

Taking the height of the water barometer at 34 feet, then, when $h' = 34$ feet, there will be a perfect vacuum round the jet at AB, and for this condition $h = \frac{4 \times 34}{3} = 45\frac{1}{3}$ feet. For a greater value of h than this the jet will break up, and the mouthpiece will not discharge full bore.

399. Borda's Mouthpiece.—The reason for the contraction of a jet issuing from an orifice being that the water entering the orifice flows towards it in various directions inclined to the axis of the orifice, it is obvious that the greater the angle between the extreme stream lines, the greater the contraction of the jet. In the case of a simple orifice in a flat plate the angle between the extreme stream lines is 180°. Evidently the maximum contraction will occur when the angle between the extreme stream lines is 360°, which is the case in *Borda's mouthpiece.* This mouthpiece consists of a thin tube projecting into a tank, as shown in Fig. 755. The jet contracts within the mouthpiece to a diameter LN. Let A be the area of the section of the mouthpiece, and a the area of the contracted section of the jet. Let XX, the free surface of the water in the tank, be at a height h above the axis of the mouthpiece. The entrance to the mouthpiece being removed from the walls of the tank, it may be assumed that the motion of the water does not affect the pressure on the walls, which will therefore follow the hydrostatic law, and, excepting the portion EF of the wall exactly opposite to the mouthpiece, the horizontal pressures on the walls will balance one another. The resultant pressure on EF is whA, and this will also be the resultant horizontal force on the water entering the mouthpiece.

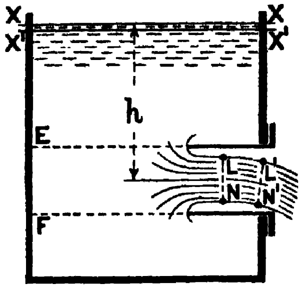


FIG. 755.

Consider the mass of water between XX and LN. Let this mass move into the position X'X'L'N' in t seconds, then since the momentum of the mass X'X'LN does not alter, the change in the momentum of the mass considered is the difference in the momenta of the masses XXX'X' and LNN'L'. But the momentum of XXX'X' is entirely vertical, therefore the change in momentum in a horizontal direction is equal to the momentum of LNN'L', and this is due to the action of the force whA.

The mass of LNN'L' is $\frac{avtw}{g}$, and its momentum is $\frac{av^2tw}{g}$, where v is the velocity of the water in the contracted jet. The impulse of the force whA is whAt. Hence equating impulse to change of momentum $wh\text{A}t = \frac{av^2tw}{g}$, therefore $\frac{a}{\text{A}} = \frac{gh}{v^2}$. But $v^2 = 2gh$ very nearly, therefore $\frac{a}{\text{A}} = \frac{1}{2}$, or the coefficient of contraction for Borda's mouthpiece is $\frac{1}{2}$. Various authorities have obtained values of a/A by direct experiment varying from 0·515 to 0·555, which confirms the foregoing theory.

400. Fluid Friction.—Fluid friction is the resistance experienced when a body moves through a fluid, or when a fluid moves over the

surface of a body. The following laws of fluid friction have been established on the results of numerous experiments by Froude * and others. (1) The frictional resistance is independent of the fluid pressure. (2) The frictional resistance depends on the amount of surface of contact between the fluid and the body, and in general it may be taken as proportional to the area of contact surface. (3) The frictional resistance is proportional to the nth power of the relative velocity of the fluid and body, where n is equal to 1 for very small velocities, but for velocities which occur in practical hydraulics n varies from about 1·7 to about 2·2, and has an average value of 2. (4) At very small velocities the frictional resistance is independent of the nature of the surface of the body, but at ordinary velocities the frictional resistance increases very rapidly with the roughness of the surface of the body. (5) The frictional resistance is proportional to the density of the fluid.

401. Wetted Perimeter—Hydraulic Mean Depth.—That part of the boundary of the cross section of a channel or pipe which is in contact with the water in it is called the *wetted perimeter*, and the area of the cross section of the stream divided by the wetted perimeter is called the *hydraulic mean depth*, or the *hydraulic mean radius*, or the *hydraulic radius*. In this work the hydraulic mean depth, or hydraulic mean radius, will be denoted by m. For example, in a channel of rectangular section (Fig. 756), having a breadth b and depth of water d, $m = \frac{bd}{b+2d}$.

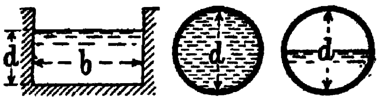


FIG. 756. FIG. 757. FIG. 758.

In a circular pipe (Fig. 757) of diameter d, running full, $m = \frac{\pi}{4}d^2/\pi d = \frac{d}{4}$.

In the same pipe, running half full (Fig. 758), $m = \frac{\pi}{8}d^2\Big/\frac{\pi}{2}d = \frac{d}{4}$, as for the full pipe.

Some writers restrict the term hydraulic mean depth to channels, and apply the term hydraulic mean radius to circular pipes.

402. Usual Velocities of Water in Pipes.—The usual velocity in water mains is less than 5 feet per second. Unwin gives the formula $v = 1{\cdot}45d + 2$ as the expression of a fair rough rule for the velocity of water in pipes used in town's supply, where v is the velocity of the water in feet per second, and d is the diameter of the pipe in feet. A velocity of 10 feet per second is fairly common in the pipes of centrifugal pumps. Velocities greater than 15 feet per second are very unusual in pipes. High velocities involve great loss of energy in friction when the pipes are long, and since the loss of energy per lb. of water delivered is greater the smaller the pipe, the velocity should be lower the smaller the pipe.

403. Critical Velocity of Water in Pipes.—Professor Osborne Reynolds made most interesting experiments on the flow of water in pipes with apparatus roughly shown in Fig. 759. AB is a tank 6 feet long, 18 inches deep, and 18 inches wide, containing water. CD is a glass tube provided with a trumpet-shaped mouthpiece at C, and pro-

* For the results of Froude's experiments see *British Association Reports*, 1875.

jecting horizontally into the tank from an iron pipe EF at one end. Water flows from the tank through the glass tube and thence through the iron pipe. The iron pipe descends vertically to about 7 feet below the tank, and at its lower end it is provided with a cock, by means of which the rate of flow through the glass tube CD may be regulated.

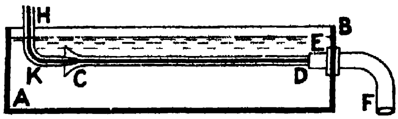


FIG. 759.

A glass tube HK communicates with a reservoir containing deeply-coloured water and terminates at its lower end in a pipette, the axis of which coincides with the axis of the tube CD. A jet of coloured water may thus be sent into the glass tube CD to flow with the water going through that tube.

At velocities below a certain velocity, called the *critical velocity*, the jet of coloured water travels in a straight unbroken stream along the axis of CD, but when the critical velocity is exceeded the coloured stream breaks up within CD, and when photographed with an electric spark it is seen that the coloured water is whirling and eddying, showing that the motion of the water within the tube is no longer steady and in parallel stream lines, but sinuous or turbulent.

In the experiments described above, the water is still before entering the experimental tube. Professor Osborne Reynolds experimented with other apparatus, in which he caused turbulent water to flow through a long smooth pipe, and he found that below a certain critical velocity the turbulent motion became non-sinuous, but this critical velocity was much lower than the critical velocity first referred to. The first critical velocity is called the *higher critical velocity*, and the second is called the *lower critical velocity*. For example, in a smooth pipe 1 inch in diameter, with the water at 0° C., the higher critical velocity is about 3 feet per second, while the lower critical velocity is only about $\frac{1}{2}$ foot per second.

The critical velocities vary inversely as the diameter of the pipe, and they are lowered by raising the temperature of the water.

For further particulars of Osborne Reynolds' researches see the *Transactions of the Royal Society*, 1884, or Dunkerley's *Hydraulics*, vol. i. chap. vii.

404. Loss of Energy or Head due to Friction in a Pipe.—At velocities below the critical velocity, the motion being non-sinuous, the experiments of Osborne Reynolds showed that the loss of energy was directly proportional to the velocity, directly proportional to the length of the pipe, and inversely proportional to the square of the diameter of the pipe, or $h' \propto \frac{vl}{d^2}$, where h' is the loss of head, v the velocity of the water, l the length, and d the diameter of the pipe. But when the critical velocity was exceeded, the motion being then sinuous or turbulent, the loss of energy was proportional to the 1·72 power, or nearly as the square, of the velocity, directly proportional to the length, and inversely proportional to the diameter of the pipe, or $h' \propto \frac{v^{1 \cdot 72} l}{d}$.

In practical cases the velocity is greater than the critical velocity, and the pipes in use have varying and uncertain degrees of roughness, so that

the amount of turbulence in the water is a varying and uncertain quantity. The consequence is that there is no exact theory of the loss of energy in practical cases, and the formulæ in use are therefore to a large extent empirical.

Experiment has shown that in practical cases the loss of energy is approximately proportional to the square of the velocity of the water and to the amount of the wetted surface, and inversely proportional to the area of the cross section of the stream. The wetted surface is sl, where s is the wetted perimeter, hence the loss of energy is approximately proportional to $\frac{sl}{A}v^2$, where A is the area of the cross section of the stream, and $\frac{s}{A}=\frac{1}{m}$ is the reciprocal of the hydraulic mean radius.

The head or energy due to the velocity v is $\frac{v^2}{2g}$, and the loss of head may be written $h'=f\cdot\frac{l}{m}\cdot\frac{v^2}{2g}$, where f is a coefficient to be determined by experiment, and is called the coefficient of friction for the pipe. This coefficient f is not simply a coefficient of friction between the water and the surface of the pipe, but includes a coefficient of resistance due to eddying motions in the water itself.

For a circular pipe running full $m=d/4$, hence $h'=f\cdot\frac{4l}{d}\cdot\frac{v^2}{2g}$.

If A and B (Fig. 760) are two sections of a pipe at a distance l apart, the heights of A and B above datum being h_1 and h_2 respectively, and the pressures P_1 and P_2. If the pipe be of uniform section, then the velocity v will be the same throughout. The total energy or head of the water at A is $\frac{P_1}{w}+\frac{v^2}{2g}+h_1$, and the total energy or head at B is $\frac{P_2}{w}+\frac{v^2}{2g}+h_2$. The loss of energy or head between A and B is

$$h'=f\cdot\frac{l}{m}\cdot\frac{v^2}{2g}.\quad \text{Hence } \frac{P_1}{w}+\frac{v^2}{2g}+h_1=\frac{P_2}{w}+\frac{v^2}{2g}+h_2+h',$$

and $h'=\frac{P_1-P_2}{w}+(h_1-h_2)$, which suggests the experimental method of finding h'. Knowing h', if v is determined, f can then be found for the particular pipe experimented with.

405. Darcy's Formula.—Darcy found from numerous and careful experiments on the flow of water in pipes up to 20 inches in diameter that the coefficient f varied with the velocity of the water, and also with the diameter of the pipe. Since the variation in the velocity in ordinary cases is comparatively small, its effect on the value of f may generally be neglected, but the range of pipe diameter in practice being considerable, Darcy allowed for it in the formula $f=0\cdot005\left(1+\frac{1}{12d}\right)$ for new clean pipes, where d is the diameter of the pipe in feet. For old and incrusted pipes Darcy found the value of f to be double that for new clean pipes.

406. Hydraulic Gradient.—Referring to Fig. 760, A and B are two sections of a straight uniform pipe at a distance l from one another,

h_1 and h_2 are the heights of A and B above datum, and P_1 and P_2 are the pressures at A and B. If pressure tubes, called *piezometers*, be inserted in the pipe at A and B, the water will rise in these tubes to height P_1/w and P_2/w, as shown. The line CD joining the tops of the water-pressure columns is called the *hydraulic gradient* or *virtual slope* of the pipe, and h', the difference between the levels of C and D, is called the *virtual fall* of the pipe AB. The hydraulic gradient or virtual slope is measured by the ratio h'/l, and is denoted by i.

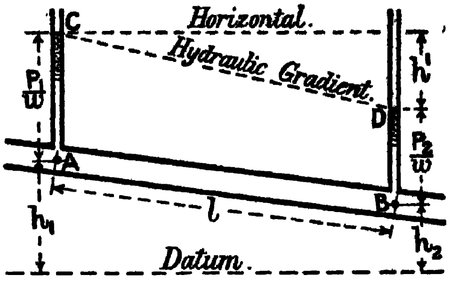


FIG. 760.

The virtual fall h' is evidently given by the equation

$$h' = h_1 - h_2 + \frac{P_1 - P_2}{w},$$

and is equal to the loss of head between A and B. In Art. 404 it was shown that h' is given by the equation $h' = f \cdot \frac{l}{m} \cdot \frac{v^2}{2g}$.

Since the loss of head is proportional to the length of pipe, it follows that for a straight uniform pipe the hydraulic gradient is a straight line. When the pipe is not straight, points in the line of hydraulic gradient may be determined from the equation $h' = f \cdot \frac{l}{m} \cdot \frac{v^2}{2g}$, taking successive values of l, the length of pipe, between the points considered.

In water mains the curvature in the direction of the length is generally small, and its effect on the hydraulic gradient is generally neglected. For example, in Fig. 761 is shown a pipe AEB leading from a tank or reservoir at A to another at B at a lower level. The hydraulic gradient is shown straight, and with the amount of curvature shown on the pipe, this will be found to be approximately true. It will be noticed that the upper end of the line of hydraulic gradient is below the level of the water in the tank A, which is accounted for by the loss of head at the entrance to the pipe at A.

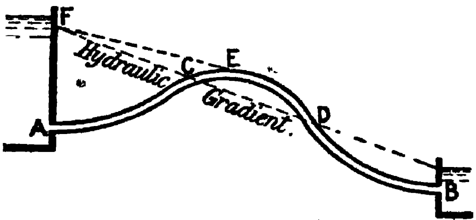


FIG. 761.

Another point illustrated by Fig. 761 is that a part CD of the pipe is above the hydraulic gradient, which shows that in that part of the pipe the pressure is less than atmospheric. If there is a leaky joint in CD air will rush in, and while the pipe from A to E will run full with a hydraulic gradient FE, the pipe from E to B will not run full, and the discharge will be reduced. Water pipes should therefore be arranged, if possible, so as to lie below the hydraulic gradient.

When a valve or other obstruction occurs in a pipe there is a sudden

fall in the hydraulic gradient at the obstruction, due to the loss of head caused by it.

407. Chézy's Formula for Flow in Pipes.—From the equation

$$h'=f\cdot\frac{l}{m}\cdot\frac{v^2}{2g},\ \text{of Art. 404,}\ v=\sqrt{\frac{2g}{f}}\cdot\sqrt{m\frac{h'}{l}}=c\sqrt{mi},$$

where c stands for $\sqrt{\frac{2g}{f}}$.

This formula, $v=c\sqrt{mi}$, is generally called the Chézy formula for the flow in pipes.

408. Examples on Flow of Water in Pipes.—The following examples show the application of the formulæ which have been discussed to practical cases.

(1) A pipe 18 inches in diameter and 6 miles long connects a storage reservoir A with a service reservoir B. The difference between the levels in the two reservoirs is 100 feet. It is required to find the rate of discharge through the pipe and the hydraulic gradient.

Let v = velocity of water through the pipe in feet per second.

The loss of head at the entrance to the pipe is $0{\cdot}45\frac{v^2}{2g}$ (Art. 396).

The loss of head due to friction in the pipe is

$$f\cdot\frac{4l}{d}\cdot\frac{v^2}{2g}=0{\cdot}005\left(1+\frac{1}{12\times1{\cdot}5}\right)\cdot\frac{4\times6\times5280}{1{\cdot}5}\cdot\frac{v^2}{2g}=445{\cdot}87\frac{v^2}{2g}.$$

The head equivalent to $\frac{v^2}{2g}$ is lost at the outlet into the lower reservoir.

$$\text{Total loss of head}=\frac{v^2}{2g}(0{\cdot}45+445{\cdot}87+1)=447{\cdot}32\frac{v^2}{2g}.$$

Hence $447{\cdot}32\frac{v^2}{2g}=100$, from which $v=3{\cdot}79$.

Discharge in cubic feet per second $=\frac{\pi}{4}\times1{\cdot}5^2\times3{\cdot}79=6{\cdot}697$.

Discharge in gallons per day

$$=6{\cdot}697\times60\times60\times24\times6{\cdot}23=3{,}604{,}808.$$

Applying the equation of energy to 1 lb. of water at the surface of the water in A and to the same quantity at a point in the pipe near to the entrance, and taking the level of the water in B as the datum level,

$$100=\frac{P}{w}+\frac{v^2}{2g}+0{\cdot}45\frac{v^2}{2g}.\quad\text{Hence}\ 100-\frac{P}{w}=\frac{1{\cdot}45v^2}{2g}=0{\cdot}32\ \text{foot.}$$

The hydraulic gradient may now be drawn as a straight line joining a point in the surface of the water in B immediately over the outlet end of the pipe, with a point immediately over the inlet end of the pipe and 0·32 foot below the surface of the water in A.

It is obvious that in examples of this kind, where the pipe is very long, the only important loss of head is that due to friction in the pipe, and the other losses may generally be neglected.

(2) Required the diameter of a pipe to connect two reservoirs which are 10 miles apart. The discharge is to be at the rate of 4,000,000 gallons per day, and the available head is 350 feet.

$$\text{Discharge in cubic feet per second} = \frac{4000000 \times 10}{62 \cdot 3 \times 24 \times 60 \times 60} = \frac{1000000}{623 \times 216}.$$

Let d = diameter of pipe in feet, and v = velocity of water through the pipe in feet per second.

$$\frac{\pi}{4} d^2 v = \frac{1000000}{623 \times 216}, \text{ and } v = \frac{1000000}{623 \times 54 \pi d^2}.$$

Neglecting all losses except that due to friction in the pipe, then

$$350 = f \cdot \frac{4l}{d} \cdot \frac{v^2}{2g} = f \cdot \frac{4 \times 10 \times 5280}{d} \cdot \frac{1000000^2}{623^2 \times 54^2 \pi^2 d^4 \times 2g},$$

from which $d^5 = 839f$.

If f be put equal to $0 \cdot 005\left(1 + \frac{1}{12d}\right)$, then $d^6 = 4 \cdot 195\left(d + \frac{1}{12}\right)$, which is an awkward equation to solve, and it is simpler to assume a value for f, say in this case 0·006, then $d^5 = 839 \times 0 \cdot 006 = 5 \cdot 034$, and by logarithms $d = 1 \cdot 38$.

Using this approximate value of d, a more approximate value of f is determined, namely, $f = 0 \cdot 005\left(1 + \frac{1}{12 \times 1 \cdot 38}\right) = 0 \cdot 0053$.

A more approximate value of d is then $d = \sqrt[5]{839 \times 0 \cdot 0053} = 1 \cdot 35$ feet.

(3) Reservoirs A and B (Fig. 762) discharge into a reservoir C through pipes AD, BD, and DC, as shown. The lengths of the pipes AD, BD, and DC are 10,000 feet, 6000 feet, and 8000 feet respectively, and their diameters are 18 inches, 12 inches, and 21 inches respectively. The water levels of B and C are 40 feet and 100 feet respectively below the water level of A. It is required to find the rates of flow from A and B.

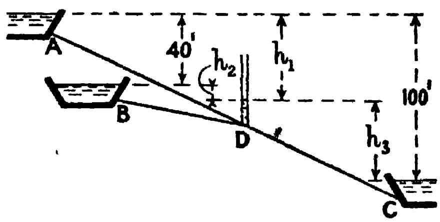


FIG. 762.

Let Q_1, Q_2, and Q_3 be the rates of flow through AD, BD, and DC respectively, in cubic feet per second.

If h is the loss of head in feet in a pipe of diameter d feet and length l feet, v the velocity of flow in feet per second, and Q the rate of discharge in cubic feet per second, then

$$h = f \cdot \frac{4l}{d} \cdot \frac{v^2}{2g}, \quad v = \sqrt{\frac{g}{2f}} \cdot \sqrt{\frac{hd}{l}},$$

and

$$Q = \frac{\pi}{4} d^2 v = \frac{\pi}{4} \sqrt{\frac{g}{2f}} \sqrt{\frac{hd^5}{l}}.$$

An average value of f for the pipes in this example may be taken at about 0·0054, then $Q = 43\sqrt{\frac{hd^5}{l}}$, and $h = \frac{lQ^2}{43^2 d^5}$.

In order that the water may flow from B towards D, it is evident that if the pipe BD be closed the loss of head h between A and D must be greater than 40 feet.

$$h = \frac{10000Q^2}{43^2 \times 1 \cdot 5^5}, \text{ and } 100 - h = \frac{8000Q^2}{43^2 \times 1 \cdot 75^5},$$

where Q is the rate of flow through AD and DC when the pipe BD is closed. From these two equations, $h = 73$ feet, nearly. Therefore if the pipe BD be open, water will flow from B towards D.

Now let h_1, h_2, and h_3 be the losses of head between A and D, B and D, and between D and C respectively, then

$$h_1 = \frac{10000Q_1^2}{43^2 \times 1 \cdot 5^5}, \quad h_2 = h_1 - 40 = \frac{6000Q_2^2}{43^2 \times 1^5},$$

$$h_3 = 100 - h_1 = \frac{8000Q_3^2}{43^2 \times 1 \cdot 75^5}, \text{ and } Q_1 + Q_2 = Q_3 .$$

These equations are sufficient to determine Q_1 and Q_2, but their solution is somewhat cumbersome, and it is really simpler and quicker to proceed by approximation, as follows :—

$$Q_1 = 43\sqrt{\frac{h_1 \times 1 \cdot 5^5}{10000}} = 1 \cdot 185 \sqrt{h_1},$$

$$Q_2 = 43\sqrt{\frac{h_2 \times 1^5}{6000}} = 0 \cdot 555 \sqrt{h_2},$$

$$Q_3 = 43\sqrt{\frac{h_3 \times 1 \cdot 75^5}{8000}} = 1 \cdot 948 \sqrt{h_3}.$$

Select values of h_1 (which must lie between 40 and 73) and calculate the values of Q_1, Q_2, and Q_3 from the equations just given, until values are obtained which make $Q_1 + Q_2 = Q_3$. The work may be tabulated as follows :—

h_1	60	61	62	63	62·1	62·2	62·25
$\sqrt{h_1}$	7·746	7·810	7·874	7·937	7·880	7·887	7·890
h_2	20	21	22	23	22·1	22·2	22·25
$\sqrt{h_2}$	4·472	4·583	4·690	4·796	4·701	4·712	4·717
h_3	40	39	38	37	37·9	37·8	37·75
$\sqrt{h_3}$	6·325	6·245	6·164	6·083	6 156	6·148	6·144
Q_1	9·179	9·255	9 331	9 405	9·338	9 346	**9·350**
Q_2	2·482	2·544	2·603	2·662	2·609	2·615	**2·618**
Q_3	12·321	12·165	12·007	11·850	11·992	11 976	11·969
$Q_1 + Q_2$	11·661	11·799	11·934	12·067	11·947	11·961	11·968

The answers required, as found above, are $Q_1 = 9 \cdot 350$, and $Q_2 = 2 \cdot 618$.

For practice, and to illustrate more fully the working of this example, the student would do well to find the answers directly by solving the equations already given.

409. Power Transmitted through a Pipe.—CASE I. The pipe (Fig. 763) is provided with a nozzle at its delivery end, as for a Pelton wheel, and there is a valve by means of which the area through the nozzle

may be varied. Diameter of pipe $= d$ feet, length of pipe $= l$ feet, total head at nozzle $= h$ feet, and $v =$ velocity of water through the pipe in feet per second. Neglect the loss of energy at the entrance to the pipe and at the nozzle.

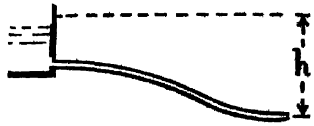


FIG. 763.

Energy delivered at the nozzle per lb. of water per second $= h - f \cdot \frac{4l}{d} \cdot \frac{v^2}{2g}$.

Total energy delivered at the nozzle per second

$$= \frac{\pi}{4} d^2 vw\left(h - f \cdot \frac{4l}{d} \cdot \frac{v^2}{2g}\right).$$

Horse-power delivered at nozzle $= \mathrm{H} = \frac{\pi d^2 vw}{4 \times 550}\left(h - f \cdot \frac{4l}{d} \cdot \frac{v^2}{2g}\right)$

To find when H is a maximum

$$\frac{d\mathrm{H}}{dv} = \frac{\pi d^2 w}{4 \times 550}\left(h - 3f \cdot \frac{4l}{d} \cdot \frac{v^2}{2g}\right).$$

Hence H is a maximum when $3f \cdot \frac{4l}{d} \cdot \frac{v^2}{2g} = h$, or when $v = \sqrt{\frac{ghd}{6fl}}$.

The maximum horse-power is therefore $= \frac{\pi d^2 wh}{6 \times 550}\sqrt{\frac{ghd}{6fl}}$.

If there were no friction, the horse-power delivered would be

$$\mathrm{H}_1 = \frac{\pi d^2 vwh}{4 \times 550}.$$

The efficiency is therefore $\frac{\mathrm{H}}{\mathrm{H}_1} = 1 - f \cdot \frac{4l}{hd} \cdot \frac{v^2}{2g}$.

When H is a maximum, the efficiency is $\frac{2}{3}$.

If P is the pressure of the water in lbs. per square foot just before passing the valve at the nozzle, a the area of the cross section of the pipe, a_1 the area through the valve or through the contracted nozzle, and v_1 the velocity of the water through the contracted nozzle, then

$$\frac{\mathrm{P}}{w} + \frac{v^2}{2g} = \frac{v_1^2}{2g} = \frac{a^2}{a_1^2} \cdot \frac{v^2}{2g}, \qquad \text{also } \frac{\mathrm{P}}{w} + \frac{v^2}{2g} + f \cdot \frac{4l}{d} \cdot \frac{v^2}{2g} = h,$$

$$\text{therefore } \frac{a^2}{a_1^2} \cdot \frac{v^2}{2g} = h - f \cdot \frac{4l}{d} \cdot \frac{v^2}{2g}, \quad \text{and } \frac{a_1}{a} = v\sqrt{\left(\frac{d}{2ghd - 4flv^2}\right)}.$$

When H is a maximum, $v = \sqrt{\frac{ghd}{6fl}}$ then $\frac{a_1}{a} = \sqrt{\frac{d}{8fl}}$.

EXAMPLE.—Let $d = 0{\cdot}5$ feet, $l = 400$ feet, $h = 300$ feet, and $f = 0{\cdot}006$.

Then $$\mathrm{H} = \frac{\pi d^2 vw}{4 \times 550}\left(h - f \cdot \frac{4l}{d} \cdot \frac{v^2}{2g}\right) = 0{\cdot}0667v(100 - 0{\cdot}0994v^2).$$

H is a maximum when $v = \sqrt{\frac{ghd}{6fl}} = 18{\cdot}31$ feet per second.

Maximum horse-power $= 0{\cdot}0667 \times 18{\cdot}31(100 - 0{\cdot}0994 \times 18{\cdot}31^2) = 81{\cdot}4$.

Efficiency per cent. $= 100\left(1 - \frac{2flv^2}{hdg}\right) = 100 - 0{\cdot}0994v^2$.

The horse-power and the efficiency per cent. are shown plotted on a velocity base in Fig. 764.

The maximum possible velocity is when the area through the nozzle is equal to the area through the pipe, then $h - f \cdot \frac{4l}{d} \cdot \frac{v^2}{2g} = \frac{v^2}{2g}$, and $v = 30{\cdot}93$ feet per second. At this maximum velocity the horse-power is $10{\cdot}17$, and the efficiency $4{\cdot}9$ per cent.

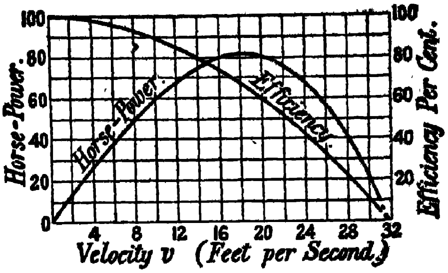


FIG. 764.

When the horse-power delivered is a maximum, a_1, the necessary area through the nozzle, is $a\sqrt{\frac{d}{8fl}} = 0{\cdot}161\ a$.

CASE II. The pressure energy of the water is so great compared with the kinetic energy that the latter may be neglected, also the effect of variation in the level of the pipe may be neglected.

Let p_1 and p denote the pressures in lbs. per square inch at points A_1 and A in the pipe at a distance l feet apart. Let H_1 = horse-power entering the portion A_1A of the pipe at A_1, H = horse-power delivered at A, v = velocity of water in feet per second, and d = diameter of pipe in feet.

Loss of energy between A_1 and A per lb. of water passing

$$= \frac{144(p_1 - p)}{w} = f \cdot \frac{4l}{d} \cdot \frac{v^2}{2g}, \text{ therefore } p = p_1 - \frac{w}{144} \cdot f \cdot \frac{4l}{d} \cdot \frac{v^2}{2g}.$$

$$H_1 = \frac{144 p_1 \pi d^2 v}{4 \times 550}, \text{ and } H = \frac{144 p \pi d^2 v}{4 \times 550} = \frac{\pi d^2 v}{4 \times 550}\left(144 p_1 - wf \cdot \frac{4l}{d} \cdot \frac{v^2}{2g}\right).$$

To find when H is a maximum,

$$\frac{dH}{dv} = \frac{\pi d^2}{4 \times 550}\left(144 p_1 - 3wf \cdot \frac{4l}{d} \cdot \frac{v^2}{2g}\right), \text{ hence H is a maximum when}$$

$$3wf \cdot \frac{4l}{d} \cdot \frac{v^2}{2g} = 144 p_1, \text{ or when } v = \sqrt{\frac{24 p_1 g d}{wfl}}.$$

The maximum value of H is therefore $= 0{\cdot}483\sqrt{\frac{p^3 d^5}{fl}}$.

All the power is lost in friction in the pipe, and $H = 0$, when

$$wf \cdot \frac{4l}{d} \cdot \frac{v^2}{2g} = 144 p_1, \text{ or } v = \sqrt{\frac{72 p_1 g d}{wfl}}.$$

The efficiency is $\frac{H}{H_1} = 1 - \frac{wflv^2}{72 p_1 g d}$.

When H is a maximum, the efficiency is $\frac{2}{3}$.

EXAMPLE.—Let $p_1 = 1120$ lbs. per square inch, $l = 1$ mile $= 5280$ feet, $d = 0{\cdot}5$ foot, and $f = 0{\cdot}006$,

$$H = \frac{\pi d^2 v}{4 \times 550}\left(144 p_1 - wf \cdot \frac{4l}{d} \cdot \frac{v^2}{2g}\right) = v(57{\cdot}58 - 0{\cdot}0875 v^2).$$

H is a maximum when $v = \sqrt{\frac{24p_1gd}{wfl}} = 14{\cdot}8$ feet per second.

Maximum value of $H = 14{\cdot}8\,57{\cdot}58 - 0{\cdot}0875 \times 14{\cdot}8^2) = 568$.

H is zero when $v = 0$, and when $v = \sqrt{\frac{72p_1gd}{wfl}} = 25{\cdot}6$ feet per second.

Efficiency per cent.

$$= 100\left(1 - \frac{wflv^2}{72p_1gd}\right) = 100 - 0{\cdot}152v^2.$$

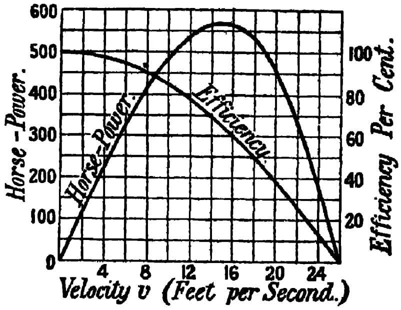


FIG. 765.

The horse-power and the efficiency per cent. are shown plotted on a velocity base in Fig. 765. In hydraulic mains the velocity of the water seldom exceeds 5 feet per second, and in the example just considered the efficiency at this velocity is 96·2 per cent.

410. Flow of Water in Channels.—When water flows in an open channel, or when it flows in a pipe or closed channel without filling the pipe or channel, the water will have a free surface, and the hydraulic gradient will be the longitudinal slope of the free surface, and, since in most cases the depth of the water will be uniform in the direction of flow, the hydraulic gradient will be the same as the longitudinal slope of the channel.

Reasoning as in Art. 404 on the loss of head due to friction in a pipe, it follows that for a channel (Fig. 766) the loss of head $h' = f \cdot \frac{l}{m} \cdot \frac{v^2}{2g}$, and

$$v = \sqrt{\frac{2g}{f}}\sqrt{\frac{mh'}{l}} = c\sqrt{mi},$$ which is the Chézy formula.

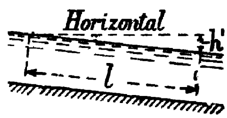


FIG. 766.

It must be kept in mind that the coefficient of friction f and the coefficient c, which is a function of f, depends on the roughness of the surface of the channel, and also on the form and slope of the channel.

411. Bazin's Channel Formula.—The eminent French hydraulic engineer Bazin examined the results of a very large number of experiments on the flow of water in channels of varied forms and dimensions, and in 1897 he published a formula based on these results.

The Bazin channel formula is $$v = \left[\frac{157{\cdot}6}{1 + \frac{\gamma}{\sqrt{m}}}\right]\sqrt{mi},$$

where γ is a coefficient depending on the roughness of the channel. The unit of length in the formula is the foot.

The formula may be written $v = c\sqrt{mi}$, where c is the quantity in the large bracket above.

A few examples of the value of γ are given below :—

Character of the Wetted Surface.	γ
Very smooth.—Smooth cement or planed wood	0·109
Smooth.—Planks, brick, or cut masonry	0·290
Rough.—Rubble masonry	0·833
Very rough.—Ordinary earth canals	2·355

412. Kutter's Channel Formula.—Two Swiss engineers, Ganguillet and Kutter, devised a formula, generally known by the name of the latter, which has been largely used, notwithstanding the fact that it is somewhat cumbrous. The use of the formula is, however, facilitated by published tables. The formula is

$$v=\left[\frac{41\cdot 6+\dfrac{1\cdot 811}{n}+\dfrac{0\cdot 00281}{i}}{1+\left(41\cdot 6+\dfrac{0\cdot 00281}{i}\right)\dfrac{n}{\sqrt{m}}}\right]\sqrt{mi},$$

where n is a coefficient depending on the roughness of the channel.

A few examples of the value of n are given below :—

Character of Wetted Surface.	n
Well planed timber	0·009
Smooth cement, or coated clean pipes	0·010
Rough planks	0·012
Ashlar, good brickwork, or iron pipes in ordinary condition . .	0·013
Rough brickwork, or incrusted iron pipes	0·015
Rough rubble in cement, canals in very firm gravel	0·020
Rivers or canals in good order	0·030

413. Depth for Maximum Discharge in a Channel of Circular Section.—Let r = radius of section of channel (Fig. 767), and θ = angle subtended at the centre by the wetted perimeter. Assume that c in the formula $v=c\sqrt{mi}$ is constant for different depths of stream.

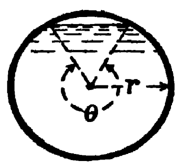


Fig. 767.

Wetted perimeter $= r\theta$.

Area of section of stream $=\dfrac{r^2}{2}(\theta-\sin\theta)$.

Therefore $m=\dfrac{r}{2}\left(\dfrac{\theta-\sin\theta}{\theta}\right)$.

And the discharge $=\dfrac{r^2}{2}(\theta-\sin\theta)c\sqrt{\dfrac{r}{2}\left(\dfrac{\theta-\sin\theta}{\theta}\right)i}$

$$=\frac{\sqrt{(\theta-\sin\theta)^3}}{\sqrt{\theta}}\cdot\frac{r^2c}{2}\sqrt{\frac{ri}{2}}.$$

The discharge will be a maximum when $\dfrac{\sqrt{(\theta-\sin\theta)^3}}{\sqrt{\theta}}$ is a maximum.

$$\text{Let } y = \frac{\sqrt{(\theta - \sin\theta)^3}}{\sqrt{\theta}} = (\theta - \sin\theta)^{\frac{3}{2}}\theta^{-\frac{1}{2}}$$

$$\frac{dy}{d\theta} = \frac{3}{2}(\theta - \sin\theta)^{\frac{1}{2}}(1 - \cos\theta)\theta^{-\frac{1}{2}} - \frac{1}{2}\theta^{-\frac{3}{2}}(\theta - \sin\theta)^{\frac{3}{2}}$$

$$= \frac{3}{2}\left(\frac{\theta - \sin\theta}{\theta}\right)^{\frac{1}{2}}(1 - \cos\theta) - \frac{1}{2}\left(\frac{\theta - \sin\theta}{\theta}\right)^{\frac{3}{2}}.$$

y is a maximum when $\frac{dy}{d\theta} = 0$, that is, when

$$\frac{3}{2}\left(\frac{\theta - \sin\theta}{\theta}\right)^{\frac{1}{2}}(1 - \cos\theta) = \frac{1}{2}\left(\frac{\theta - \sin\theta}{\theta}\right)^{\frac{3}{2}},$$

which reduces to $\sin\theta = \theta(3\cos\theta - 2)$.

The value of θ, found by trial or by plotting, which satisfies this equation, is 308° to the nearest degree.

The depth of water in the channel is then

$$r + r\cos\left(180 - \frac{\theta}{2}\right) = r(1 + \cos 26°) = 1{\cdot}899r,$$

or practically 0·95 of the diameter.

Exercises XXVIIIb.

1. A horizontal pipe 3 inches in diameter suddenly enlarges to a pipe 4 inches in diameter. Water is flowing from the smaller to the larger pipe at the rate of 90 gallons per minute. What is the loss of energy at the enlargement, in ft.-lbs. per minute?

2. If the water flows through the pipes of the preceding exercise at the same rate as before, but in the opposite direction. What is the loss of energy at the sudden contraction, in ft.-lbs. per minute? Assume that the coefficient of contraction (k) of the stream on entering the smaller pipe is 0·7.

3. A pipe of 3 inches diameter conveying water is suddenly enlarged to 5 inches diameter. A **U**-tube containing mercury is connected to two points, one on each side of the enlargement, at points where the flow is steady. Find the difference in level in the two limbs of the **U** when water flows at the rate of ½ cubic foot per second from the small to the large section, and *vice versâ*. The specific gravity of the mercury is 13·6. [U.L.]

4. State briefly the laws of fluid friction deduced from the experiments of Froude. Taking skin friction to be 0·4 lb. per square foot at 10 feet per second, find the skin resistance in pounds of a ship of 12,000 square feet immersed surface, at 15 knots. Also the horse-power to overcome skin friction. [Inst.C.E.]

5. The friction of a thin plate when moved edgewise through water is found by experiment to be ¼ lb. per square foot of surface in contact with the water, when the velocity of rubbing is 600 feet per minute, and that it varies as the square of the velocity of rubbing. How many ft.-lbs. of work per minute will be expended in overcoming the skin friction in the case of a ship steaming at 18½ knots, if the immersed surface of the ship when floating at her load line is 27,620 square feet? If this skin friction is 70 per cent. of the total resistances encountered by the ship, what is the total horse-power usefully expended in propelling the ship? [B.E.]

6. Calculate the hydraulic mean depth for (1) a channel having a bottom width of 6 feet, side slopes of 2 vertical to 1 horizontal, and a depth of water 5 feet; (2) a channel whose section is an arc of a circle of 4 feet radius, the greatest depth of water being 2 feet.

7. In a water main 3 feet in diameter the velocity of the water is 3 feet per second. Find the head lost in friction in feet per mile, using 0·005 as the coefficient of friction.

8. Find the head lost in friction in a pipe 15 inches in diameter and 4 miles long when the discharge is 2,000,000 gallons in twenty-four hours. Take 0·0054 as the coefficient of friction.

9. A set of pumping engines has to force 3,000,000 gallons of water per day through a pipe 18 inches in diameter and 5 miles long to a height of 210 feet. Taking the coefficient friction as 0·0053, what is the effective horse-power of the engines?

10. Two reservoirs, 10 miles apart, are connected by a pipe 3 feet in diameter, the difference in their water levels being 40 feet. If the inlet valve to the lower reservoir is partially closed, so that the water rises in a vertical tube let into the pipe on the inlet side of the valve 20 feet above the level of the water in the reservoir, what would be the discharge of the pipe? [Inst.C.E.]

11. Assuming a coefficient of friction equal to 0·006, what must be the diameter of a pipe 12 miles long to discharge 40,000 gallons of water per hour, the available head being 600 feet?

12. A pipe, 9 inches diameter and 1 mile long, connects two reservoirs. The pipe has a slope of 1 in 80. The level of the water is 25 feet above the inlet end, and 6 feet above the outlet end. Neglecting all losses except skin friction, find the discharge, and draw the hydraulic gradient. Determine the pressure head in the pipe at a distance of half a mile from the inlet. The coefficient of friction may be taken as 0·007. [Inst.C.E.]

13. A pipe 30 inches in diameter branches into two pipes of equal diameter whose combined area equals that of the 30-inch pipe. Compare the loss of head in a mile of the latter pipe with that in a mile of the two pipes, the rate of flow being 4 feet per second. [Inst.C.E.]

14. Two reservoirs are connected by a pipe 1 mile long and 10 inches diameter, the difference in the water surface levels being 25 feet. The value of c in the formula $v=c\sqrt{mi}$ is 120, feet and seconds being the units. Determine the flow through the pipe in gallons per hour, and find by how much the discharge would be increased if for the last 2000 feet a second pipe 10 inches diameter is laid alongside the first and coupled to it so that the water flows equally along the two pipes. [U.L.]

15. A pipe consists of half a mile of 12-inch and half a mile of 6-inch pipe, and slopes at 1 in 100. The discharge is 2 cubic feet per second. Find the difference in pressure head at the two ends of the pipe. [Inst.C.E.]

16. A line of piping has, in the upper portion of its length, a diameter of 15 inches for a length of 5000 feet, and an inclination of 4 per 1000. A tapering pipe then reduces the diameter to 12 inches, which remains constant for a length of 2000 feet, throughout which length the inclination is 3 per 1000. Find the rate of discharge in cubic feet per second when the pipe is fully charged and is delivering freely at its termination. The equation of discharge may be assumed as $Q=42\sqrt{\frac{hd^5}{l}}$, where Q denotes cubic feet per second, h the head lost in length l, and d the diameter in feet. [Inst.C.E.]

17. A pipe AB is fully charged with water at A. Two smaller pipes BC and BD convey the water from B to two points C and D. The length and diameter respectively of AB are 10,000 feet and 15 inches; of BC, 10,000 feet and 12 inches; of BD, 10,000 feet and 9 inches. Points C and D are respectively 50 feet and 80 feet below A. At all points the piping is under pressure except at C and D, where the water issues freely. Find the discharge at C and D, using the equation of discharge in the preceding exercise. [Inst.C.E.]

18. A reservoir A supplies water to two other reservoirs B and C (Fig. 768). The difference of level between the surfaces of A and B is 75 feet, and between A and C 97·5 feet. A common 8-inch cast-iron main supplies for the first 850 feet to the point D. A 6-inch main of length 1400 feet is then carried on in the same straight line to B, and a 5-inch main of length 630 feet branches off at D and goes to C. The entrance to the 8-inch main is bell-mouthed, and losses at the pipe exits to the reservoirs and at the junction of the pipes may be neglected. Find the quantity of water discharged per minute into the reservoirs B and C. Take the coefficient of friction as 0·01. [U.L.]

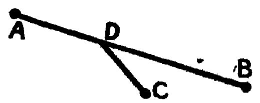


FIG. 768.

19. A 4-inch fire main is connected to a storage tank, the length of the pipe being 800 feet. If the main ends in a nozzle $1\frac{1}{4}$ inches in diameter, and if the head of water in the storage tank is 150 feet above the nozzle, to what height will the nozzle be able to deliver water? The coefficient of friction is 0·006. [U.L.]

20. A pipe 8 inches in diameter and 1000 feet long leads from a reservoir, and terminates in a nozzle open to the atmosphere. The nozzle is 600 feet below the free surface of the water in the reservoir. Determine the diameter of the nozzle when the kinetic energy of the jet is a maximum. Take the coefficient of friction as 0·006.

21. Referring to the preceding exercise, calculate the velocity of the water in the pipe and in the jet, also the horse-power of the jet and the efficiency, for nozzles of 1, 2, 3, 4, 5, 6, 7, and 8 inches diameter, and plot the results on a base representing the diameters of the nozzles.

22. Prove that when power is transmitted hydraulically through a pipe the maximum horse-power is transmitted when one-third of the original head is wasted in friction. You may assume that the loss of head due to friction is proportional to the square of the velocity.

What will the maximum horse-power be if the diameter of the pipe is 6 inches, its length 1200 feet, the original pressure 700 lbs. per square inch, and the coefficient of friction 0·0075? [U.L.]

23. Calculate the percentage loss of horse-power per mile, when power is hydraulically transmitted in cast-iron pipes, 6 inches in diameter, for velocities of flow of 120, 150, and 180 feet per minute, and for pressures of 250, 750, and 1250 lbs. per square inch.

Draw curves to show the results of your calculations, and from your curves obtain the total loss of horse-power when the 6-inch pipe is 4500 yards in length, the velocity of flow 175 feet per minute, and the pressure 1000 lbs. per square inch. Use the formula—loss of head $= 0{\cdot}03lv^2/2gd$ (feet second units). [B.E.]

24. One hundred horse-power is to be transmitted to a distance of 5 miles with a loss of 15 per cent. of the head due to an accumulator pressure of 750 lbs. per square inch. The beginning and end of the pipe are at the same elevation. Find the diameter of the pipe. The equation of discharge in Exercise 16 may be used in this case, but with a coefficient of 36 instead of 42. [Inst.C.E.]

25. Some hydraulic machines are served with water under pressure by a pipe 1000 feet long, the pressure at the machines being 600 lbs. per square inch. The horse-power developed by the machines is 300, and the friction horse-power in the pipes 120. Find the necessary diameter of the pipe, taking the loss of head in feet as $0{\cdot}03\frac{l}{d}\cdot\frac{v^2}{2g}$, and 0·43 lb. per square inch as equivalent to 1 foot head. Also determine the pressure at which the water is delivered by the pump.

What is the maximum horse-power at which it would be possible to work the machines, the pump pressure remaining the same? [U.L.]

26. The cross sections of four channels are shown in Fig. 769. They are all

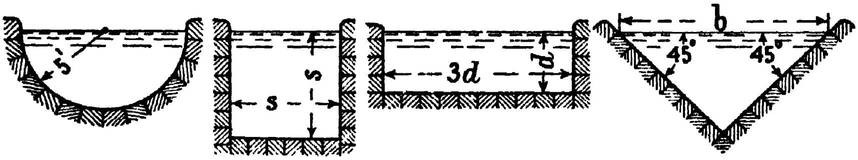


Fig. 769.

equally smooth, they have the same slope and the same rate of discharge. Find the dimensions s, d, and b.

27. If the faces of an open channel are plane and they are tangential to the surface of a cylinder whose axis is in the free surface of the water in the channel, show that the hydraulic mean depth is equal to half the radius of the cylinder.

28. A semicircular channel 10 feet in diameter flows full of water. Compare its discharge with that of a rectangular channel of the same cross sectional area 9 feet wide, lined with the same material, and having the same inclination. [Inst.C.E.]

29. A channel, with a bottom width of 30 feet, and side slopes of 2 horizontal to 1 vertical, flows full of water to a depth of 7 feet. Find the velocity of flow in, and the discharge of, the channel, the inclination being 1 in 5000, and c in the Chézy formula 100. [Inst.C.E.]

30. A channel with cement sides is 4 feet wide at the bottom, and its sides slope at 2 vertical to 1 horizontal. The slope of the channel is 1 in 500. What will be the depth of the water for a discharge of 12,000 gallons per minute. Take the coefficient of friction as 0·006. [U.L.]

31. A channel in which the water is to run 3 feet deep has to discharge 60 cubic feet per second, the velocity of flow being 2·5 feet per second. The sides slope at 1·5 vertically to 1 horizontally, and the value of c in the formula $v=c\sqrt{mi}$ may be taken as 110, feet and seconds being units. Find the width of the channel at the bottom, and the hydraulic inclination necessary. [U.L.]

32. Apply Kutter's formula to find the rate of discharge in cubic feet per second of a channel having a bed width of 20 feet, side slopes of $1\frac{1}{2}$ horizontal to 1 vertical, depth 6 feet, and longitudinal slope 1 in 5000. Take n, the coefficient of roughness, $=0{\cdot}02$.

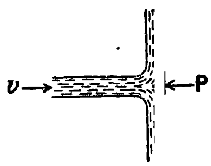
33. Water flows in a pipe without filling it. Show that the velocity of flow for a given slope is a maximum when the wetted perimeter subtends an angle θ at the centre given by the equation $\theta=\tan\theta$, and that $\theta=257\frac{1}{2}$ degrees nearly.

34. The cross section of a closed channel is a square with a diagonal vertical. s is the side of the square, and y is the depth of the water line below the apex. Show that for maximum discharge $y=0{\cdot}127s$, and that for maximum velocity $y=0{\cdot}414s$.

35. A cast-iron pipe 18 inches in diameter is laid with a slope of 1 in 1000. Water flows through this pipe with a depth of 13·5 inches. Taking c in the formula $v=c\sqrt{mi}$ as 125, find the discharge in gallons per hour.

414. Impact of a Jet on a Flat Vane.—CASE I. *Direction of jet perpendicular to vane. Vane at rest* (Fig. 770).—A = sectional area of jet. v = velocity of jet before impact. W = weight of liquid reaching vane per second. w = weight of unit of volume of liquid. P = total normal pressure on vane due to impact of jet.

Since the motion of the liquid in the direction in which the jet is moving is entirely destroyed, the loss of momentum per second in that direction is $\frac{Wv}{g}$, and therefore



$$P=\frac{Wv}{g}=\frac{wAv^2}{g}=2wA\cdot\frac{v^2}{2g}=2wAh,$$

FIG. 70.

where h is the head due to the velocity v. But wAh is the static pressure on an area A due to a head h. Therefore the total dynamic pressure due to the impact of the jet on the vane is equal to twice the static pressure on an area A due to a head h. In other words, the total dynamic pressure of the jet issuing under a head h will balance a static pressure on an area due to a head $2h$. This may be demonstrated by the apparatus shown in Fig. 771, where B is a tank containing water to a constant height h above the axis of a tube projecting from the side of the tank. C is another tank containing water to a height $2h$ above the axis of a projecting tube of the same size and shape as that projecting from B. A flat plate D is suspended loosely against the mouth of the tube on C, and is held there by the force of the jet from B, as shown. Experimentally, the head in C will be slightly

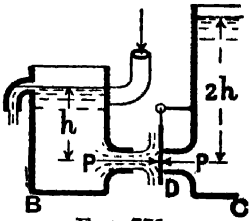


FIG. 771.

less than $2h$ on account of the loss of energy in the jet from B due to friction.

As to the distribution of the pressure on a flat plate struck normally by a jet, this is shown approximately in Fig. 772, where the intensity of the pressure due to the impact of the jet is plotted on the back of the plate. At the centre the pressure h, in feet of water, is slightly less than the head due to the velocity v, that is to say, h is slightly less than $v^2/2g$.

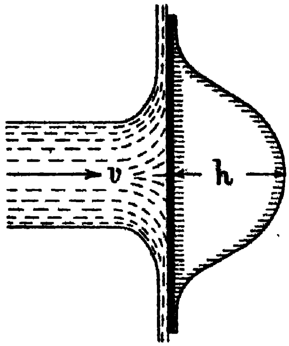


FIG. 772.

CASE II. *Same as Case I., except that the vane is moving in the same direction as the jet with a velocity* v_1.—Loss of momentum of water impinging on vane per second in direction of motion of jet $= \frac{W}{g}(v - v_1)$, therefore

$$P = \frac{W}{g}(v - v_1).$$

But $W = wA(v - v_1)$, therefore $P = \frac{wA}{g}(v - v_1)^2$.

Useful work done per second $= Pv_1 = \frac{wAv_1}{g}(v - v_1)^2$.

Kinetic energy of jet per second $= \frac{wAv^3}{2g}$.

$$\text{Efficiency} = \frac{wAv_1}{g}(v - v_1)^2 \div \frac{wAv^3}{2g} = \frac{2v_1(v - v_1)^2}{v^3}.$$

For a given value of v the efficiency will be a maximum when $v_1(v - v_1)^2$ is a maximum.

Let $\quad y = v_1(v - v_1)^2 = v_1v^2 - 2vv_1^2 + v_1^3.$

$$\frac{dy}{dv_1} = v^2 - 4vv_1 + 3v_1^2.$$

y is a maximum when $\frac{dy}{dv_1} = 0$, *i.e.* when $v_1 = v$ or $\frac{v}{3}$. Obviously $v_1 = \frac{v}{3}$ is the result to take. Therefore the velocity of the vane should be one-third of the velocity of the jet for the highest efficiency.

$$\text{Maximum efficiency,} = \frac{\frac{2v}{3}\left(v - \frac{v}{3}\right)^2}{v^3} = \frac{8}{27}, \text{ or } 29{\cdot}6 \text{ per cent.}$$

CASE III. *Direction of jet makes an angle* θ *with vane. Vane at rest* (Fig. 773).—Loss of momentum per second in direction of normal to vane $= \frac{W}{g}v \sin \theta$.

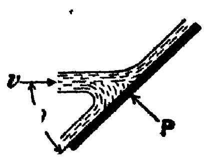


FIG. 773.

Therefore $\qquad P = \frac{W}{g}v \sin \theta.$

CASE IV. *Same as Case III., except that the vane is moving parallel to itself with a velocity v_1 in a direction making an angle ϕ with the vane* (Fig 774).—Loss of momentum per second in direction of normal to vane

$$= \frac{W}{g}(v \sin\theta - v_1 \sin\phi) = P.$$

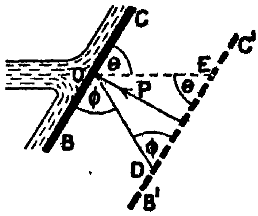


FIG. 774.

Let the vane BC move to B′C′ in 1 second. OD $= v_1$, and OE = velocity of vane in direction of jet.

OE $\sin\theta =$ OD $\sin\phi$. Therefore

$$\text{OE} = \frac{\text{OD}\sin\phi}{\sin\theta} = \frac{v_1\sin\phi}{\sin\theta}.$$

Relative velocity of jet and vane in direction of jet $= v - \dfrac{v_1\sin\phi}{\sin\theta}$.

Therefore $$W = wA\left(v - \frac{v_1\sin\phi}{\sin\theta}\right) = \frac{wA}{\sin\theta}(v\sin\theta - v_1\sin\phi),$$

and $$P = \frac{wA}{g\sin\theta}(v\sin\theta - v_1\sin\phi)^2.$$

Useful work done per second

$$= Pv_1\sin\phi = \frac{wAv_1\sin\phi}{g\sin\theta}(v\sin\theta - v_1\sin\phi)^2.$$

Kinetic energy of jet per second $= \dfrac{wAv^3}{2g}$.

$$\text{Efficiency} = \frac{wAv_1\sin\phi}{g\sin\theta}(v\sin\theta - v_1\sin\phi)^2 \div \frac{wAv^3}{2g}$$

$$= \frac{2v_1\sin\phi}{v^3\sin\theta}(v\sin\theta - v_1\sin\phi)^2.$$

In the same way as in Case II. this efficiency can be shown to be a maximum when $v_1 = \dfrac{v\sin\theta}{3\sin\phi}$.

The maximum efficiency $= \dfrac{8}{27}\sin^2\theta$.

Case IV. is the general case from which the others may easily be deduced. For example, Case II. may be deduced from Case IV. by putting θ and ϕ each equal to 0.

In the foregoing demonstrations the losses due to friction and the production of eddies have been neglected.

415. Impact of a Jet on a Succession of Vanes.—In the preceding Article the jet was supposed to impinge on a single vane, and it was seen that the amount of water arriving at the moving vane was less than the amount delivered by the nozzle. If, however, a series of vanes come in turn in front of the jet, each vane entering the jet at the same point, the vanes will receive the whole of the water discharged by the nozzle, and the useful work done will be increased, and the efficiency therefore raised. For example, consider Case II. of the preceding Article with a succession of vanes instead of one.

The weight of water reaching the vanes is now $W = wAv$, and the total pressure on the vanes is $P = \frac{W}{g}(v - v_1) = \frac{wAv}{g}(v - v_1)$.

Useful work done per second $= Pv_1 = \frac{wAvv_1}{g}(v - v_1)$.

Kinetic energy of jet per second $= wAv \cdot \frac{v^2}{2g}$.

Efficiency $= \frac{wAvv_1}{g}(v - v_1) \div wAv \cdot \frac{v^2}{2g} = \frac{2v_1}{v^2}(v - v_1)$.

For a given value of v the efficiency will be a maximum when $v_1(v - v_1)$ is a maximum. Let $y = v_1(v - v_1)$, then $\frac{dy}{dv_1} = v - 2v_1$, therefore the efficiency is a maximum when $v = 2v_1$, and the maximum efficiency is $\frac{v}{v^2}(v - \frac{1}{2}v) = \frac{1}{2}$, or 50 per cent.

The action of a series of vanes will perhaps be better understood by reference to Fig. 775. This does not represent a practical contrivance, and it is designed to illustrate the principle only. A frame, carrying a series of vanes at intervals q apart, travels parallel to the jet, and each vane in turn is swung into the jet at the same point. The vanes are perpendicular to the axis of the jet. At (a) the first vane has just come in front of the jet. At (b) the second vane has just come into action, cutting the jet in two. The forward part of the jet will continue moving until its rear end B overtakes the vane in front of it. At (c) B, which is moving faster than the vanes, is overtaking the vane in front of it, and at (d) B has overtaken the vane in front of it, and that vane therefore ceases to act.

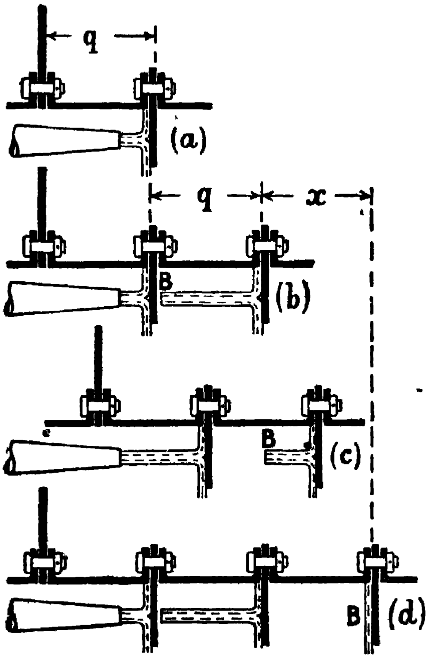


Fig 775.

Referring to (b), let x be the distance which the front vane will have to travel before it ceases to act after the second vane has come into action. Then $q + x$ is the distance travelled by a point in the jet while a vane travels the distance x. Hence $\frac{q+x}{x} = \frac{q}{x} + 1 = \frac{v}{v_1}$, and the number of vanes in action at one time $= \frac{q+x}{q} = 1 + \frac{x}{q} = \frac{v}{v - v_1}$.

The total pressure on *one* vane is, by Art 414, $\frac{wA}{g}(v - v_1)^2$. Therefore

the total pressure on all the vanes is $\frac{wA}{g}(v-v_1)^2 \times \frac{v}{v-v_1} = \frac{wAv}{g}(v-v_1)$, as already shown in another way. Fig. 775 is drawn for the case where $v = 2v_1$.

It is easy to show that in the general case, Case IV. of the preceding Article, but with a succession of vanes instead of one, the total normal pressure on all the vanes in action at one time is

$$P = \frac{wAv}{g}(v \sin \theta - v_1 \sin \phi),$$

and that the efficiency is $\frac{2v_1 \sin \phi}{v^2}(v \sin \theta - v_1 \sin \phi)$, also that the maximum efficiency is $\frac{1}{2} \sin^2 \theta$ when $v \sin \theta = 2v_1 \sin \phi$.

416. Impact of a Jet on a Cup.—The axis of the jet is supposed to coincide with the axis of the cup, and the effect of friction will be neglected.

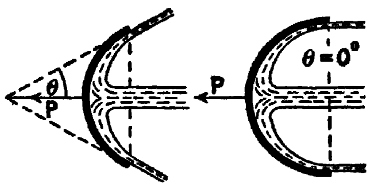


FIG. 776.

CASE I. *Cup at rest.*—The water will leave the cup in a direction tangential to the surface at the lip of the cup, as shown in Fig. 776, and the velocity of the water as it leaves the cup will have the same magnitude v as the velocity of the jet, but its direction will have been turned through an angle $180° - \theta°$.

Loss of momentum of water per second in the direction in which it is moving before striking the cup $= \frac{Wv}{g}(1 + \cos \theta)$.

Therefore $P = \frac{Wv}{g}(1 + \cos \theta) = \frac{wAv^2}{g}(1 + \cos \theta)$.

If $\theta = 0$, $P = \frac{2wAv^2}{g}$.

CASE II. *Cup moving in same direction as jet with velocity v_1.*—Relative velocity of jet and cup $= v - v_1$, and this will be the relative velocity of water and cup as the water leaves the cup. Hence the loss of momentum of water per second in the direction in which it is moving before striking the cup is $\frac{W}{g}(v - v_1)(1 + \cos \theta)$, and this is equal to P.

But $W = wA(v - v_1)$, therefore $P = \frac{wA}{g}(v - v_1)^2(1 + \cos \theta)$.

Useful work per second $= Pv_1 = \frac{wAv_1}{g}(v - v_1)^2(1 + \cos \theta)$.

Efficiency $= \frac{wAv_1}{g}(v - v_1)^2(1 + \cos \theta) \div wAv \cdot \frac{v^2}{2g} = \frac{2v_1}{v^3}(v - v_1)^2(1 + \cos \theta)$.

The efficiency will be a maximum when $v = 3v_1$.

Maximum efficiency $= \frac{8}{27}(1 + \cos \theta) = \frac{16}{27} \cos^2 \frac{\theta}{2}$.

417. Reaction of a Jet.—When a jet of cross sectional area A issues from a vessel with a velocity v, the momentum given to it per second is

$\frac{Wv}{g} = \frac{wAv^2}{g}$, and this requires that a force $P = \frac{wAv^2}{g}$ shall act on the jet in the direction of its motion. As a consequence of this there must be another force F equal and opposite to P acting on the vessel, as shown in Fig. 777, and unless an external force be applied to the vessel the latter will move under the action of the force F.

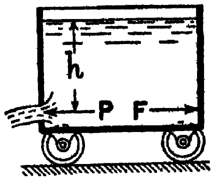


FIG. 777.

If the vessel moves under the action of the force F in the direction of that force with a velocity v_1, the magnitude of F and P remain the same, namely,

$$F = P = \frac{wAv^2}{g}.$$

Useful work done per second $= Fv_1 = \frac{wAv^2v_1}{g}$.

Total energy = useful work + lost work

$$= \frac{wAv^2v_1}{g} + wAv\frac{(v - v_1)^2}{2g} = wAv\frac{v^2 + v_1^2}{2g}.$$

$$\text{Efficiency} = \frac{wAv^2v_1}{g} \div wAv\frac{v^2 + v_1^2}{2g} = \frac{2vv_1}{v^2 + v_1^2}.$$

418. Deviation of a Jet in one Direction by a Vane without Shock.—In the examples on the impact of a jet on a vane which have hitherto been considered, the jet has struck the vane and been deviated abruptly. A consequence of this abrupt deviation is a shock, and therefore a loss of energy in agitating the water. The full force of the impact may, however, be obtained and the shock avoided by so shaping the vane that the jet on meeting it glides along its surface and is deviated gradually.

CASE I. *Vane at rest* (Fig. 778).—At B, where the jet first meets the vane, the direction of the surface of the vane coincides with the direction of the jet, that is, the jet meets the vane tangentially. The jet is then gradually deviated by the curved surface of the vane, and leaves it in a direction tangential to the vane at C.

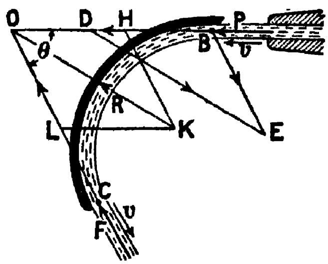


FIG. 778.

The velocity of the water at B is equal to v in the direction BD, and the velocity at C is equal to v in the direction of the tangent to the vane at C. Draw BE parallel to the tangent to the vane at C, and make BE and BD each equal to v. Join DE. Then DE is the change in the velocity of the water, in magnitude and direction, while it passes over the vane. If θ is the interior angle between the tangents to the vane at B and C, then $DE = 2\,v\cos\frac{\theta}{2}$. The change in the momentum of the water per second in passing from B to C is $\frac{W}{g} \cdot \overline{DE} = \frac{2Wv}{g}\cos\frac{\theta}{2}$, and this is equal to R, the resultant force on the vane due to the impact of the

jet. The line of action of R passes through O, the intersection of the axes of the jets at B and C.

The foregoing result may be obtained in another way. At B the impulse of the jet on the vane is $P=\frac{Wv}{g}$ in the direction BO. At C the reaction of the jet on the vane is $F=\frac{Wv}{g}$ in the direction CO. The resultant of these two forces, obtained by the parallelogram of forces OHKL, is $R=\frac{2Wv}{g}\cos\frac{\theta}{2}$.

CASE II. *Vane moving parallel to itself in a given direction with a velocity* v_1 (Fig. 779).—The jet moving in the direction BD with velocity v meets the vane BC at B. The vane is moving in the direction BE with velocity v_1. Make $BD=v$, and $BE=v_1$. Complete the parallelogram BEDH. Then $BH=v_r$ is the direction and magnitude of the relative velocity of the water and vane; therefore in order that there may be no shock at entrance, BH must be the direction of the tangent to the vane at B.

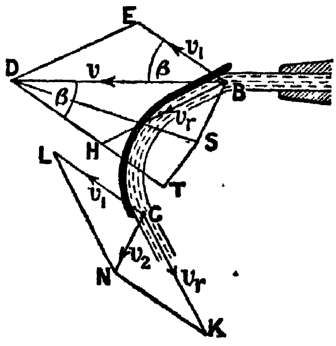


FIG. 779.

The water moves over the vane with the relative velocity v_r, leaving the vane at C, where it has a velocity v_r in the direction CK tangential to the vane at C, and a velocity v_1 in the direction CL parallel to BE. Make $CK=v_r$, and $CL=v_1$. Complete the parallelogram CKNL. The diagonal $CN=v_2$ is the direction and magnitude of the absolute velocity of the water leaving the vane at C.

Draw BS parallel and equal to CN. Join DS. Then DS is the change, in magnitude and direction, of the velocity of the water while passing over the vane. If R is the resultant force on the vane due to the impact of the jet, then $R=\frac{W}{g}\cdot\overline{DS}$, where W is the weight of water impinging upon the vane per second. Draw ST perpendicular to and meeting DH produced at T. Then if P is the component of R in the direction of the motion of the vane, $P=\frac{W}{g}\cdot DT$.

CN, the absolute direction in which the water leaves the vane, should be perpendicular to CL, the direction of motion of the vane. CN has then no component in the direction CL. The component of CN in the direction CL in the case of a revolving vane is called the *velocity of whirl* at exit, and for maximum efficiency this should be zero.

If CN is perpendicular to CL, then BS and ST are in the same straight line, and $DT=v\cos\beta$, where β = angle BDT = angle DBE. Then $P=\frac{W}{g}v\cos\beta$. But $W=wA(v-v_1\cos\beta)$, where A is the area of the section of the jet, and $v_1\cos\beta$ is the velocity of the vane in the direction of the motion of the jet. Hence $P=\frac{wA}{g}(v-v_1\cos\beta)v\cos\beta$.

$$\text{Useful work per second} = \mathrm{P}v_1 = \frac{w\mathrm{A}}{g}(v - v_1 \cos\beta)v_1 v \cos\beta.$$

$$\text{Energy of jet per second} = w\mathrm{A}v \cdot \frac{v^2}{2g} = \frac{w\mathrm{A}v^3}{2g}.$$

$$\text{Efficiency} = \frac{w\mathrm{A}}{g}(v - v_1 \cos\beta)v_1 v \cos\beta \div \frac{w\mathrm{A}v^3}{2g} = \frac{2v_1}{v^2}(v - v_1\cos\beta)\cos\beta.$$

For given values of v and β the efficiency will be a maximum when $v_1(v - v_1\cos\beta)$ is a maximum, that is, when $v_1 = \frac{v}{2\cos\beta}$. Hence the maximum efficiency is $\frac{1}{2}$, or 50 per cent. It is obvious that when $v_1 = \frac{v}{2\cos\beta}$, $v_r = v_1$, and $v_2 = 0$.

It is evident that unless β is a small angle a single vane of limited length BC could only remain in action for a very short time, and while the vane is receiving water, that part of it upon which the jet is impinging must be straight and parallel to BH in order that there may be no shock at entrance.

For a succession of vanes, with CN perpendicular to CL, the total pressure on all the vanes in action at one time is $\mathrm{P} = \frac{\mathrm{W}}{g} v \cos\beta$, where W is now the total weight of water delivered by the jet per second, and is equal to $w\mathrm{A}v$. Therefore $\mathrm{P} = \frac{w\mathrm{A}v^2 \cos\beta}{g}$, and the useful work per second is $\mathrm{P}v_1 = \frac{w\mathrm{A}v_1 v^2 \cos\beta}{g}$. It would therefore seem that the useful work increases indefinitely with v_1, but if the vanes are driven by the jet, the useful work cannot exceed the energy of the jet. Hence the maximum useful work $= \frac{w\mathrm{A}v^3}{2g}$; the efficiency is then unity, and $v_1 = \frac{v}{2\cos\beta}$.

419. Action of a Jet on a Revolving Vane.—A case of great importance in connection with turbines and certain forms of water wheels is that in which the vane upon which the water impinges is part of a revolving wheel. Referring to Fig. 780, O is the axis of the wheel which is perpendicular to the plane of the figure; the acting surface of the vane is also perpendicular to that plane. The inner and outer edges of the vane are at distances r_1 and r_2 from the axis of the wheel. In what follows the wheel is assumed to be moving with uniform angular velocity. The linear velocities of the inner and outer edges of the vane are c_1 and c_2 respectively. Evidently $c_1/r_1 = c_2/r_2$. At entrance the axis of the jet makes an angle θ_1 with c_1. The absolute velocity of the water at entrance is v_1 Completing the parallelogram of velocities at B_1, the relative velocity u_1 of the water and vane at

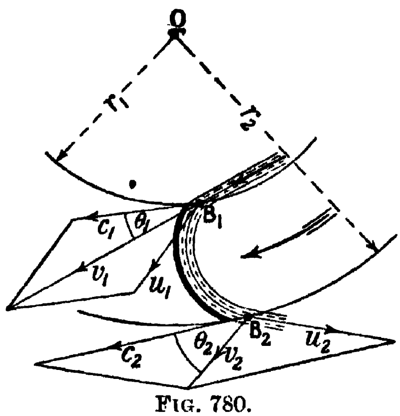


FIG. 780.

entrance is found, and its direction determines the direction of the tangent to the vane at entrance, so that there shall be no shock there. At exit the relative velocity u_2 is in the direction of the tangent to the vane at that point. v_2, the absolute velocity of the water at exit, is the diagonal of the parallelogram, having c_2 and u_2 for adjacent sides. The angle between v_2 and c_2 is θ_2.

Consider a small portion of the water of mass m. At entrance the velocity of this mass in the tangential direction c_1 is $v_1 \cos\theta_1$; this is the *velocity of whirl* at entrance. At exit the velocity of whirl is $v_2 \cos\theta_2$. Taking moments about O, the angular momentum of the mass at entrance is $mv_1r_1 \cos\theta_1$, and at exit its angular momentum is $mv_2r_2 \cos\theta_2$. Hence the loss of angular momentum of the mass in passing over the vane is $m(v_1r_1 \cos\theta_1 - v_2r_2 \cos\theta_2)$. If W is the weight of water impinging on the vane per second, then the angular momentum lost by the water per second is $\frac{W}{g}(v_1r_1 \cos\theta_1 - v_2r_2 \cos\theta_2)$. If there is a succession of vanes, then W is the weight of water supplied by the jet per second, and the turning moment on the wheel, due to the action of the water on the vanes, is $M = \frac{W}{g}(v_1r_1 \cos\theta_1 - v_2r_2 \cos\theta_2)$, since the angular momentum gained by the wheel is equal to that lost by the water. In the foregoing discussion the effect of friction has been neglected.

If ω is the angular velocity of the wheel, then the work imparted to the wheel per second is

$$M\omega = \frac{W}{g}(v_1r_1\omega \cos\theta_1 - v_2r_2\omega \cos\theta_2) = \frac{W}{g}(v_1c_1 \cos\theta_1 - v_2c_2 \cos\theta_2).$$

Exercises XXVIIIc.

1. A jet of water 2 inches in diameter, and having a velocity of 30 feet per second, impinges upon a fixed flat plate. Find the total pressure on the plate due to the impact of the jet, (*a*) when the plate is perpendicular to the axis of the jet, (*b*) when the plate is inclined at 30° to the axis of the jet.

2. A jet of water 3 inches in diameter, and having a velocity of 40 feet per second, strikes a flat vane which is perpendicular to the axis of the jet. Determine the total pressure on the vane, (*a*) when it is fixed, (*b*) when it is moving in the same direction as the jet with a velocity of 15 feet per second.

3. A fixed nozzle discharges 2 cubic feet of water per second. The jet, which has a cross section of 10 square inches, impinges on a flat vane which is moving in the same direction as the jet with a velocity of 10 feet per second. Find the work done on the vane in horse-power.

4. A series of flat vanes come in turn into a jet of water 4 inches in diameter. The vanes when in action are perpendicular to the axis of the jet, and they are driven forward by the jet with a velocity v_1 feet per second. The velocity of the jet is 50 feet per second. On a base representing values of v_1 from 0 to 50, plot the horse-power delivered to the vanes. State the value of the maximum horse-power, and the corresponding value of v_1.

5. A jet of water has a sectional area of 20 square inches, and delivers 1869 gallons of water per minute. The jet impinges at right angles on a flat vane, which is driven in a direction inclined at 30° to the axis of the jet with a velocity of 12 feet per second. Find the work done on the vane in ft.-lbs. per second, and the efficiency.

6. Taking the data of the preceding exercise, except that the jet impinges on the vane at an angle of 30° to its normal, find the work done on the vane in ft.-lbs. per second, and the efficiency.

7. Same as Exercise 5, except that there is a succession of vanes at equal distances apart.

8. A jet of water 1 inch in diameter, coming from a reservoir at a height of 200 feet, strikes a fixed hemispherical cup so that the direction of its motion is reversed. Find the force it exerts upon the cup, assuming that the jet has 90 per cent. of the full velocity due to its head. [Inst.C.E.]

9. A jet of water 2 inches in diameter, moving with a velocity of 40 feet per second, strikes the interior of a cup. The axis of the jet coincides with the axis of the cup. The interior surface of the cup is part of the surface of a sphere whose radius is 6 inches, and the depth of the cup is 3 inches. Find the total pressure on the cup, (*a*) when the cup is fixed, (*b*) when the cup is moving in the same direction as the jet with a velocity which makes the work done per second on the cup a maximum.

10. Taking the data of the preceding exercise, except that there is a succession of cups instead of one cup. Find the work done on the cups in ft.-lbs. per second when the efficiency is a maximum, and determine the maximum efficiency.

11. A small steam-boat is provided with two jet propellers, one on each side of the vessel. The combined area of the two jets is 2 square feet. The water for the jets is taken from the sea and driven astern, below the water line, by a centrifugal pump. The velocity of the jets in relation to the vessel is 25 feet per second, and the speed of the vessel is 9 knots. Determine the resistance to the motion of the vessel, also the horse-power developed in the cylinders of the engine, assuming that the useful work done by the jets in propelling the vessel is 40 per cent. of the work done in the cylinders. Take the weight of 1 cubic foot of sea water = 64 lbs., and 1 knot = 6080 feet per hour.

12. The cross section of a jet of water is a rectangle 6 inches wide and 1 inch deep. This jet impinges upon a vane without shock. The cross section of the vane is a quadrant of a circle. The velocity of the jet is 30 feet per second. Find the component of the total pressure on the vane in the direction of the motion of the jet, (*a*) when the vane is fixed, (*b*) when the vane is moving in the same direction as the jet with a velocity of 15 feet per second.

13. AB and AC are two lines inclined at 30°. A jet of water moves in the direction AC with a velocity of 24 feet per second, and a vane in the direction AB with a velocity of 12 feet per second. Show how to find the form of a vane so that the water may come on it tangentially, and leave it in a direction perpendicular to the direction of motion of the vane. Determine the pressure on the vane in the direction of motion due to each pound of water striking the vane. [Inst.C.E.]

14. Indicate how a vane, moving with a velocity of 25 feet per second in a horizontal direction, must be shaped in order to abstract the maximum amount of energy from a jet of water impinging upon it at an angle of 45° to the horizontal with double the above velocity. What pressure would be exerted on the vane per cubic foot of water impinging per second? [Inst.C.E.]

15. A jet of water, area 1 square inch, velocity 160 feet per second, has its axis inclined at 15° to the direction of motion of a bucket upon which it impinges, the velocity of the bucket being 70 feet per second. Find the direction and magnitude of the total pressure and the pressure in the direction of motion, if there is no loss due to shock at entrance, and no velocity of whirl at exit from the bucket. Find the maximum possible hydraulic efficiency of a wheel provided with such buckets, and find also the speed corresponding. [U.L.]

16. The rim of a turbine is going at 50 feet per second; 100 lbs. of fluid enter the wheel each second, with a velocity in the direction of the rim's motion of 60 feet per second, leaving it with no velocity in the direction of the wheel's motion. What work is done per second upon the wheel? [B.E.]

17. A wheel having curved vanes is driven by a jet of water delivered on to the vanes, as shown in Fig. 780, p. 482. $r_1 = 2$ feet, $r_2 = 2\frac{1}{2}$ feet. The jet delivers 2 cubic feet of water per second. The absolute velocities of the water at entrance and exit are 100 feet per second and 10 feet per second respectively. If $\theta_1 = 20°$, and $\theta_2 = 85°$, what tangential resistance will this wheel overcome at uniform speed and at a radius of 10 inches, neglecting friction?

18. A locomotive going at 40 miles per hour scoops up water from a trough. The outlet to the tank is 8 feet above the mouth of the scoop, and the delivery pipe has an area of 150 square inches. If half the available head at entrance is wasted, find the velocity at which the water is delivered into the tank, and the number of tons lifted in a trench 500 yards long. What, under these conditions, is the increased resistance to the motion of the train; and what is the minimum speed of the train at which water can be delivered to the tank?

CHAPTER XXIX

WATER WHEELS AND TURBINES

420. Water Wheels and Turbines are prime movers, which utilise the potential and kinetic energy of water. In one class of water wheels the wheel acts by the direct weight of the water delivered to it. In a second class the wheel acts partly by the weight of the water and partly by the impulse due to the weight and velocity of the water striking the wheel. In a third class the action is entirely by impulse. In a fourth class the action is entirely due to the reaction of the moving water on the wheel.

In a water wheel there are usually a considerable number of buckets or vanes placed round the periphery, and the water is delivered to the wheel on a part of its circumference, filling or striking one or a few buckets only at one time.

In a turbine the revolving wheel has numerous buckets or vanes, which are all supplied with water simultaneously. Turbines have almost entirely superseded the slow-moving and cumbrous vertical water wheels. Turbines occupy less space, and are cheaper to construct than the older vertical wheels of the same power; they are also highly efficient, and suitable for large or small falls.

421. Overshot Wheels.—An overshot water wheel is shown in Fig. 781. The water is led to the wheel by a *head race*, and the quantity entering the buckets is regulated by a sluice A, which is operated by hand, or controlled by a governor driven by the wheel. The water enters the buckets at or near the top of the wheel, and acts almost entirely by its weight, descending in the buckets on about one half of the wheel. The buckets empty themselves, when near their lowest position, into the *tail race*. A small part of the effort on the wheel is due to the impulse of the water as it enters the buckets.

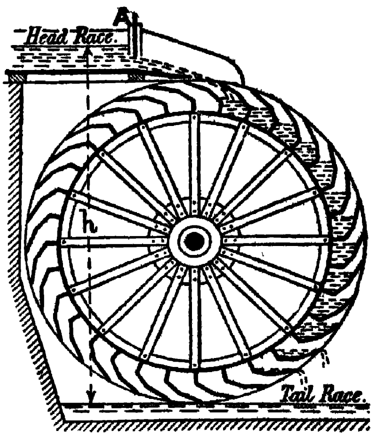


FIG. 781.

If h is the total fall in feet, and Q the number of cubic feet of water delivered to the wheel per second, and w the weight of 1 cubic foot of water, then the available horse-power is $\dfrac{wQh}{550}$.

To utilise as much as possible of the available power an overshot wheel must have a diameter nearly equal to the fall h, but to obtain

sufficient velocity of water the top of the wheel requires to be about 2 feet below the head race. Hence for high falls the overshot wheel is of large diameter. Wheels over 70 feet in diameter have been used.

The velocity of the buckets is from 3 to 6 feet per second, or about half the velocity of the entering water. The efficiency of overshot water wheels is from 70 to 85 per cent. when well designed and properly constructed. It is interesting to notice that the hydraulic efficiency of the overshot wheel is greater at lower loads when the buckets carry less water, because then the buckets do not begin to empty until a greater part of the descent has been made.

422. Breast Wheels.—The feature which gives its name to the *breast wheel* is the casing, apron, curb, or breast between the head race and tail race, which enables the buckets to retain the water for a greater portion of the fall. This breast fits as close to the wheel as is consistent with security from actual contact. Wheels with breasts are also termed *high-breast*, *breast*, and *low-breast* wheels, according as the water is delivered to the wheel above, at, or below the middle level of the wheel. Fig. 782 shows a high-breast wheel as made by Fairbairn. The regulating sluice and its seat are curved, so as to fit close to the wheel. The water passes over the top of the sluice through guide passages designed to deliver the water to the buckets without shock. The sluice is operated by a rack and pinion under the control of the governor.

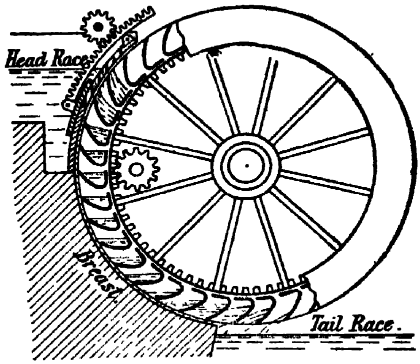


FIG. 782.

The power is taken from the wheel by a pinion gearing with a large internal toothed wheel attached to the rim of the wheel, as shown. The position of the pinion is such that the downward thrust, due to the weight of the water in the buckets, is taken by the pinion without being transmitted to the axle of the wheel, and no torque is carried by the arms, which have only to carry the weight of the wheel. The arms are comparative slender rods, and are in tension like the spokes of a bicycle wheel.

The buckets are of iron, and it will be observed that they stand out a little way from what is called the *sole* of the wheel, permitting a free circulation of air over the water in the buckets, which facilitates the discharge of the water from them when they reach the lower end of the breast. With this arrangement for the admission of air to the buckets, the latter are said to be "ventilated."

The high-breast wheel, like the overshot wheel, acts almost entirely by the weight of the water, and its efficiency is about the same.

In low-breast wheels the water acts on the wheel partly by impulse and partly by weight.

Breast and low-breast wheels have efficiencies varying from 50 to 80 per cent., being greater for large than for small wheels.

423. Undershot Wheels.—The undershot wheel acts entirely by the impulse of the water on its vanes. The older undershot wheels had radial vanes, as shown in Fig. 783, and on account of the loss of energy, due to shock, the efficiency of these wheels was only from 20 to 30 per cent., the maximum theoretical efficiency being only 50 per cent., the velocity of the vanes being then half that of the impinging stream (see Art. 415, p. 477).

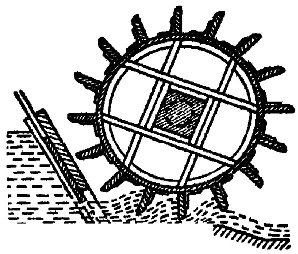


FIG. 783.

The undershot wheel was greatly improved by Poncelet, who curved the vanes, as shown in Fig. 784. In the Poncelet wheel the water enters without shock, leaves it with a small velocity in a nearly vertical direction, and during the whole time that the water is in the wheel it exerts an impulse on the vanes. The supply of water is regulated by a curved sluice A. The theory of the form of the vanes has been discussed in Articles 418 and 419, pp. 480–483, and the observations there made apply to Fig. 785, which shows the vanes of a Poncelet wheel in action, with the parallelograms of velocities as the water enters the wheel at B and leaves it at C. $BD = v$ is the direction and magnitude of the velocity of the water in the impinging stream. BE is tangential to the wheel at B, and equal to v_1, the velocity of the outer circumference of the wheel. Completing the parallelogram BEDH, the vane at B must be tangential to BH. The water glides up the vane with the relative velocity v_r, and returns, gliding down the vane, leaving it at C. At C the water has a velocity v_1 in the direction CL tangential to the wheel at C; it also has a velocity in the direction CK tangential to the vane at C, and slightly less than v_r, on account of loss by friction. Neglecting this loss, if CL be made equal to $v_1 = BE$, and $CK = v_r = BH$, and if the parallelogram CKNL be completed, then CN is the absolute velocity of the water as it leaves the wheel at C.

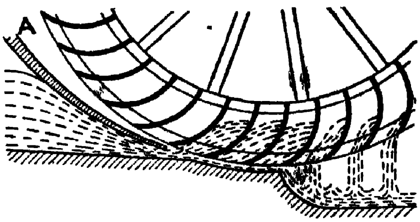


FIG. 784.

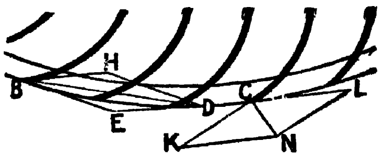


FIG. 785.

The efficiency of the Poncelet wheel is about 60 per cent.

Common undershot wheels with radial vanes should not be used for falls greater than 5 feet. Poncelet wheels are suitable for falls up to 7 feet.

A suitable diameter for undershot wheels is from two to four times the head due to the velocity of the impinging stream, and the linear velocity of the tips of the vanes should be about half that of the impinging stream.

424. Pelton Wheel.—The *Pelton wheel* is a development of the old *hurdy-gurdy*, which was introduced into the mining districts of California

about 1865. The hurdy-gurdy was a vertical wheel, having flat radial vanes, upon which a jet of water with high velocity impinged. The maximum theoretical efficiency of the hurdy-gurdy is only 50 per cent., and its actual efficiency from 25 to 35 per cent.

The substitution of curved buckets for the flat vanes was the great improvement which converted the hurdy-gurdy into the Pelton wheel. Fig. 786 shows a Pelton wheel consisting of a disc, to the periphery of

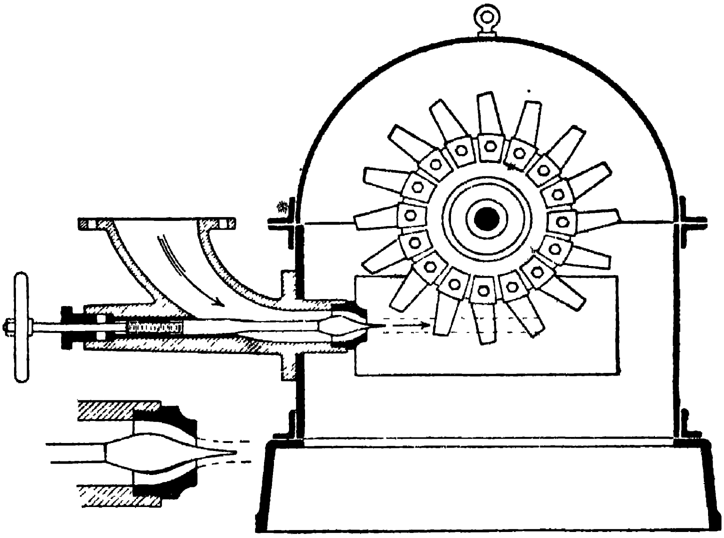


FIG. 786.

which the buckets are attached. The jet issues from a nozzle at the end of a pipe and strikes against the buckets, as shown. The form of the *Pelton bucket* is shown in Fig. 787, from which it will be seen that the jet is divided by a sharp ridge in the bucket, and is then gradually deflected through an angle slightly less than 180°. It is necessary to make the angle through which the jet is deflected less than 180°, in order that the returning stream may clear the bucket which follows. The *Doble bucket*, shown in Fig. 788, is an improvement on the Pelton bucket. The improvement consists in making the two compartments of the bucket of ellipsoidal form, and in cutting away a part of the outer lip to clear the jet as the bucket comes into action.

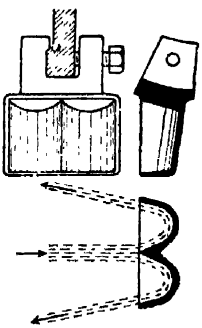


FIG. 787.

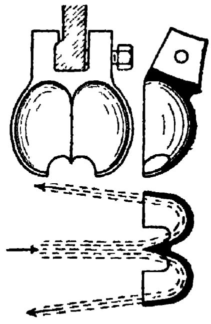


FIG. 788.

The disc of the wheel may be of cast-iron or steel, and the buckets may be of cast-iron or hard bronze.

Assuming a complete reversal of the jet, it is evident that if the velocity of the buckets is half that of the jet the absolute velocity of the

water on leaving the buckets is zero, and the hydraulic efficiency is unity. The actual efficiency of Pelton wheels is from 70 to 90 per cent.

The Pelton wheel is suitable for situations where a comparatively small amount of water at a high pressure or under a great head is available.

There are difficulties connected with the governing of Pelton wheels which may be here referred to. The opening through the nozzle may be varied by a curved stopper at the end of a screwed rod, which works in a nut, as shown in Fig. 786. The area through the nozzle, for a given energy of jet, depends, however, on the friction of the supply-pipe as well as on the head of water, and is determined in the manner discussed in Art. 409, p. 467. The central stopper, or needle as it is sometimes called, may be operated by a governor driven from the shaft of the wheel. A sudden throttling of the jet due to the action of the governor when there is a sudden reduction in the power required causes a sudden check on the flow of the water in the supply-pipe, and if this pipe is long the result is a water-hammer action, which may unduly strain the pipe. This difficulty may be got over by providing a spring-loaded relief valve. In another system of governing the nozzle is at the end of a short pipe, so jointed as to permit of the jet being deflected so that only part of it strikes the buckets. A difficulty with this system of governing, however, is that a very considerable force is required to deflect a jet moving at a high velocity.

The power of a Pelton wheel may be increased by having two or more nozzles, instead of one, directing jets in tangential directions at different parts of the circumference of the wheel.

425. Girard Impulse Wheel.—One form of the *Girard impulse wheel* is shown in Fig. 789. This wheel is mounted on a horizontal shaft A. The water enters through a pipe B, which bends over and

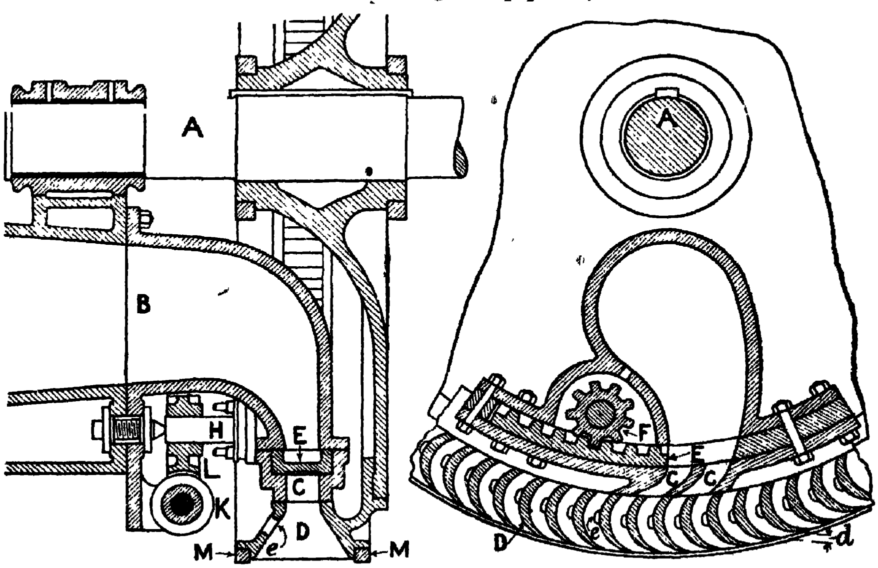


FIG. 789.

terminates opposite to one or more guide passages C, which direct the water on to the vanes D of the wheel. The quantity of water entering

the wheel is regulated by a sluice E, which has teeth on its upper face gearing with a pinion F, which is secured to the shaft H. A worm K gears with a worm wheel L, which is fixed to the shaft H. The worm K is fixed to a shaft operated by a governor or by hand.

The rim of the particular wheel illustrated runs at a high speed, over 100 feet per second, and it is strengthened by steel hoops M shrunk on. In some wheels of this type these steel hoops are made of much larger section than shown in Fig. 789, in order to increase the fly-wheel action, preventing a too rapid change of speed with change of load.

Owing to the greater obliquity of the vanes at exit than at entrance, the distance d between two consecutive vanes at exit is less than the distance between them at entrance, and to prevent the choking of the passage by the water the passage is widened transversely towards the circumference of the wheel, as shown in the left-hand view in Fig. 789.

In impulse wheels the water flows over the vanes under atmospheric pressure, and to ensure free access of air ventilating holes e are made through the sides at the back of the vanes, as shown.

426. Speed, Power, and Efficiency of Girard Impulse Wheel.—Referring to Fig. 790, r_1 and r_2 are the inner and outer radii respectively of the wheel. $c_1 = B_1C_1$ is the tangential velocity of the wheel at radius r_1. $c_2 = B_2C_2$ is the tangential velocity of the wheel at radius r_2. Obviously $c_1/r_1 = c_2/r_2$. $v_1 = B_1V_1$ is the absolute velocity of the water as it enters the wheel. $v_2 = B_2V_2$ is the absolute velocity of the water as it leaves the wheel.

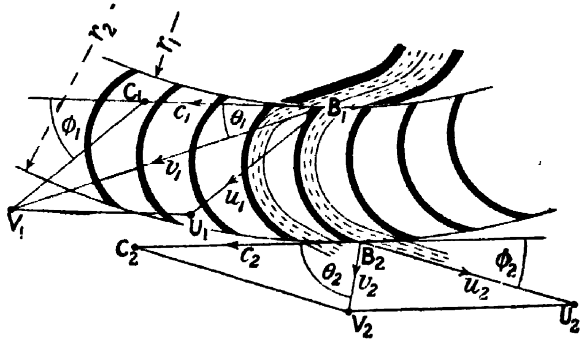


FIG. 790.

θ_1 = angle $C_1B_1V_1$. θ_2 = angle $C_2B_2V_2$. $B_1C_1V_1U_1$ and $B_2C_2V_2U_2$ are the parallelograms of velocities at entrance and exit respectively. $u_1 = B_1U_1$ is the relative velocity at entrance, and B_1U_1 is the direction of the tangent to the vane at entrance. $u_2 = B_2U_2$ is the relative velocity at exit, and B_2U_2 is the direction of the tangent to the vane at exit. ϕ_1 = angle $C_1B_1U_1$. ϕ_2 = the supplement of the angle $C_2B_2U_2$.

As the water in entering and passing through the wheel is under atmospheric pressure, the velocity v_1 depends only on the effective head at B_1, and is to be calculated from the formula $v_1 = \sqrt{2gH_1}$, where H_1 is the effective head.

If W is the weight of water entering the wheel per second, then, neglecting friction, the energy given to the wheel per second is $\frac{W}{2g}(v_1^2 - v_2^2)$. But by Art. 419, p. 482, the energy given to the wheel per second is also equal to $\frac{W}{g}(v_1c_1 \cos \theta_1 - v_2c_2 \cos \theta_2)$.

Therefore $v_1^2 - v_2^2 = 2(v_1c_1 \cos\theta_1 - v_2c_2 \cos\theta_2)$.

But $v_1^2 = c_1^2 + u_1^2 + 2c_1u_1 \cos\phi_1$,

and $v_2^2 = c_2^2 + u_2^2 - 2c_2u_2 \cos\phi_2$,

also $v_1 \cos\theta_1 = c_1 + u_1 \cos\phi_1$,

and $v_2 \cos\theta_2 = c_2 - u_2 \cos\phi_2$.

Substituting these equivalents in the equation

$$v_1^2 - v_2^2 = 2(v_1c_1 \cos\theta_1 - v_2c_2 \cos\theta_2),$$

the result is $u_1^2 - u_2^2 = c_1^2 - c_2^2$.

It is evident that the efficiency of the wheel will be greater the smaller v_2 is, and v_2 will be smaller the smaller ϕ_2 is. But since there must be a sufficient area of passage between the vanes at exit, ϕ_2 cannot be made indefinitely small. For a given value of ϕ_2 the velocity v_2 will have nearly its minimum value when u_2 is equal to c_2, and if u_2 be made equal to c_2, this leads to very simple relations between the various quantities. For since $u_1^2 - u_2^2 = c_1^2 - c_2^2$, it follows that if $u_2 = c_2$, then $u_1 = c_1$, and $\phi_1 = 2\theta_1$. Hence $c_1 = \frac{v_1}{2\cos\theta_1}$, and the angular speed of the wheel is $\omega = \frac{c_1}{r_1} = \frac{\;}{2r_1 \cos\theta_1}$.

The energy given to the wheel per second $= \frac{W}{2g}(v_1^2 - v_2^2)$, but since $u_2 = c_2$, $v_2 = 2c_2 \sin\frac{\phi_2}{2} = \frac{r_2v_1 \sin\frac{\phi_2}{2}}{r_1 \cos\theta_1}$, therefore energy per second $= \frac{Wv_1^2}{2g}\left\{1 - \left(\frac{r_2 \sin\frac{\phi_2}{2}}{r_1 \cos\theta_1}\right)^2\right\}$, and the horse-power of the wheel is this expression divided by 550.

The energy in the water per second as it enters the wheel is $\frac{Wv_1^2}{2g}$.

Hence the efficiency of the wheel is $1 - \left(\frac{r_2 \sin\frac{\phi_2}{2}}{r_1 \cos\theta_1}\right)^2$. The efficiency is therefore greater the smaller the angles ϕ_2 and θ_1.

In practice the angle θ_1 is generally between 20° and 25°, and ϕ_2 is generally between 15° and 20°. The ratio of r_2 to r_1 is generally between 1·15 and 1·25.

Taking friction into account the efficiency is about 80 per cent., and the efficiency is not reduced by diminishing the sluice opening when there is a reduction in the load.

The axis of the wheel may be either horizontal or vertical.

427. Jet Reaction Wheels.—The simplest form of the jet reaction wheel is that generally known as *Barker's mill*. Fig. 791 shows a Barker's mill constructed of ordinary wrought-iron or steel tubing. The vertical central tube AB has jointed to it two horizontal tubular arms CD and EF, which are opposite to one another. These arms are closed

at their outer ends. A "bend" H leading from the tank K enters a short distance into the tube AB, and serves as a bearing for AB, permitting the latter to rotate freely. A footstep bearing is arranged at the lower end of AB, as shown.

Water flows from the tank K through H into AB, and thence into the arms CD and EF. In CD and EF are orifices L and M, through which the water issues in jets perpendicular to the arms and horizontal, as shown. The reactions of the jets on the arms cause the latter to rotate, driving the central tube AB. The power developed may be taken off at the pulley N, which is secured to AB. Leakage at the joint at the upper end of AB is prevented by a simple gland and stuffing-box, as shown.

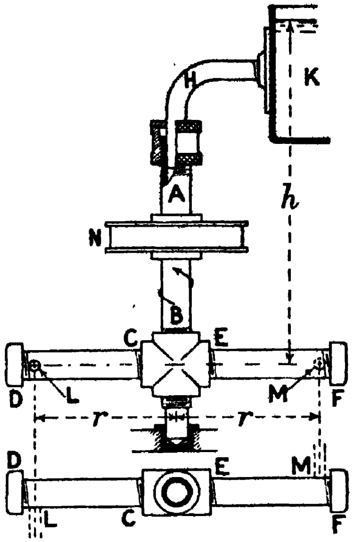


FIG. 791.

In *Whitelaw's turbine*, sometimes called the *Scotch turbine*, instead of the straight arms of the Barker's mill, there are casings of a more or less spiral form leading the water to the orifices, the casings contracting as they approach the orifices.

Jet reaction wheels are not now in practical use, but they are interesting from the student's point of view.

The theory of the jet reaction wheel is as follows. Referring to Fig. 791, let

h = static head of water at orifices.
r = distance of orifices from axis of wheel.
c = linear velocity of arms at radius r.
u = velocity of jets relative to arms.
W = weight of water passing through wheel per second.

The pressure exerted by the water in the neighbourhood of the orifices is due to the static head h and to the centrifugal force of the revolving water in the arms. The head, due to the centrifugal pressure at the orifices, is $\frac{c^2}{2g}$, and the total head at the orifices is therefore $h+\frac{c^2}{2g}$. Hence $u=\sqrt{2gh+c^2}$, neglecting losses.

The reaction of the jets on the arms at radius r is $\frac{W}{g}(u-c)$, and the work imparted to the wheel per second is $\frac{W}{g}(u-c)c$.

The efficiency is $\frac{W}{g}(u-c)c \div Wh = \frac{(u-c)c}{gh}$.

If $u=\sqrt{2gh+c^2}$, then $h=\frac{u^2-c^2}{2g}$, and the expression for the efficiency becomes $\frac{2c}{u+c}$.

A loss which is inevitable is the kinetic energy in the water as it leaves the wheel, and this amounts to $\frac{W(u-c)^2}{2g}$ per second.

428. Classification of Turbines.—A turbine consists of two main parts, the *wheel* or *runner*, and the stationary guides. The wheel consists of two plates or rings called *crowns*, between which lie numerous *vanes*. The guides direct the water on to the vanes of the wheel.

Turbines may be divided, according to the manner in which the water acts on the moving vanes, into two classes, namely, *impulse turbines* and *reaction turbines*. In impulse turbines the water does not fill the passages between the wheel vanes, and there being free access of air to these passages, the velocity of the water as it enters them is that due to the head. Also the energy of the water as it enters the wheel is entirely kinetic. In reaction turbines the water completely fills the passages between the guides and between the wheel vanes, and the velocity of the water at the entrance to the wheel may be greater or less than that due to the head there. Also the energy of the water as it enters the wheel is partly kinetic and partly pressure energy.

An impulse turbine must discharge into the atmosphere, and must therefore be clear of the tail race, but a reaction turbine may be completely immersed or *drowned*.

Another classification of turbines is according to the direction in which the water flows through the wheel. This leads to four classes. (1) *Outward flow* turbines, in which the direction of flow is radial and outwards. (2) *Inward flow* turbines, in which the direction of flow is radial and inwards. (3) *Parallel flow* or *axial flow* turbines, in which the direction of flow is parallel to the axis of the wheel. (4) *Mixed flow* turbines, in which the direction of flow is partly radial and partly axial, changing from one to the other on the vanes inside the wheel.

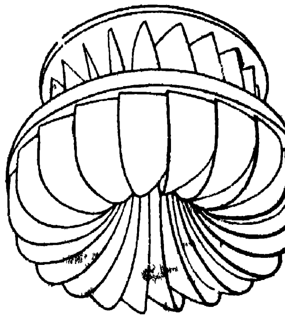


FIG. 792.

The various types are also frequently referred to by the names of the engineers who were identified with their introduction or improvement, as the *Fourneyron* turbine (radial outward flow), the *Francis* turbine (radial inward flow), and the *Jonval* turbine (parallel flow). Mixed flow turbines, in which the water enters in a radial inward direction and leaves in an axial direction, are largely used in America, and this type is often called the *American* turbine. Fig. 792 shows the wheel or runner of the *Victor* (American) turbine.

The *Girard* turbines are impulse turbines, and they may have either radial or axial flow.

429. Formulæ for Reaction Turbines.—The notation to be used in this Article is partly shown on Figs. 793, 794, and 795, which represent outward flow, inward flow, and parallel flow turbines respectively, and is the same as was used for the impulse wheel, Art. 426, p. 490.

Values of the Angles.—The angles θ_1 and ϕ_2 are assumed in designing a turbine. θ_1 varies from 10° to 25° in inward flow turbines, and from

15° to 25° in outward flow and parallel flow turbines. ϕ_2 varies from 10° to 25°.

Areas of Passages.—A_1 and A_2 = areas of guide passages and wheel

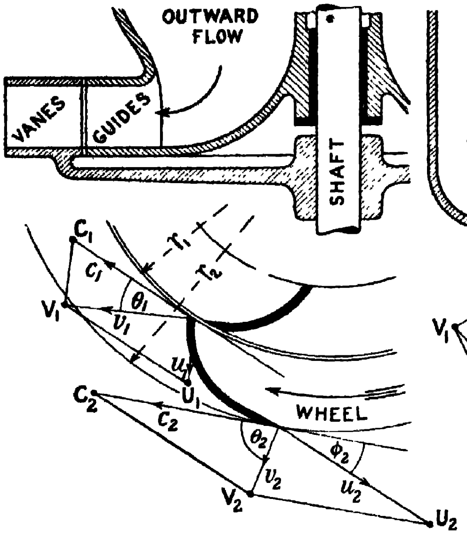


FIG. 793.

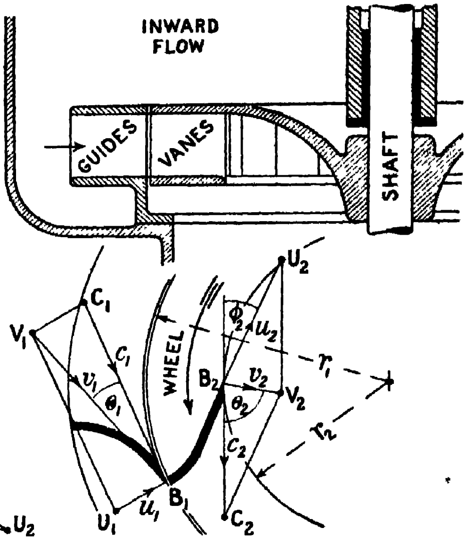


FIG 794.

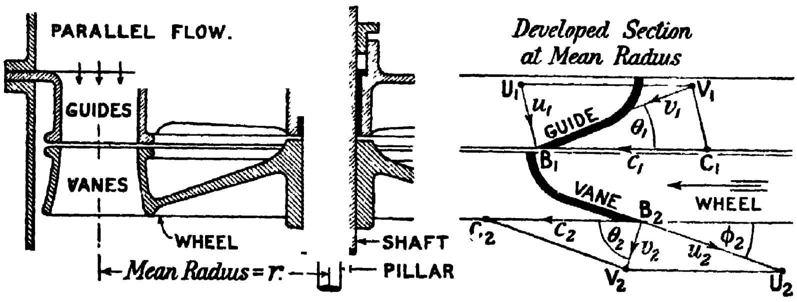


FIG. 795.

passages respectively at exit, measured at right angles to direction of flow.

If n is the number of passages, and b the distance between the crowns, then, referring to Fig. 796, the area of the passages referred to above is $A = ndb$. If t denotes the thickness of the guides or vanes, then, approximately,

$$\frac{n(d+t)}{2\pi r} = \sin\theta, \text{ and } A = (2\pi r \sin\theta - nt)b.$$

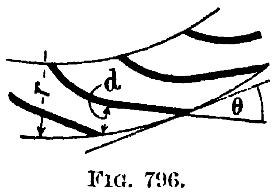


FIG. 796.

The ratio $A_1 \div A_2$ may be assumed in commencing the design of a turbine. $A_1 \div A_2$ varies from 0·5 to 1 in outward flow turbines, and from 0·6 to 1·5 in inward flow turbines. In parallel flow turbines $A_1 \div A_2$ is usually about 1.

Ratio of Radius r_1 to Radius r_2.—The ratio $r_1 \div r_2$ is also assumed. For outward flow $r_1 \div r_2$ varies from 0·7 to 0·85, and for inward flow from 1·2 to 2. In parallel flow turbines take $r_1 = r_2 = r$, the mean radius.

Result of Continuity of Flow.—Since the water completely fills the passages in flowing through them, it follows that $v_1 A_1 = u_2 A_2$.

Velocities of Whirl.—At entrance to wheel the velocity of whirl is $v_1 \cos \theta_1$, and at exit $v_2 \cos \theta_2$.

Work Imparted to Wheel.—If W = weight of water passing through the wheel per second, then by Art. 419, p. 482, the work imparted to the wheel per second is

$$\frac{W}{g}(c_1 v_1 \cos \theta_1 - c_2 v_2 \cos \theta_2).$$

Efficiency.—The energy available per second is Wh, where h is the available head of water. Hence the efficiency is

$$E = \frac{1}{gh}(c_1 v_1 \cos \theta_1 - c_2 v_2 \cos \theta_2).$$

The efficiency varies from 75 to 85 per cent., and may be taken, when unknown, at 80 per cent.

Velocity of Flow from Guide Passages (v_1).—First assume that the velocity of whirl at exit $v_2 \cos \theta_2 = 0$.

The angle θ_2 is then 90°, and $u_2 = \dfrac{c_2}{\cos \phi_2}$.

Also $v_1 = \dfrac{A_2}{A_1} u_2 = \dfrac{A_2}{A_1} \cdot \dfrac{c_2}{\cos \phi_2}$, but $c_2 = \dfrac{r_2}{r_1} c_1$.

Therefore $v_1 = \dfrac{A_2}{A_1} \cdot \dfrac{r_2}{r_1} \cdot \dfrac{c_1}{\cos \phi_2}$, and $c_1 = \dfrac{A_1}{A_2} \cdot \dfrac{r_1}{r_2} \cdot v_1 \cos \phi_2$.

Work imparted to wheel per second $= \dfrac{W}{g} c_1 v_1 \cos \theta_1$.

Efficiency $E = \dfrac{1}{gh} \cdot c_1 v_1 \cos \theta_1 = \dfrac{v_1^2}{gh} \cdot \dfrac{A_1}{A_2} \cdot \dfrac{r_1}{r_2} \cdot \cos \theta_1 \cos \phi_2$.

Hence $v_1 = \sqrt{\dfrac{E}{2\dfrac{A_1}{A_2} \cdot \dfrac{r_1}{r_2} \cos \theta_1 \cos \phi_2}} \sqrt{2gh} = K_1 \sqrt{2gh}$,

where $K_1 = \sqrt{\dfrac{E}{2\dfrac{A_1}{A_2} \cdot \dfrac{r_1}{r_2} \cos \theta_1 \cos \phi_2}}$ may be called the *coefficient of velocity*.

If instead of assuming that $\theta_2 = 90°$ it be assumed that $u_2 = c_2$, then it may be left as an exercise to the student to show that

$$K_1 = \sqrt{\frac{E}{2\dfrac{A_1}{A_2}\left(\dfrac{r_1}{r_2} \cos \theta_1 + \dfrac{A_1}{A_2} \cos \phi_2 - \dfrac{A_1}{A_2}\right)}}.$$

Wheel Speed.—Assuming $\theta_2 = 90°$, it has been shown that

$$c_1 = \frac{A_1}{A_2} \cdot \frac{r_1}{r_2} \cdot v_1 \cos \phi_2, \text{ therefore } c_1 = \frac{A_1}{A_2} \cdot \frac{r_1}{r_2} \cdot \cos \phi_2 K_1 \sqrt{2gh} = K_2 \sqrt{2gh},$$

$$\text{where} \qquad K_2 = \frac{A_1}{A_2} \cdot \frac{r_1}{r_2} \cos \phi_2 \sqrt{\frac{E}{2\frac{A_1}{A_2} \cdot \frac{r_1}{r_2} \cdot \cos \theta_1 \cos \phi_2}}$$

$= \sqrt{\dfrac{E}{2 \cos \theta_1} \cdot \dfrac{A_1}{A_2} \cdot \dfrac{r_1}{r_2} \cos \phi_2}$ may be called the *coefficient of wheel speed.*

If instead of assuming $\theta_2 = 90°$ it be assumed that $u_2 = c_2$, then it follows that $K_2 = \dfrac{r_1}{r_2}\sqrt{\dfrac{E}{2\left(\dfrac{A_2}{A_1} \cdot \dfrac{r_1}{r_2} \cos \theta_1 + \cos \phi_2 - 1\right)}}$.

Effective or Brake Horse-power $= \dfrac{WEh}{550}$.

430. Use of Suction Tube for Reaction Turbines.—Since a reaction turbine works full of water, it is not necessary that it should be placed at the level of the tail water in order to utilise the full head. A reaction turbine may with advantage be placed at a height, less than the height of the water barometer, above the tail water, provided that it discharges into a pipe which, running full, opens under the tail water. The advantages of this arrangement are that a shorter shaft is necessary, and the turbine is more accessible.

Exercises XXIX.

1. The effective horse-power of a vertical water wheel is 28, and its efficiency is 70 per cent. If the total fall is 20 feet, how many gallons of water must be delivered to the wheel per minute?

2. The head race of a vertical water wheel is 5 feet wide, and the water in it is 6 inches deep, and has a velocity of 10 feet per second. The total fall is 30 feet, and the efficiency of the wheel is 75 per cent. What is the effective horse-power of the wheel?

3. The stream impinging on the vanes of a common undershot water wheel passes through a sluice opening 6 inches deep and 5 feet wide. The head of water is 4 feet 6 inches Taking the coefficient of discharge for the sluice opening at 0·62, and the efficiency at 30 per cent., what is the useful horse-power of the wheel?

4. If in a Poncelet wheel the water enters in a direction bisecting the angle θ between the tangents to the wheel and vane at the tip of the latter, and if the points of entrance and exit are at the same level, show that, so far as the action of the water on the vanes is concerned, the efficiency is equal to $1 - \tan^2 \frac{\theta}{2}$, the friction of the water on the vanes being neglected.

5. The centres of the buckets of a Pelton wheel move in a circle 3 feet in diameter. The actual head of water for the jet is 2000 feet, and the diameter of the jet is ½ inch. The wheel makes 1000 revolutions per minute, and develops 80 horse-power, using 28 cubic feet of water per minute. Determine, (*a*) the resultant efficiency, (*b*) the loss of head estimated at the jet, and (*c*) the ratio of the mean velocity of the buckets to the actual velocity of the jet.

6. In a series of brake tests of a small Pelton wheel the following tabulated results were obtained :—

W	6·5	5·7	5·4	4·7	4·4	3·5	2·9	2·1	1·0	0·0
N	960	1360	1480	1800	1900	2240	2520	2800	3280	3580

where W = effective load in lbs. on brake lever at 12 inches from axis of wheel, and N = speed of wheel in revolutions per minute. The weight of water used in each test was 41·5 lbs. per minute, and the pressure of the water was 700 lbs. per square inch in the pipe behind the orifice. The diameter of the orifice was 0·0835 inch. Complete the above table by adding the brake horse-power and the efficiency per cent. Plot the brake horse-power and efficiency on a speed base. Scales.—Horse-power, 2 inches to 1 horse-power; efficiency, 1 inch to 20 per cent.; speed, 1 inch to 500 revolutions per minute. State the maximum brake horse-power and the maximum efficiency.

7. Show that the efficiency of a Pelton wheel is a maximum, neglecting frictional and other losses, when the velocity of the cups equals half the velocity of the jet. 25 cubic feet of water are supplied per second to a Pelton wheel through a nozzle, the area of which is 44 square inches. The velocity of the cups is 41 feet per second. Determine the horse-power of the wheel, taking a reasonable value for the efficiency. [Inst.C.E.]

8. A Pelton wheel is to run at 900 revolutions per minute. The head of water is 720 feet, and the maximum water supply is 15 cubic feet per minute. Determine the diameter of the wheel, the diameter of the nozzle, and the maximum power developed, assuming an over-all efficiency of 0·8. [U.L.]

9. Explain why it may happen that when the opening through the nozzle of a Pelton wheel has a certain area the power of the wheel may be diminished by increasing and also by decreasing the opening.

10. The following particulars relate to a Girard impulse wheel. Using the notation of Art. 426, p. 490, $\theta_1 = 19°$, $\phi_1 = 36°$, $\phi_2 = 15°$, $r_1 = 4$ feet, $r_2 = 4·6$ feet. Total head = 500 feet. Absolute velocity of water at entrance = 85 per cent. of theoretical velocity due to total head. Volume of water entering wheel per second = 8 cubic feet. Determine the velocities v_1, c_1, c_2, u_1, u_2, and v_2 in feet per second, also the angle θ_2, the speed in revolutions per minute, and the horse-power of the wheel, neglecting losses in the wheel itself.

11. In a Girard impulse wheel, using the notation of Art. 426, p. 490, $\theta_1 = 20°$, $\phi_1 = 40°$, $\phi_2 = 15°$, $r_1 = 2$ feet, $r_2 = 2·5$ feet, $v_1 = 80$ feet per second, and water passing through wheel per second = 25 cubic feet. Determine the velocities c_1, c_2, u_1, u_2, and v_2 in feet per second, also the angle θ_2, the speed in revolutions per minute, and the horse-power of the wheel, neglecting losses in the wheel itself.

12 A simple reaction wheel of the Barker's mill type is supplied with water at a head of 10 feet. The combined areas of the orifices amount to 40 square inches, and the velocity of the centres of the orifices is 24 feet per second. Find the horse-power if the net efficiency is 60 per cent., and find also the hydraulic efficiency. [U.L.]

13. Certain experiments with a jet reaction wheel showed that the maximum efficiency was obtained when $c = \sqrt{2gh}$ (using the notation of Art. 427, p. 491). Taking the coefficient of velocity for the orifices as 0·95, calculate the maximum efficiency and the percentage of the energy due to the head h which is carried away by the water leaving the wheel.

14. A parallel flow impulse turbine works under a head of 64 feet. The water is discharged from the wheel in an axial direction with a velocity due to a head of 4 feet. The circumferential speed of the wheel at its mean diameter is 40 feet per second. Neglecting all frictional losses, determine the mean vane and guide angles. [U.L.]

15. The rim of an inward flow turbine moves at a speed of 30 feet per second, and the vanes are there at right angles to the rim. Water enters the rim with a radial velocity of 5 feet per second. If the water is to enter without shock, what must be the angle between the rim and the guide blades? Find the weight of water entering per second if the circumferential area of all the openings of the rim is 2·4 square feet. [B.E.]

16. In an inward flow turbine the water enters the inlet circumference, 2 feet diameter, at 60 feet per second, and at 10° to the tangent to the circumference. The water leaves the inner circumference, 1 foot diameter, with a radial velocity of 5 feet per second. The peripheral velocity of the inlet surface of the wheel is 50 feet per second. Find the angles of the vanes at the inlet and outlet surface. [Inst.C.E.]

17. An inward flow turbine wheel works under a head of 60 feet, and makes 380 revolutions per minute. The diameter of the outer circumference of the wheel is 24 inches, and of the inner circumference 12 inches. The velocity of the water entering the wheel is 44 feet per second, and the angle it makes with the tangent to the wheel is 10°. Assuming the radial velocity of flow through the wheel to be constant, and that the water leaves the wheel in a radial direction, determine the direction of the tangent to the vane of the wheel at the inlet and outlet. Sketch a suitable form of vane. Determine the hydraulic efficiency of the turbine.

18. Using the notation of Art. 429 and the result proved in Art. 446, p. 516, apply Bernoulli's theorem to show that in a radial flow reaction turbine

$$\frac{v_1^2}{2g}+\frac{u_2^2-u_1^2}{2g}-\frac{c_2^2-c_1^2}{2g}=h,$$

where h is the available or effective head at the inlet surface.

Show also that in an axial flow reaction turbine

$$\frac{v_1^2}{2g}+\frac{u_2^2-u_1^2}{2g}=h+h_1,$$

where h_1 is the depth of the wheel.

19. The supply of water for an inward flow reaction turbine is 500 cubic feet per minute, and the available head is 40 feet. The vanes are radial at the inlet, the outer radius is twice the inner, the constant velocity of flow is 4 feet per second, and the revolutions are 350 per minute. Find the velocity of the wheel, the guide and vane angles, the inner and outer diameters, and the width of the bucket at inlet and outlet. [U.L.]

CHAPTER XXX

PUMPS

431. Distinction between a Piston, a Bucket, and a Plunger.—A *piston* is generally a cylindrical piece which slides backwards and forwards inside a hollow cylinder. A piston may be moved by the action of fluid pressure upon it, as in a steam-engine or as in certain types of water pressure motors. A piston may, however, be used to give motion to a fluid, as in certain types of pumps. A piston is usually attached to a rod called a *piston-rod.* Numerous forms of packing are used to prevent leakage past the piston. In the piston shown in Fig. 797 *cup-leathers* are

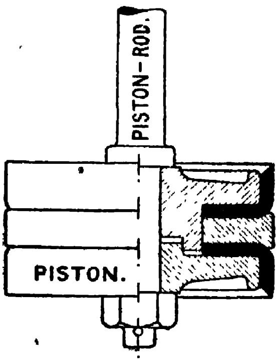


FIG. 797.

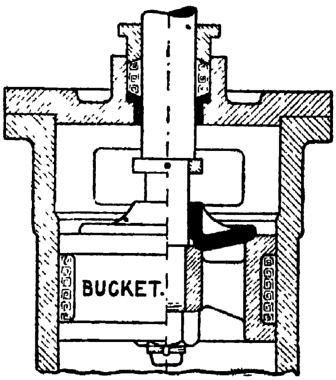


FIG. 798.

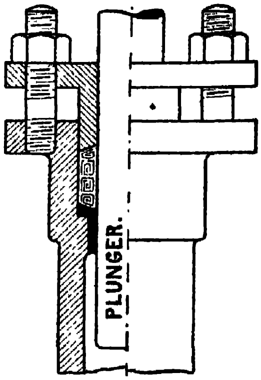


FIG. 799.

used for packing; the pressure of the water acting on the inside of the cup presses the leather outwards against the cylinder.

A *bucket* (Fig. 798) is a piston provided with one or more valves which permit of the fluid passing through it in one direction.

A *plunger* (Fig. 799) may be looked upon as a piston having the same diameter as its piston-rod.

432. Bucket Pump.—Referring to Fig. 800, AB is a cylinder or barrel, in which is made to reciprocate a bucket C. A pipe DE, called the *suction pipe*, leads from the lower end of the barrel and dips into the water which the pump is required to raise. A pipe FH, called the *delivery* pipe, leads from the top of the barrel to the vessel into which the water is to be delivered. There are three valves, all opening upwards, one in the bucket, one at the top of the suction pipe, called the *suction valve,* and one at the bottom of the delivery pipe, called the *delivery valve.*

The action of the pump is as follows. The bucket being at the bottom of its stroke, and the barrel and pipes full of air at atmospheric

pressure, the bucket is pulled upwards, the valve in it being kept shut by its own weight and the excess pressure of the air above it. As the space between the bucket and the suction valve increases, the air in that space expands and its pressure falls. This enables the pressure of the air in the suction pipe to lift the suction valve, and a portion of that air then flows into the barrel below the bucket. The pressure of the air in the suction pipe therefore falls below the pressure of the atmosphere, and in consequence water is forced into the suction pipe from below by the pressure of the atmosphere outside until the water stands at such a height that the pressure at E due to that column of water, and the pressure of the air above it, is equal to the pressure of the atmosphere. During the downward stroke of the bucket the air beneath it is compressed, the suction valve having closed, and when the compression is sufficient, the bucket valve opens and a portion of the air beneath the bucket passes through it into the space above. In the next upward stroke the air beneath the bucket is still further rarefied, and the water is forced by the pressure of the atmosphere to a greater height in the suction pipe. This goes on until the water gets into the barrel. The bucket in descending then enters the water, part of which passes through the bucket to the space above. The whole space below the bucket within the barrel and suction pipe is now full of water, and subsequent up strokes of the bucket lift the water higher and higher, until it reaches the top of the delivery pipe. After this, during each up stroke, a volume of water equal to the volume swept through by the bucket is discharged through the delivery pipe.

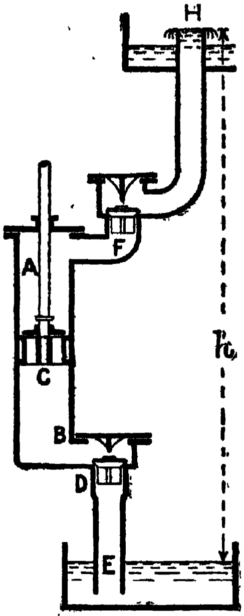


FIG. 800.

Since the water beneath the bucket is held up by the pressure of the atmosphere it is evident that the bucket in its highest position must not be at a greater height above E than the height of the water barometer. For a pressure of 14·7 lbs. per square inch the height of the water barometer is 34 feet. The height of the bucket above the level of the water at E is called the *suction head*. In practice the suction head is generally not more than about 26 feet.

It may be observed that in the pump just described the delivery valve is not absolutely necessary, but during the down stroke of the bucket it acts as a check on the suction valve in holding up the column of water.

433. Force required to Work a Bucket Pump.—Once the barrel and pipes of the pump are fully charged with water it is evident that, neglecting the volume of the pump-rod, the volume of water delivered during each up stroke of the bucket is equal to the volume swept through by the bucket in one stroke. Let a = area of bucket in square feet; l = length of stroke in feet; h = total height through which the water is raised, in feet; P = force (in lbs.) required to lift the bucket, neglecting friction and the weight of the bucket and bucket-rod.

Weight of water raised in one up stroke $= 62{\cdot}3al$.

Work done in one up stroke $= 62{\cdot}3alh = Pl$.

Therefore $P = 62{\cdot}3ah$. That is, the pull on the pump-rod is equal to the weight of a column of water, whose base is equal to the area of the bucket, and whose height is the total head. Hence when friction is neglected, P is independent of the diameters of the suction and delivery pipes.

During the downward stroke no water is raised, and only friction has to be overcome. Strictly speaking, a volume of water is discharged during the down stroke equal to the additional volume of pump-rod entering the barrel, but in the pump under consideration this may be neglected.

Considering the effect of the pump-rod, if a_1 = effective area of bottom of bucket $= 0{\cdot}7854d^2$, where d is the diameter of the barrel, a_2 = effective area of top of bucket (a_2 is less than a_1 by the area of the section of the rod), h_1 = suction head, h_2 = delivery head, then $P = 62{\cdot}3(a_1h_1 + a_2h_2)$.

434. Plunger Pump.—Fig. 801 shows a short stroke plunger pump provided with ball valves. S is the suction valve, and D the delivery valve. The action of this pump during the out or suction stroke is the same as that under the bucket of the bucket pump when the bucket is ascending. During the in or delivery stroke the air within the pump is compressed and a portion of it is discharged through the delivery valve, and when the pump becomes charged with water a volume of water equal to the displacement of the plunger is discharged through the delivery valve during each delivery stroke. By the *displacement of the plunger* is meant the volume equal to the area of the cross section of the plunger multiplied by the length of its stroke.

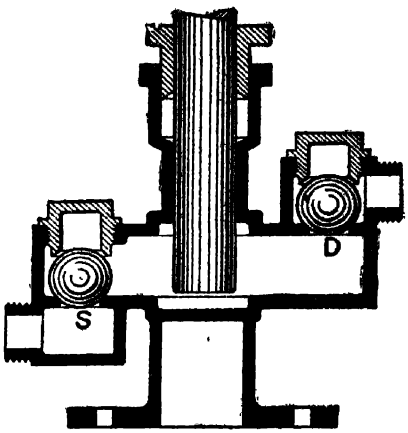


FIG. 801.

A duplex pump, consisting of two plunger pumps side by side and delivering into the same pipe, is shown in Fig. 802. The two plungers

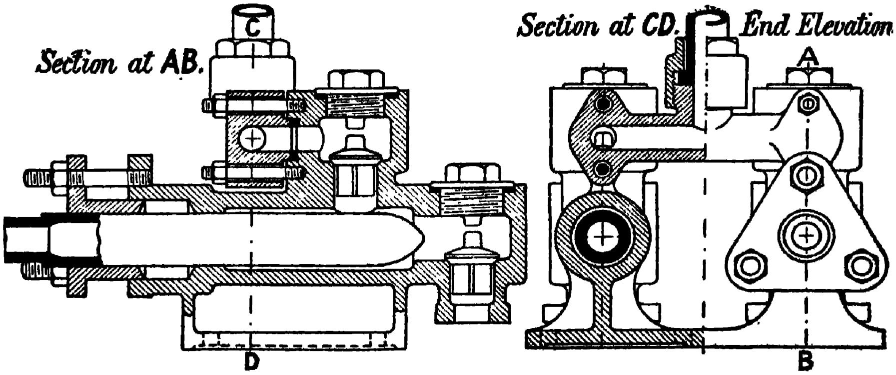


FIG. 802.

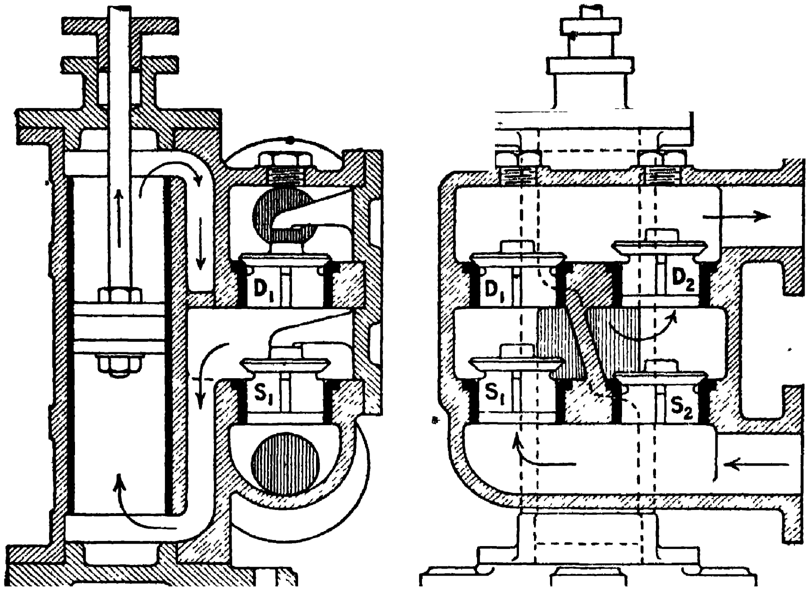


FIG. 803.

are driven so that the delivery stroke of the one takes place during the time of the suction stroke of the other. The result is a continuous flow of water in the main delivery pipe. The particular pump shown in Fig. 802 is used for delivering water at a high pressure, such as is required by hydraulic machines. Another form of high pressure pump is described in Art. 437, p. 503.

435. Double-Acting Piston Pump.—Fig. 803 shows one form of *double-acting piston pump*. There are two suction valves S_1 and S_2, and two delivery valves D_1 and D_2. The action of this pump will be readily understood by an inspection of the illustration. There is a continuous flow of water through the suction pipe, and also through the delivery pipe.

436. Combined Plunger and Bucket Pump.—A plunger pump is single-acting, discharg-

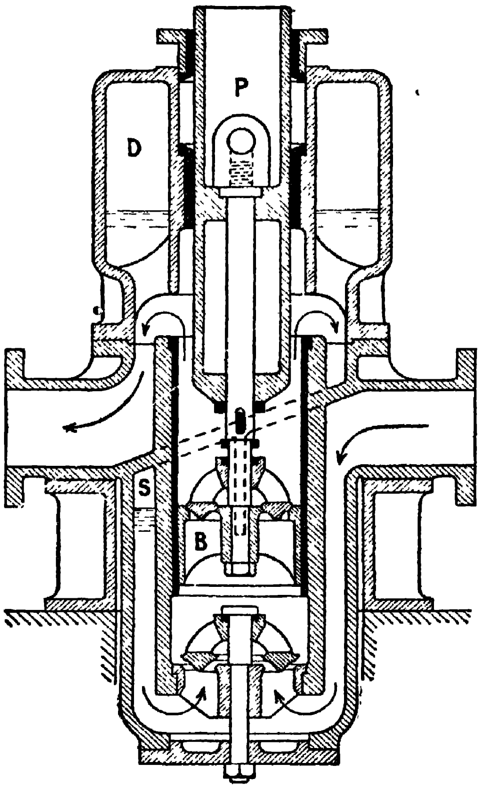


FIG. 804.

ing water during the *inward* stroke. A bucket pump is also single-acting, discharging water during the *outward* stroke. By combining these two a double-acting pump is obtained, and this has two valves only. Fig. 804 shows a compact form of combined plunger and bucket pump,* designed by Mr. Arthur Rigg. P is the plunger, and B the bucket. D and S are the delivery and suction air chambers respectively. The valves are of the annular seated ring type, provided with rubber-packed stop sockets.

The area of the cross section of the plunger is half that of the barrel. Hence, during the up stroke, half of the water raised by the bucket goes to fill the space left by the plunger, the other half going to the delivery pipe. During the down stroke the plunger displaces the other half of the water raised by the bucket.

437. Continuous Delivery Pump for High Pressures.—For charging

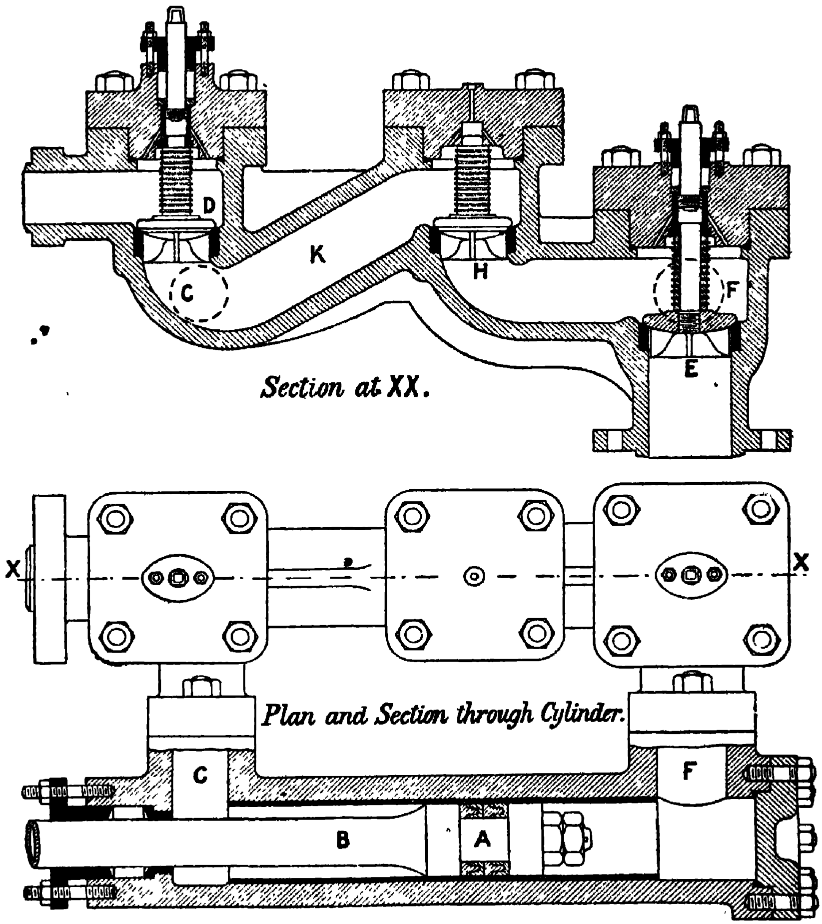


FIG. 805.

hydraulic acumulators, in which the pressure of the water may be from 700 to 7000 lbs. per square inch, the type of pump shown in Fig. 805 is

* *The Mechanical Engineer*, January 18, 1908.

generally used. The piston A has an area twice that of the piston-rod B. During the outward stroke the water to the left of the piston is discharged through the passage C and valve D to the accumulator, and water at the same time enters by the suction valve E and passage F, and fills the space to the right of the piston. During the inward stroke the water to the right of the piston is discharged through the passage F and valve H into the passage K, but only half of this water goes through the valve D to the accumulator; the other half goes by the passage C to the annular space in the barrel or cylinder to the left of the piston. A volume of water, equal to half the volume swept through by the piston, is evidently discharged to the accumulator during each stroke. The valve D is not absolutely necessary, but it acts as a check on the others when the pump is not working. To make the valves close promptly they are loaded with springs, which consist of rubber rings separated by metallic washers.

438. Air and Vacuum Chambers.—In a pump of the single-acting type water is delivered during alternate strokes only, and the flow through the delivery pipe is therefore intermittent. The result of this is that in the neighbourhood of the delivery valve there is a great fluctuation of pressure due to the inertia of the water, and a consequent series of shocks. To remedy this defect an *air chamber* A (Figs. 806 and 807) is placed over or near the delivery valve D.

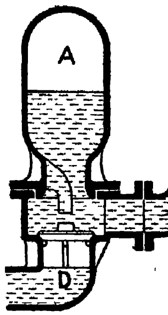


FIG. 806.

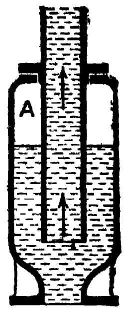


FIG. 807.

The theory of the action of the air chamber is as follows. Referring to Fig. 808, the base line is a time base. The height of the straight line MN above the base represents the static pressure of the water due to the head in the delivery pipe. The height of the line marked "total

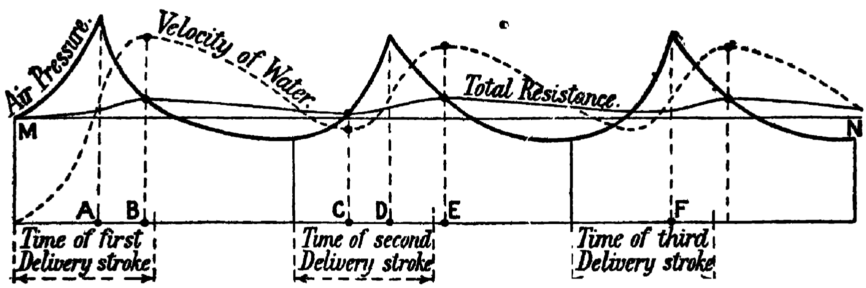


FIG. 808.

resistance" above MN represents the additional pressure required to overcome the friction in the delivery pipe. Suppose that the pump starts from rest. The water in the delivery pipe being at rest, the pressure of the air in the air chamber is equal to the static pressure of the water. At the beginning of the first delivery stroke the delivery valve opens, and the water, flowing through, finds two passages open to it,

one leading into the delivery pipe, and the other leading into the air chamber. To flow into the delivery pipe the entering water must exert a pressure greater than that due to the head of water in that pipe in order to overcome the inertia of the column of water and the friction in the pipe, but to enter the air chamber the resistance is practically only that due to the air pressure in it, and this pressure is only the static pressure due to the head in the delivery pipe. Hence the water coming from the pump barrel, taking the path of least resistance, enters the air chamber. As the water enters the air chamber the air in it is compressed and the pressure rises gradually. This gradually increasing air pressure acts of course on the column of water in the delivery pipe, and sets it in motion gradually without shock. Up to the time A (Fig. 808) all the water going into the delivery pipe has come direct from the pump barrel, but the greater portion of the water coming from the pump barrel has gone into the air chamber. At the time A the velocity has increased until the flow through the delivery pipe is equal to the discharge from the pump, and after this a diminishing pressure is sufficient to keep up the delivery. The air now forces water from the air chamber, and in doing so it increases in volume and falls in pressure. At the time B the air pressure has fallen until it just equals the resistance. Up to this point the driving force on the water has been greater than the resistance, and therefore the velocity of the water has been increasing, and is now a maximum. After the time B the water continues to flow from the air vessel, although the air pressure is now less than the resistance, because of the kinetic energy in the moving water. At the beginning of the second delivery stroke the water from the pump has again two passages open to it. A quantity sufficient to keep up the flow at the now reduced velocity will go into the delivery pipe, but to send a greater quantity would mean increasing the velocity, and therefore increasing the pressure above that in the air chamber, hence the remainder of the water enters the air chamber, and the pressure increases gradually. At the time C the air pressure is just equal to the resistance. Between B and C the air pressure, which is the driving force, has been less than the resistance, and the velocity has therefore been diminishing, and will have reached a minimum at C. Between C and D the air pressure increases, and at D the flow through the delivery pipe is again equal to the discharge from the pump. At E the velocity is again a maximum, and at F the flow through the delivery pipe is again equal to the discharge from the pump, and so on.

It is seen, therefore, that the air chamber makes the flow of water through the delivery pipe continuous, and shocks due to sudden changes of pressure are eliminated.

The positions of the points A, B, C, etc., will depend on the character of the motion of the bucket or plunger, the volume of air in the air chamber, and the friction in the delivery pipe.

The volume of the air chamber varies greatly in practice, being from two to six times the displacement of the bucket or plunger per stroke, and sometimes as much as ten times.

An air chamber is not so necessary on a double-acting pump, but it is still advantageous, because the velocity of the piston not being uniform, the discharge through the delivery valves is not uniform. The capacity

of the air chamber for a double-acting pump may, however, be less than for a single-acting one.

On the suction side of a pump the water in the suction pipe requires time to acquire, under the action of the pressure of the atmosphere, sufficient velocity to make it flow into the pump barrel and completely fill the space left by the bucket or plunger; and when the suction pipe is long, or when the speed of the pump is high, the amount of water entering the barrel during the suction stroke may not be sufficient to fill it. Again, at the end of the suction stroke the suction valve is suddenly closed, and the water in the suction pipe is suddenly brought to rest and a shock is produced. To remedy these defects a *vacuum chamber* V (Fig. 809) is placed on the suction pipe near to the suction valve S. This vacuum chamber is not entirely devoid of air. When the water is at rest the pressure of the air in the vacuum chamber, together with the pressure due to the head of water in the suction pipe, is equal to the pressure of the atmosphere.

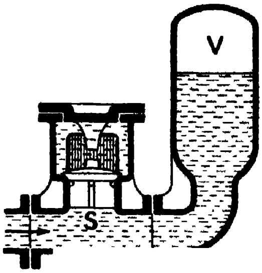


FIG. 809.

The theory of the action of the vacuum chamber is similar to that of the air chamber, already discussed, but while the air chamber converts the intermittent discharge of the pump into a continuous flow in the delivery pipe, the vacuum chamber converts a continuous flow in the suction-pipe into an intermittent flow in the pump.

The volume of the vacuum chamber may be about half that of the air chamber.

439. Pump Valves.—In nearly all pumps in which valves are essential, the valves are operated automatically by the pressure of the water passing through them. The valves permit the water to pass freely in one direction, but when the actuating force is removed, the valves close and prevent the return of the water. Kinematically, these valves are the same as the pawl which permits a ratchet wheel or rack to move in one direction only.

There are many designs of valves in use in pumps, but it will only be necessary to refer to a few of them here. A simple *conical direct lift valve*

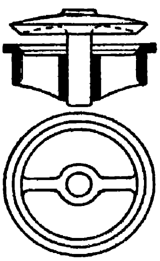


FIG. 810.

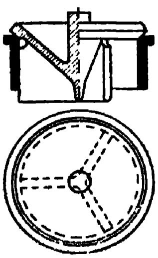


FIG. 811.

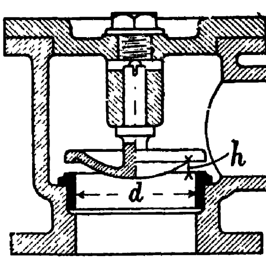


FIG. 812.

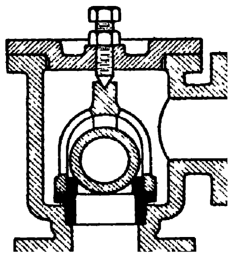


FIG. 813.

is shown in Fig. 810. The body of this valve is a slightly arched disc with a conical edge, which forms the *face* of the valve. The valve face beats on a corresponding conical *seat*, formed on a bush or on a part of

the valve casing. The valve is guided as it rises or falls by means of the central stem, forming part of the valve body, which slides in a hole in a bridge stretching across the opening below the seat. The amount of lift of the valve is determined by a stop on the casing above the valve. A modification of this type of valve is shown in Fig. 811. Here the body is made conical so as to direct the flow of water more gradually towards the opening, and thus reduce shock. This valve is guided by three wings cast on it, which slide in the opening below the seat. In valves with conical faces and seats, the slant side of the cone is usually inclined at 45° to its axis.

The valve shown in Fig. 812 differs from the one shown in Fig. 810, in having its face and seat flat, and in having the central stem above instead of below the valve. This stem slides in a guide forming part of the valve casing. The interior of this guide is in free communication with the interior of the casing through the small holes shown at the top, otherwise the stem would not rise and fall freely in the guide. The final grinding of the valve on its seat should be done when the guide is in position. The lift of the valve is limited by the collar on the stem striking the lower end of the guide.

Fig. 813 shows a *ball valve*. The ball is guided and its lift determined by the cage surrounding it.

In the valves just described the width of the seat may be as small as $\frac{1}{32}$ inch, and it is sometimes as much as $\frac{1}{2}$ inch. The narrower the seat, the easier is it to make the valve tight, but the area of the seat must be sufficient to prevent the crushing of the material of the valve or seat. These valves are generally made of brass or gun-metal.

Referring to Fig. 812, where the seat is flat, d is the diameter of the valve, and h its lift. The lateral opening through the valve is πdh, and the area through the seat is $\frac{\pi}{4}d^2$, hence when these two are equal, $h = \frac{1}{4}d$. The valve is therefore full open when the lift is one quarter of the

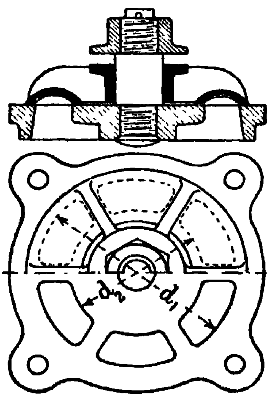


FIG. 814.

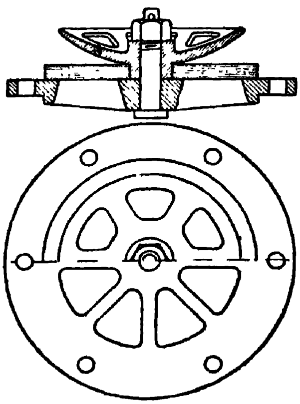


FIG. 815.

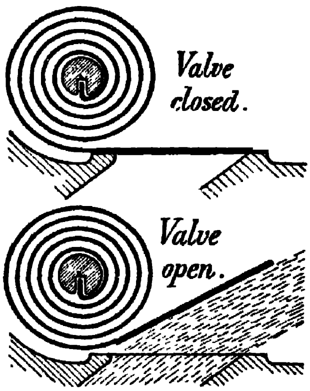


FIG. 816.

diameter. In practice the lift of single-beat metal valves working on metal seats, and actuated by the fluid, is generally considerably less than one quarter of the diameter, especially when the speed of the pump and

the pressure of the water are high. In the valves shown in Figs. 810 and 811, the area through the seat is less than $\frac{\pi}{4}d^2$, on account of the presence of the arrangements for guiding the valve. These valves are therefore full open when the lift is something less than $\frac{1}{4}d$.

An annular valve which has two seats is shown in Fig. 814. If d_1 and d_2 are the diameters of the annular opening between the seats, and h the lift of the valve, then, neglecting the effect of the ribs between the seats, the valve is full open when $\frac{\pi}{4}(d_1^2 - d_2^2) = \pi(d_1 + d_2)h$, that is, when $h = \frac{1}{4}(d_1 - d_2)$. If $d_2 = \frac{1}{2}d_1$, then $h = \frac{1}{8}d_1$.

The *india-rubber disc valve* is shown in Fig. 815. The thickness of the india-rubber is generally $\frac{3}{8}$ inch to $\frac{1}{2}$ inch for small valves, and may be as much as $\frac{7}{8}$ inch in the largest sizes. The area of the seat or grating in contact with the india-rubber should be sufficient to prevent the pressure between them exceeding 40 lbs. per square inch. The perforated guard limits the angular lift of the disc to about 30°.

The *Gutermuth valve*, shown in Fig. 816, is an ingenious form of flap valve. This valve is made from a sheet of special bronze of high tenacity. Part of the sheet forms a spiral coil, the inner end of which is turned over and enters a slot in a spindle. The flat or uncoiled part of the sheet forms the valve proper, and this is thicker than the rest. The projecting ends of the spindle are rigidly held in bearings, so that the flap is always in its correct position over the port. Before clamping down, the spindle is rotated until the spring of the coil is of the necessary stiffness. The advantages claimed for this valve are, (1) the port is entirely uncovered with a relatively small deflection of the metal of the valve, (2) quite a small force exerted by the water is sufficient to deflect the valve, (3) the valve closes promptly when the flow ceases.

440. Fluctuation of Delivery in Crank-driven Pumps.—In many cases the plunger or piston of a pump is driven through a crank and connecting-rod, the crank being fixed to a shaft which has uniform angular velocity. In other cases the plunger or piston is connected directly to the piston-rod of a steam cylinder, and there is a crank shaft whose crank is also connected to the piston-rod by a connecting-rod. On the crank shaft is a heavy fly-wheel, which causes the angular velocity of the shaft to be fairly uniform.

In all such cases the velocity of the plunger or piston varies during each stroke in a well-defined manner, and the curve which represents the variation of the plunger or piston velocity may be constructed as fully explained in Art. 260, p. 300. The velocity of the water through the delivery valve at any instant will evidently be proportional to the velocity of the plunger or piston at that instant. Hence a plunger- or piston-velocity diagram will also be a *rate of delivery diagram*. This diagram may be drawn on a *stroke base* or, preferably, on a *time base*.

First consider a simple plunger pump having one plunger. During the suction stroke there is no discharge through the delivery valve, but during the delivery stroke the variation of the velocity of discharge is shown by the plunger-velocity diagram. The result for one revolution of the crank is shown at (*a*), Fig. 817, on a time base.

Next consider a double-acting piston pump. Here the delivery from one side of the piston takes place at the same time as the suction on the other side, and the variation in the rate of delivery through the delivery valves for one revolution of the crank is as shown at (*b*), Fig. 817, on a time base.

The student should have no difficulty in constructing the rate of

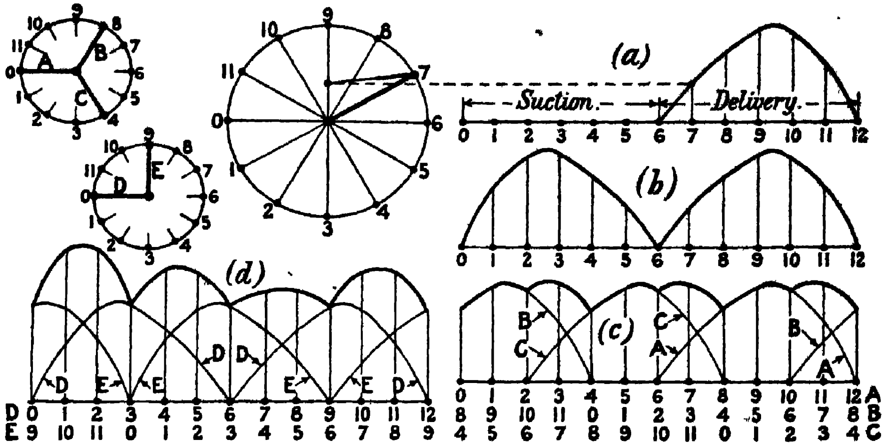


FIG. 817.

delivery diagrams for cases where there are two or more plungers or pistons driven from the same crank shaft through cranks making known angles with one another, there being one delivery pipe for all.

The resultant rate of delivery curve for a three-plunger pump is shown at (*c*), Fig. 817. The plungers are supposed to be all of the same size, and to be driven through cranks A, B, and C, which make angles of 120° with one another.

At (*d*), Fig. 817, is shown the resultant rate of delivery curve for a double-acting piston pump having two pistons driven through two cranks D and E, which are at right angles to one another. The displacements of the pistons per stroke are assumed to be equal.

441. Direct Driven Steam Pumps.—The shocks and irregularity in delivery which are almost inseparable from crank-driven pumps are to a large extent obviated in pumps in which a water piston is driven direct from a steam piston. In the latter type of pump the steam and water pistons are at opposite ends of the same piston-rod, and there is no fly-wheel and no crank shaft. The motion of the pistons being controlled mainly by the steam and water pressures, the pistons can more readily follow the moving water, the velocities of the pistons and water adapting themselves to one another without shock.

In a pump the head of water in the delivery pipe is usually constant, and therefore when the water is in motion with fairly uniform velocity the resistance is fairly uniform. Hence the pressure of the steam on the steam piston must not vary to any great extent, unless means are adopted to store up energy during one part of the stroke and restore it during another. A fly-wheel will do this effectively, permitting the steam to be

used expansively, and therefore more economically. Hence the advantage of fly-wheel pumps.

Many contrivances, other than the fly-wheel, have been tried to enable the direct driven steam pump to use the steam expansively. One, which has been used with great success on large pumping engines, will now be described. This is the compensating cylinders of the Worthington pumping engine. Fig. 818 shows a triple-expansion pumping engine having three steam cylinders and a water cylinder in line, the three steam pistons and

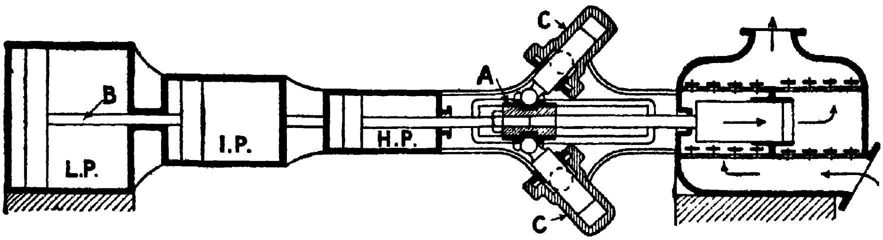


FIG. 818.

the water piston moving together. In the particular engine considered, the high pressure piston and the water piston are at opposite ends of a piston-rod connected to a cross-head A. The intermediate and low pressure pistons are connected by a central piston-rod B. The low pressure piston is connected directly to the cross-head A by two other piston-rods which pass through the front end of the low pressure cylinder, but these rods pass outside the intermediate and high pressure cylinders. CC are the compensating cylinders, which are mounted on trunnions to permit them to oscillate. The compensating cylinders are provided with rams,

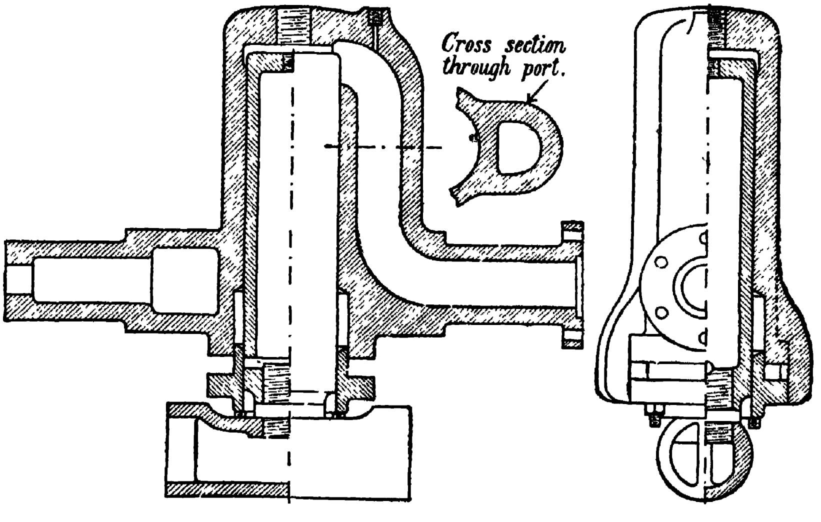


FIG. 819.

on the outer ends of which are formed gudgeons, which work in bearings on the cross-head A.

Detailed illustrations of a compensating cylinder and its ram are shown in Fig. 819.

The compensating cylinders contain water, which is in free communication with an accumulator under air pressure. During the first half of a stroke of the pistons the excess work done by the steam is used to push the rams into the compensating cylinders, thereby compressing the air in the accumulator, and during the second half of the stroke the work done during the first half in compressing the air is restored, the rams being forced out, and, as they now slope the other way, they assist in driving forward the cross-head.

The action of these compensating cylinders presents an interesting problem which is worthy of careful study by the student. Dimensions and further particulars relating to the engine just described * will now be given, so that the problem may be fully worked out.

The dimensions are as follows. Diameters of steam cylinders, 14, 22, and 38 inches. Diameter of water cylinder, $10\frac{1}{2}$ inches. Stroke of all pistons, 24 inches. Diameter of rams of compensating cylinders, 5 inches. Distance between axes of trunnions, $29\frac{1}{2}$ inches. Distance between axes of gudgeons, 13 inches. The pressure in the compensating cylinders is 515 lbs. per square inch.

The indicator diagrams taken from the steam cylinders are given in Fig. 820, while Fig. 821 shows these diagrams corrected to show *effective*

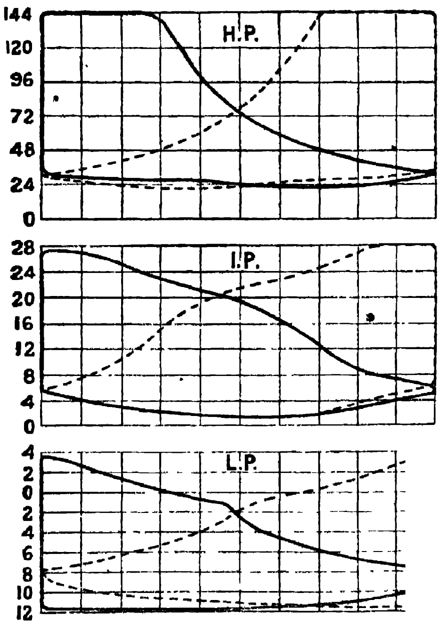


FIG. 820.

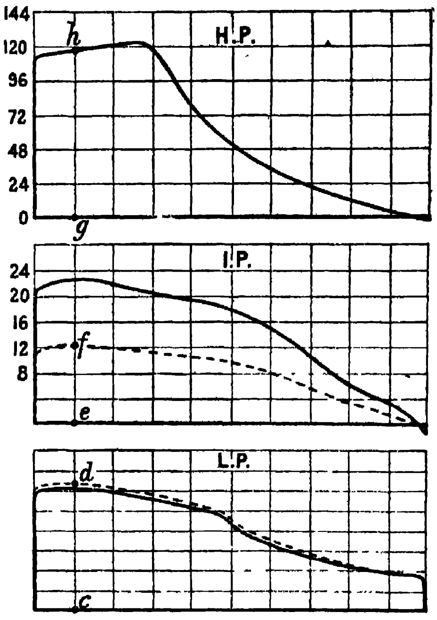


FIG. 821.

pressures per square inch on the several pistons for one stroke. The dotted lines on the intermediate and low pressure diagrams in Fig. 821 represent the pressures on the intermediate and low pressure pistons per

* Kindly supplied by the Worthington Pump Co., London.

square inch of high pressure piston to the same scale as the high pressure diagram. These various conversions were discussed in Art. 272, p. 314. Adding together the corresponding ordinates of these corrected diagrams would give the total effective pressure on the three pistons *per square inch of high pressure piston.*

It remains to construct the diagram which shall show the horizontal thrust on the cross-head due to the action of the compensating cylinders. This is shown by the dotted curve DE (Fig. 822) constructed on the base XX. To construct the curve DE, the first step is to find the pressure on the rams per square inch of high pressure piston. This amounts to

$$\frac{0{\cdot}7854 \times 5^2 \times 515}{0{\cdot}7854 \times 14^2} = 65{\cdot}7 \text{ lbs.}$$

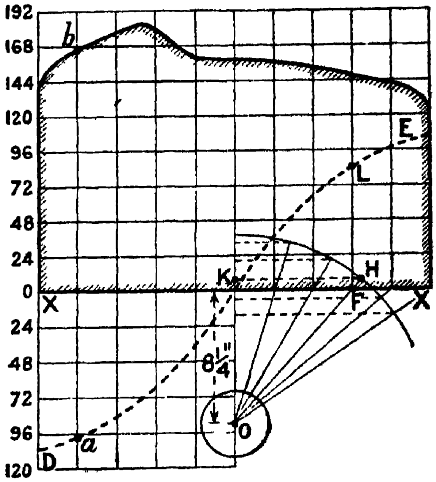


FIG. 822.

per square inch for each ram, neglecting the effect of the piston-rod. As there are two rams, the total effect is equal to 131·4 lbs. per square inch of high pressure piston on one ram. Referring now to Fig. 822, O is the axis of the trunnion of one ram, and XX is the line of stroke of its gudgeon. When the gudgeon is at F, the axis of the ram is OF. On OF make OH equal to 131·4 lbs. to the pressure scale of the high pressure diagram. Draw the horizontal line HK to meet the vertical line OK at K, then HK is the horizontal thrust exerted on the cross-head by the rams. Make the ordinate FL equal to HK. L is a point on the curve DE. For the first half of the stroke the horizontal thrust of the rams on the cross-head is reckoned as negative, since it is opposing the motion of the cross-head.

The upper boundary line of the final diagram, shown shaded in Fig. 822, is determined by laying off the sum of the respective ordinates of the corrected curves in Fig. 821 above the curve DE in Fig. 822. For example, $ab = cd + ef + gh$.

The ordinates of the final diagram represent the resultant horizontal driving force on the pump piston *per square inch of high pressure piston.* The total resultant driving force is of course obtained by multiplying the force shown by the shaded diagram by the area of the high pressure piston. In constructing these diagrams, the effect of the various piston-rods in diminishing the effective areas of the pistons has been neglected.

Direct driven steam pumps are generally of the duplex type, there being two sets of steam and water cylinders side by side with one common delivery pipe.

442. Centrifugal Pumps.—The ordinary form of centrifugal pump consists of a *wheel*, *disc*, or *runner*, provided with a number of vanes, which revolves within a casing. When fully charged with water the revolving wheel carries the water round with it, and the centrifugal force

of the revolving water causes it to travel outwards from the centre to the circumference of the wheel. The suction or supply pipe leads the water to the centre of the wheel, and the delivery pipe takes it from the casing at the circumference of the wheel. The centrifugal pump is to a certain extent a reversed turbine, and the principles involved in the theory of the centrifugal pump are the same as for that of the turbine.

Fig. 823 shows a type of centrifugal pump made by Messrs. W. H. Allen, Son, & Co., of Bedford, who kindly supplied the drawings from

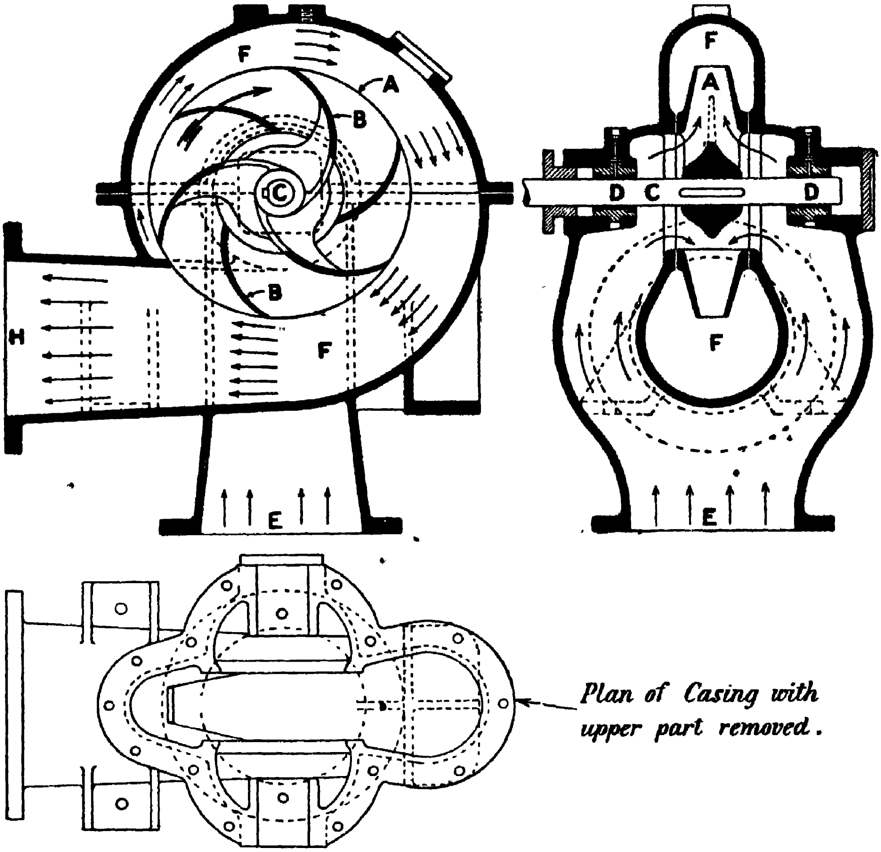


FIG. 823.

which this illustration has been prepared. A is the wheel, which is of the shrouded type, provided with six vanes B between the shrouds. For pumping ordinary water the wheel may be made of cast-iron, but for salt or brackish water a gun-metal wheel is generally used. The wheel is keyed to a steel or bronze shaft C, which runs in white metal bearings D carried by the casing. Where the shaft leaves the casing there is a gland and stuffing box to prevent leakage. The water enters from the suction pipe connected to the pump casing at E, and flows to both sides of the wheel, entering the latter at its eye or centre opening. Passing through the wheel the water flows into the expanding chamber F, and thence into

the delivery pipe, which is connected to the pump casing at H. The expanding chamber or *volute* F, which collects the water from the wheel, has a varying radial section proportioned to the quantity of water passing through it in a given time, so that the mean velocity of the water in the volute is uniform.

The object of the volute is to gradually reduce the velocity of the water after it leaves the wheel, and so convert part of its kinetic energy into pressure energy.

A centrifugal pump will not act unless it is fully charged with water. A foot valve in the suction pipe will keep the pump charged once it has been filled with water. A common method of charging large pumps is to withdraw the air by means of a steam ejector; this requires that the delivery pipe be fitted with a valve, which is closed while the ejector is acting.

Comparing the ordinary centrifugal pump with a plunger or piston pump, the former is much more efficient at low lifts, say under 30 feet. The centrifugal pump also gives a uniform delivery, and having no valves, it is much better adapted for pumping dirty water.

443. Design of Vanes of Centrifugal Pumps.—Referring to Fig. 824, r_1 and r_2 are the inner and outer radii of the wheel respectively. In practice r_2 generally lies between $2r_1$ and $3r_1$, and is frequently equal to $2r_1$. B_1B_2 represents one vane. Water enters the wheel at B_1 in the direction B_1V_1 with an absolute velocity v_1, and moving over the vane, leaves the wheel at B_2 in the direction B_2V_2 with an absolute velocity v_2. The tangential velocities of the wheel at B_1 and B_2 are c_1 and c_2 respectively. The parallelograms of velocities at B_1 and B_2 are constructed as in the case of turbines.

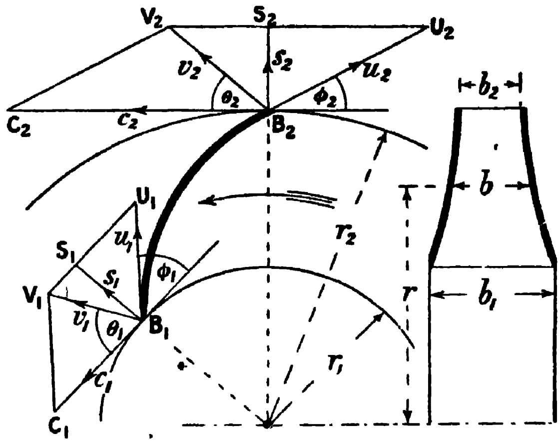


FIG. 824.

$B_1U_1 = u_1$ is the relative velocity of the water and vane at B_1, and ϕ_1 is the inclination of the vane at B_1 to the tangent to the wheel at that point. $B_2U_2 = u_2$ is the relative velocity of the water and vane at B_2, and ϕ_2 is the inclination of the vane at B_2 to the tangent to the wheel at that point.

$B_1S_1 = s_1$ is the radial velocity of the water at B_1, and $B_2S_2 = s_2$ is the radial velocity of the water at B_2. If A_1 and A_2 are the areas of the circumferential sections of the wheel at radii r_1 and r_2 respectively, then $s_2A_2 = s_1A_1$. Generally $s_2 = s_1$, then $A_2 = A_1$. If b_1 and b_2 are the breadths of the wheel at inlet and outlet respectively, $A_1 = 2\pi r_1 b_1$, and $A_2 = 2\pi r_2 b_2$. Hence if $A_2 = A_1$, $b_2 r_2 = b_1 r_1$. If the radial velocity of the water throughout the wheel is to be constant, then the breadth b at any radius r is given by the equation $br = b_1 r_1$.

$S_1V_1 = v_1 \cos \theta_1 = w_1$ is the velocity of whirl of the water as it enters the wheel, and $S_2V_2 = v_2 \cos \theta_2 = w_2$ is the velocity of whirl of the water as it leaves the wheel.

It is generally assumed that the velocity of whirl at entrance is zero, that is, the direction of motion of the water at entrance is radial, as shown in Fig. 825. In this case $\tan \phi_1 = s_1/c_1$.

The radial velocity of flow through the wheel may be from 2 to 10 feet per second, and is commonly about 5 feet per second.

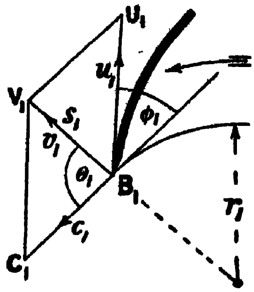


FIG. 825.

444. Work Imparted to the Water by the Wheel.—The increase in the angular momentum of 1 lb. of water in passing through the wheel of a centrifugal pump is $\frac{1}{g}(w_2r_2 - w_1r_1)$, and this is equal to T, the turning moment on the wheel. Hence if ω is the angular velocity of the wheel in radians per second, the work done per second per pound of water passing through the wheel is

$$T\omega = \frac{1}{g}(w_2r_2\omega - w_1r_1\omega) = \frac{1}{g}(w_2c_2 - w_1c_1). \quad \text{(See also Art. 419, p. 482.)}$$

If w_1 is zero, then $T\omega = \frac{w_2c_2}{g}$.

If H is the maximum theoretical head or the height to which the wheel will raise the water, neglecting all losses, then obviously $H = \frac{1}{g}(w_2c_2 - w_1c_1)$, and when $w_1 = 0$, $H = \frac{w_2c_2}{g}$.

445. Efficiencies of Centrifugal Pumps.—If v denotes the velocity of the water as it leaves the delivery pipe, then the energy or head loss on account of this velocity is $\frac{v^2}{2g}$ per lb. of water delivered. The loss of energy or head in friction in the suction and delivery pipes may be computed as in Article 401, p. 462. Let the loss due to friction in the pipes be denoted by h_1, and let h be the actual height through which the water is raised by the pump, then $H_1 = h + h_1 + \frac{v^2}{2g}$ is called the gross head or gross lift of the pump.

The ratio of the gross head H_1 to the maximum theoretical lift H (see preceding Article) is called the *hydraulic efficiency* of the pump.

Except in small lifts the term $\frac{v^2}{2g}$ is unimportant, and for small lifts the term h_1 is unimportant. The velocity v may be reduced by making the delivery end of the pipe bell-mouthed.

The *actual* or *commercial efficiency* of a centrifugal pump is the ratio of the work represented by the product of the weight of water raised, and the height to which it is raised to the work done in driving the shaft of the pump.

The kinetic energy or velocity head $\left(\frac{v_2^2}{2g}\right)$ of the water as it leaves the wheel can only be utilised for lifting the water by first converting it into

pressure energy by gradually reducing the velocity in an expanding chamber or volute. If no part of the velocity head is utilised, then the theoretical lift is reduced from $\frac{w_2c_2}{g}$ to $\frac{w_2c_2}{g} - \frac{v_2^2}{2g}$.

446. Centrifugal Head Imparted to Water by Wheel of Centrifugal Pump.—Suppose the pump to be fully charged with water, and that the wheel is rotating with angular velocity ω, but that no water is being delivered. The water within the wheel will have rotary motion only, and the centrifugal force of this water will cause the pressure at the outer circumference to be greater than that at the inner circumference.

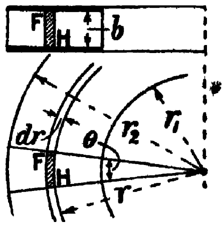


FIG. 826.

To determine the difference of pressure at the outer and inner circumferences of the wheel due to the centrifugal force of the water in the wheel, consider (Fig. 826) a wedge of this water of breadth b and angle θ, as shown. Take an element FH of this wedge at radius r and thickness dr. If w is the density of the water, then the weight of the element FH is $wbr\theta dr$, and its centrifugal force is $\frac{wbr\theta dr \cdot \omega^2 r}{g}$. To prevent FH from moving outwards in a radial direction the intensity of the pressure on its outer face must exceed the intensity of the pressure on its inner face by an amount dp, and $dp \cdot br\theta = \frac{wbr\theta dr \cdot \omega^2 r}{g}$ or $dp = \frac{w\omega^2 r dr}{g}$. The difference of the intensities of the pressures on the outer and inner ends of the wedge of water within the wheel is $\int_{r_1}^{r_2} \frac{w\omega^2 r dr}{g} = \frac{w\omega^2}{g} \cdot \frac{r_2^2 - r_1^2}{2}$, and if h_c is the head equivalent to this difference of pressure,

$$h_c = \frac{\omega^2(r_2^2 - r_1^2)}{2g} = \frac{c_2^2 - c_1^2}{2g}$$

In order that the pump may discharge water through the delivery pipe the actual head must be less than the centrifugal head.

447. Turbine Pumps.—A greater amount of the kinetic energy of the water as it leaves the wheel of a centrifugal pump may be converted into pressure energy by providing suitably designed guide passages in the chamber surrounding the wheel. The centrifugal pump then becomes a *turbine pump*. A single wheel turbine pump will raise water to a much greater height than an ordinary centrifugal pump.

448. Multi-stage Turbine Pumps.—Water may be pumped to almost any height by mounting a series of turbine pump wheels side by side on the same shaft, each wheel being provided with a suitable casing. The water enters the eye of the first wheel, and is delivered through its casing to the eye of the second wheel, and so on to the delivery pipe, which leads the water from the casing of the last wheel.

Fig. 827 shows a multi-stage turbine pump, as made by Messrs. W. H. Allen, Son, & Co., of Bedford. This pump is provided with a balancing arrangement, designed with a view to reducing the leakage of water through the clearance between the balancing piston A and the bush B. The cylinder C is attached to the pump casing, and is provided

with a facing against which the radial facing on the balancing piston A runs with a small clearance. When the radial facings of the piston and cylinder touch one another there is a small clearance between the collars and the facings of the multi-collar thrust-bearing D, thus allowing the spindle a very small axial movement. This movement also gives a very small clearance between the two balancing facings mentioned above.

When the leakage water from the periphery of the end wheel or runner enters from the annular chamber E to the chamber F it is not drained away directly, but passes between the two radial facings of the piston A and the cylinder C. The maximum clearance between these two facings is very small, and therefore only a very small quantity of

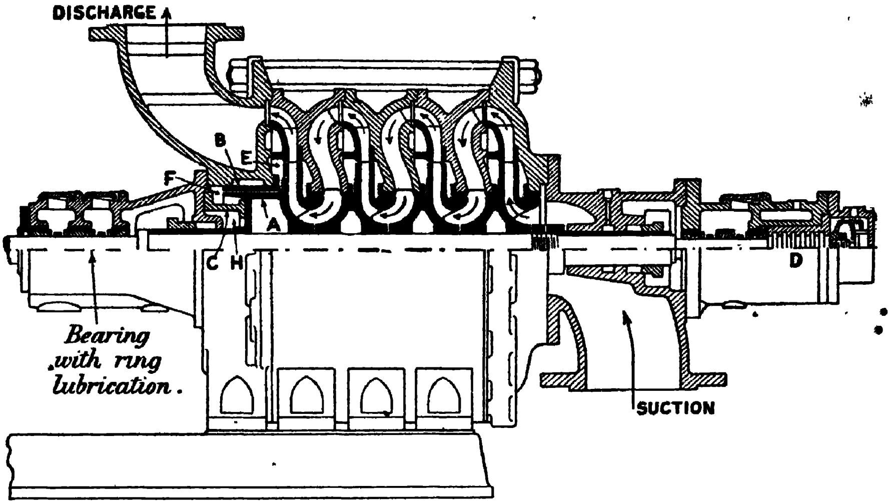


FIG. 827.

water can pass into the chamber H, thus reducing the quantity which has to be drained away.

In chamber F there will be a pressure which will vary with the head against which the pump is working, and also with the clearance between the piston A and the bush B, and between the piston A and the cylinder C. When water in the chamber F reaches a certain pressure it will force the runners and spindle to move in an axial direction, thus slightly increasing the clearance between the facings of the balancing piston and cylinder. The water in the chamber F will drain away through the increased outlet opening, and the two facings will move away from each other until the intensity of the pressure between them and also in the chamber F is so small that the thrust on the runners overcomes it, and brings the runners back into their first position, consequently the spindle will make a small reciprocating movement in an axial direction, thus materially helping to ensure perfect balance.

Exercises XXX.

1. The diameter of the barrel of a simple bucket pump is 9 inches, and the stroke of the bucket is 2 feet. The bucket-rod is driven through a connecting-rod coupled to a crank which makes 25 revolutions per minute. The total head of water is 40 feet. How many cubic feet of water are raised per minute, and what horse-power must be delivered to the crank if the efficiency of the pump is 50 per cent.?

2. A bucket pump works under a mean suction head of 20 feet and a mean delivery head of 40 feet of water. The diameters of the bucket and pump-rod are 12 inches and 2½ inches respectively, and the stroke is 3 feet. Taking into account the influence of the pump-rod, what is the pull on the rod during the up stroke? Also, what is the volume of water discharged during (*a*) one up stroke, (*b*) one down stroke?

3. The suction and delivery heads of a plunger pump are 20 feet and 50 feet respectively. The diameter of the plunger is 6 inches, and its stroke 10 inches. The plunger makes 48 double strokes per minute. Calculate the force required to work the plunger (*a*) during the suction stroke, (*b*) during the delivery stroke, assuming that the efficiency of the pump is 50 per cent. for the suction stroke, and 70 per cent. for the delivery stroke. Find also the horse-power required to work this pump.

4. In a shale mine, in order to drain one of the pits a treble-ram pump, driven by an electric motor, is employed. The rams are 9½ inches in diameter by 12-inch stroke, they each make 34·75 double strokes per minute, the height to which the water is lifted is 393 feet, and the total length of the 6-inch discharge pipe is 700 feet. Find: (i.) How many gallons of water are lifted per minute. (ii.) The useful horse-power when the pumps are running steadily. (iii.) The efficiency of the pumps if the B.H.P. of the motor is 50. (iv.) How many foot-pounds of work are done per minute in overcoming the friction in the pipe (the coefficient of friction is 0·0075). (v.) The B.H.P. required to lift the water and overcome the pipe friction. [B.E.]

5. The cylinder of a double-acting piston pump has a diameter of 9 inches, and the piston-rod has a diameter of 2 inches. The piston-rod goes through one end of the cylinder only. The suction and delivery heads are 10 feet and 50 feet respectively. Neglecting friction, calculate the force necessary to work the piston during the "in" and "out" strokes. If the mean speed of the piston is 90 feet per minute, how many gallons of water does this pump deliver per hour.

6. In a duplex Worthington pumping engine (see Fig. 818, p. 510) each of the two pump-pistons is 10½ inches diameter, and the pump-rods are 3½ inches diameter. The rate of pumping is 1,442,312 gallons per 24 hours. What is the mean speed of the pistons in feet per minute? The head is 648 feet, including friction. What is the pump horse-power? The duty of the engine is 183,300,000 ft.-lbs. per cwt. of coal, and the steam consumption is 12·1 lbs. per pump horse-power per hour. What is the weight of steam produced per lb. of coal used?

7. Reproduce the indicator diagrams given in Fig. 820, p. 511, enlarging them to, say, twice the size shown. Then, taking the particulars of the Worthington pumping engine given in Art. 441, construct, on a stroke base, the diagram whose ordinates show the effective driving force on the pump piston as described in Art. 441, but allow for the effect of the various piston-rods on the areas of the pistons. All the steam piston-rods are 2½ inches diameter. On the final diagram add a horizontal line whose height above the base is the mean height of the diagram. Also add the line whose height above the base shows the resistance of the pump obtained from the particulars given in the preceding exercise. If several students in a class should work this exercise, a number of them should construct the diagram for the forward stroke, and the others for the return stroke.

8. Show, by drawing the rate of delivery curves, that the fluctuation of delivery from a pump having two single-acting plungers driven through two cranks at right angles to one another is considerably greater than for one plunger only.

9. Draw the rate of delivery diagram for a pump of the type shown in Fig.

805, p. 503, the area of the piston-rod being half that of the piston. Then construct the rate of delivery diagram for two such pumps driven through cranks at right angles to one another. Take the length of the connecting-rods equal to 5 cranks.

10. A "three-throw" deep well pump has three pistons each 6 inches diameter, with piston-rods 2 inches diameter. Stroke of each, 24 inches. The pistons are driven through a crank shaft having three cranks making 120° with one another. The connecting-rods are so long that the pistons may be assumed to have harmonic motion. Draw the rate of delivery diagram for this pump.

11. A centrifugal pump with vanes curved back has an outer radius of 10 inches and an inlet radius of 4 inches, the tangents to the vanes at outlet being inclined at 40° to the tangent at the outer periphery. The section of the wheel is such that the radial velocity of flow is constant, 5 feet per second; and it runs at 700 revolutions per minute. Determine, (1) The angle of the vane at inlet so that there shall be no shock, (2) the theoretical lift of the pump, (3) the velocity head of the water as it leaves the wheel. [U.L.]

12. Determine the circumferential speed of the wheel of a centrifugal pump which is required to raise water to a height of 5 feet, having given that the efficiency is 0·6. The velocity of flow through the wheel is 4·5 feet per second, and the vanes are curved backwards so that the angle between their directions and a tangent to the circumference of the wheel is 20°.

13. A centrifugal pump 4 feet diameter, running at 200 revolutions per minute, pumps 5000 tons of water from a dock in 45 minutes, the mean lift being 20 feet. The area through the wheel periphery is 1200 square inches, and the angle of the vanes at outlet is 26°. Determine the hydraulic efficiency, and estimate the average horse-power. Find also the lowest speed to start pumping against the head of 20 feet, the inner radius being half the outer. [U.L.]

CHAPTER XXXI

SOME HYDRAULIC PRESSURE MACHINES

449. Packing for Hydraulic Rams and Pistons.—To prevent leakage of the water between a ram, plunger, or piston and the cylinder various forms of packing are used. The *U-leather packing* shown in Fig. 828 has been extensively used. The water leaks past the ram as far as the packing, and, entering its interior, presses one side against the recess in the cylinder and the other against the ram. The greater the pressure of the water the greater is the tendency to leak, but in the U-leather packing the force with which the leather is pressed against the ram and against the recess in the cylinder to prevent leakage is proportional to the pressure of the water. This is one great merit of the U-leather packing.

The U-leather packing is made from a disc or flat ring of leather, which is moulded between two cast-iron blocks, as shown in Fig. 829.

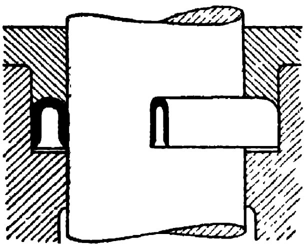


FIG. 828.

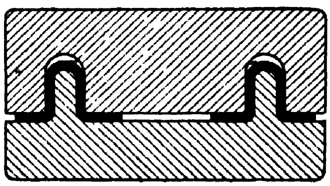


FIG. 829.

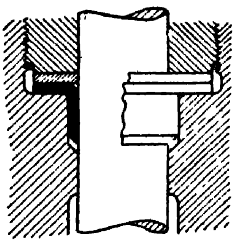


FIG. 830.

The leather is softened in hot water and placed between the blocks, which are then pressed together in a hydraulic press or by bolts and nuts, the bolts passing through the blocks. The leather is kept in the mould for about twenty-four hours, when it is removed, and after it is dried, it is trimmed to the form shown in Fig. 828.

The *hat-leather packing* is shown in Fig 830. This is more commonly used for small rams, valve spindles, etc.

The *cup-leather packing* is mainly used for pistons. A piston packed with two cup-leathers is shown in Fig. 797, p. 499. A cup-leather packing on a ram is shown in Fig. 839, p. 526.

Hat- and cup-leathers are moulded in a similar manner to the U-leather.

The leather for hydraulic packings should be the best quality oak-tanned, sole leather. It is planed to a uniform thickness of about three-sixteenths of an inch. The flesh side of the leather is made the rubbing side of the packing.

An objection to the U-leather packing is that the ram must be removed before the packing can be renewed, and this, in many cases, is very troublesome. In such cases the ordinary stuffing-box is generally used. The ordinary gland and stuffing-box for hemp or other packing is shown in Fig. 799, p. 499.

To give a smooth and non-corrosive surface to a ram, it is frequently sheathed with brass. Cylinders are also lined with brass for the same reasons and to make them non-porous at high pressures.

450. Joints and Connections for Hydraulic Pipes.—Hydraulic mains are generally made of cast-iron. For a pressure of 800 lbs. per

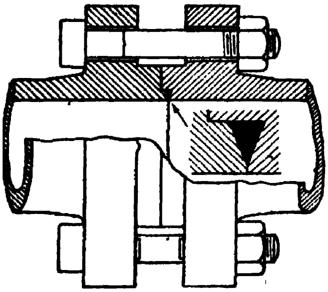


FIG. 831.

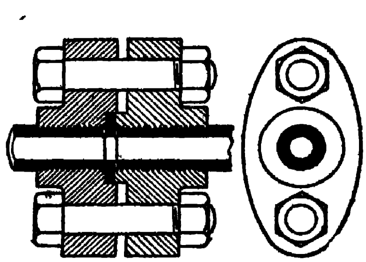


FIG. 832.

square inch the diameter of these mains does not exceed 7½ inches. The form of joint used for cast-iron hydraulic mains is shown in Fig. 831. The joint is made water-tight by a gutta-percha ring, which is forced into the V-shaped recess formed between the spigot and socket.

For smaller pipes, solid-drawn steel tubes are used. A common form of joint for these tubes is shown in Fig. 832. Strong cast-iron flanges are screwed on to the ends of the tubes. One of the two flanges has a shallow socket to receive a spigot on the other. The joint is made water-tight by a leather or gutta-percha washer placed between the spigot and socket. The end of the spigot and the bottom of the socket in contact with the leather or gutta-percha have concentric grooves turned on them, and the leather is forced into these grooves.

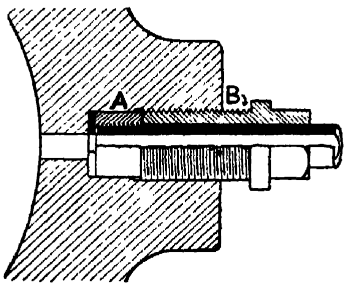


FIG. 833.

A very convenient form of joint between a steel tube and a hydraulic cylinder is shown in Fig. 833. A screwed recess is formed in a boss on the cylinder, and into this is placed, first a leather or gutta-percha washer, and then the end of the tube, on which is screwed a collar A. A gland B screwed into the recess completes the connection.

451. Hydraulic Accumulator.—Hydraulic pressure machines are usually intermittent in their action, and their demand for power is therefore very variable. In an installation of these machines the pressure water is obtained from pumps, and it is desirable that the pumps should be kept, as far as possible, working continuously. This

necessitates the introduction of a hydraulic accumulator, which shall store up the pressure water when the pumps are delivering more than is required by the machines, and give it out again when the delivery of the pumps is less than the machines require. There are two principal types of the ordinary hydraulic accumulator. In the one type there is a fixed cylinder fitted with a loaded ram, and in the other there is a fixed ram on which is fitted a loaded cylinder. An example of the fixed ram type of accumulator is shown in Fig. 834. A is the ram, and B the cylinder. When the delivery of the pumps is greater or less than is required by the machines, the water enters or leaves the cylinder at C through the ram, which is hollow. Resting on the flange at the lower end of the cylinder is a strong cast-iron base D, which carries the remainder of the load. The remainder of the load may consist of a number of blocks of cast-iron, or, as in the form shown, D carries a cylindrical casing made of steel plates, which holds scrap iron, stones, or other suitable heavy material. When at the bottom of its stroke, the load rests on the wood blocks shown. EE are two timber posts sunk in the concrete foundation at their lower ends, and fixed at their upper ends to the walls of the building containing the accumulator, or in any other way convenient. To the inside faces of these timber posts are attached steel or iron channels, in which slide blocks attached to the load casing, as shown in the elevation, and more clearly in the cross section at (*a*). In this way the load is guided as it rises and falls. A strong buffer or stop is provided to prevent the cylinder rising too high, and if there is only one accumulator, there are levers which are automatically brought into action at or near the top of the stroke, and which stop or restart the pumps.

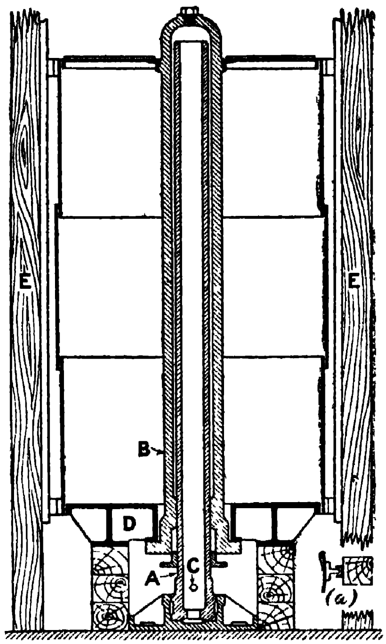


FIG 834.

In large installations where one accumulator would be inconveniently large two or more are used, in which case the second carries a heavier load than the first, and the third a heavier load than the second. Each increase of load corresponds to an increase of pressure of about 20 lbs. per square inch. The second accumulator does not come into action until the first is fully charged, and the third does not come into action until the second is fully charged. Only when the last of the series is fully charged are the pumps stopped.

If d = diameter of ram in inches, p = pressure of water in lbs. per square inch, W = total moving load in lbs., and h = stroke in feet, then, neglecting friction, $W = \frac{\pi}{4}d^2p$, and the capacity of the accumulator is $Wh = \frac{\pi}{4}d^2ph$ ft.-lbs.

452. Tweddell's Differential Accumulator.—In this accumulator there is a fixed ram ABC (Fig. 835), the lower part BC being of larger diameter than the upper part AB. The lower end of the ram is fixed in the base shown, and the upper end (not shown) is rigidly held by a bracket attached to a wall or in any other convenient way. D is the cylinder which slides up and down on the ram. The lower and upper ends of the cylinder fit the larger and smaller parts of the ram respectively, leakage being prevented by packings, as shown. The cylinder is loaded with cast-iron weights as shown, the amount of the load depending on the water pressure required. Water enters or leaves the accumulator by the pipe E, and passes along a central hole in the larger part of the ram; this central hole communicates with the water space in the cylinder by transverse holes in the ram at B. When at the bottom of its stroke the cylinder rests on the props F.

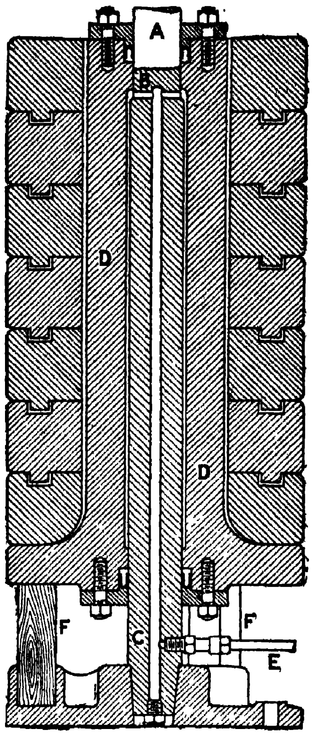


FIG. 835.

If W = total moving load, d_1 = larger diameter of ram, d_2 = smaller diameter of ram, and p = intensity of water pressure, then, neglecting friction, $\frac{\pi}{4}(d_1^2 - d_2^2)p = W$. From this it will be seen that if the difference between d_1 and d_2 be small, p will be large for a comparatively small value of W, but the capacity of the accumulator is small.

If d is the diameter of the ram of an ordinary accumulator carrying the same load W under the same pressure p, then $d = \sqrt{(d_1^2 - d_2^2)}$. The ordinary accumulator would therefore have a much more slender ram than the differential accumulator.

The differential accumulator is usually comparatively small, say $d_1 = 6$ inches, $d_2 = 5$ inches, and a stroke of about 4 feet.

The differential accumulator is used in connection with one hydraulic machine only, such as a hydraulic riveter, where the force exerted at the beginning of the operation is comparatively small, but at the end it must be large. The capacity of the differential accumulator being small, the load descends rapidly and with increasing speed, and as the operation of the hydraulic machine approaches the end, the falling load is brought quickly to rest, with the result that there is a considerable rise in the pressure of the water, and therefore a considerable increase in the force exerted by the hydraulic machine at the conclusion of its operation.

453. Intensifying Accumulator.—An accumulator in which the load on the ram is produced by low pressure water, say from the ordinary water supply, acting on a piston, is shown in Fig. 836. This is called an

intensifying accumulator, because the pressure of the water on the piston intensifies the pressure of the water on the ram. The low pres-

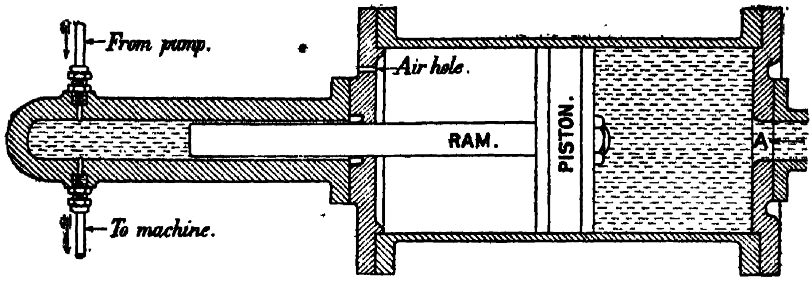


FIG. 836.

sure water enters at A. The action is obvious, and need not be further described.

If d is the diameter of the ram, D the diameter of the piston, p the pressure of the water on the ram, and P the pressure of the water on the piston, then, neglecting friction, $pd^2 = PD^2$.

454. Hydraulic Intensifiers. — When a hydraulic machine, in finishing the operation which it is designed to perform, has to exert a much greater force than is required during the earlier part of the operation, the increased force may be obtained by the use of an *intensifier*.

The intensifying accumulator described in the preceding Article may be used as an intensifier by first charging the smaller hydraulic cylinder with water, and then introducing the power water from the pumps into the larger cylinder at A. The effect of this will be to raise the pressure of the water in the smaller cylinder and in the pipes leading from it. Generally, however, the hydraulic intensifier has two rams instead of a ram and piston.

If d and d_1 are the diameters of the smaller and larger rams respectively, p the pressure of the water supplied by the pumps, and p_1 the intensified pressure, then, neglecting friction, $p_1d^2 = pd_1^2$.

A hydraulic intensifier, which may be operated to give three different higher pressures, as required, with the same pump pressure, is shown in Fig. 837.* A is the smaller ram, and this is stationary. B is a cylinder fitting over A, but it is also a ram fitting into C. While C is the cylinder for B, it is also the ram for D; and while D is the cylinder for C, it is also the ram for the fixed outer cylinder E.

On the tension rods FF are placed sleeves HH, which have lugs KK, which, when brought round into action by turning the sleeves, prevent D from rising. The sleeves also have lugs LL, which, when brought into action, prevent C from rising. MM are handles for turning the sleeves to put the lugs LL or LL and KK into or out of action.

The cylinder B and the pipes connected with it through the interior

* *American Machinist*, Nov. 5, 1904.

of the fixed ram A are charged with the water whose pressure is to be intensified, and water at the ordinary pump pressure is introduced into the outer cylinder E. The lugs on the sleeves are shown in action* in Fig. 837 preventing C and D from rising.

Let d, d_1, d_2, and d_3 be the external diameters of A, B, C, and D respectively. Let p be the ordinary pump pressure of the water, p_1 the intensified pressure when B alone rises, p_2 the intensified pressure when B and C rise together and D is fixed, and p_3 the intensified pressure when B, C, and D rise together. Then, neglecting friction,

$$\frac{\pi}{4}d^2p_1 = \frac{\pi}{4}d_1^2p \quad \text{or} \quad d^2p_1 = d_1^2p.$$

Similarly, $d^2p_2 = d_2^2p$, and $d^2p_3 = d_3^2p$.

In the particular intensifier shown in Fig. 837 the ram A and cylinder B are made of forged steel. The other cylinders are made of cast-iron. To diminish the clearance spaces all the cylinders except the outer one are bored out.

In operating a hydraulic machine which works in connection with an intensifier, the machine is first driven directly by the ordinary power water, and at the point in the operation where the greater force is required the power water is switched on to the larger cylinder of the intensifier, and at the same time the delivery of the intensifier is switched on to the machine.

455. Hydraulic Press for making Lead Pipes.—Fig. 838 shows a form of hydraulic press used for "squirting" lead pipes. A is a large ram, on the top of which is the cylinder B, containing the lead. C is a smaller fixed ram, which is hollow, and provided with a die at its lower end. This die has a hole in it of a diameter equal to the outside diameter of the lead pipe.

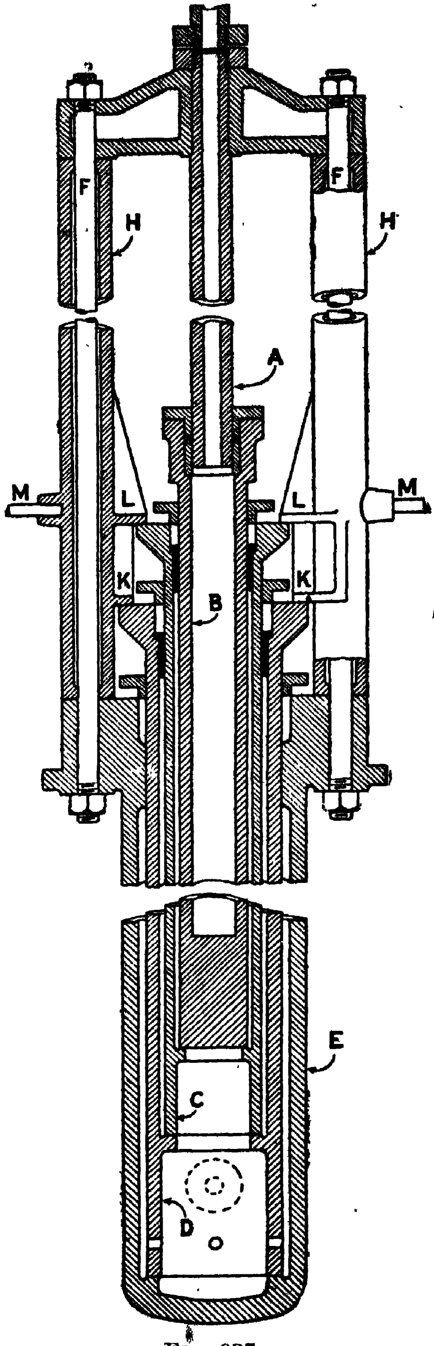


FIG. 837.

Attached to the large ram there is a straight round rod having a diameter equal to that of the inside of the lead pipe. This rod passes up through the centre of the die. The lead, in a molten state, is poured in at D, and when it has solidified, but while still hot, the large ram is forced up, and the lead is squirted through the die on the end of the smaller ram, coming out at D in the form of a pipe. An enlarged section of the part of the apparatus in the neighbourhood of the die is shown at (*a*).

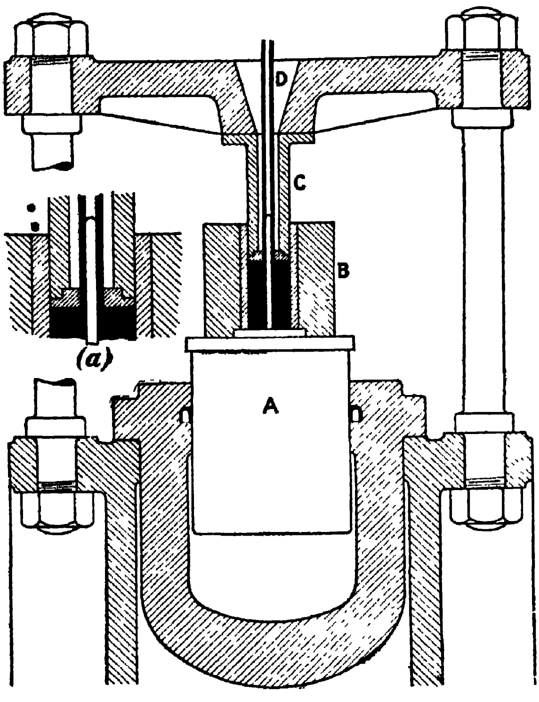


FIG. 838.

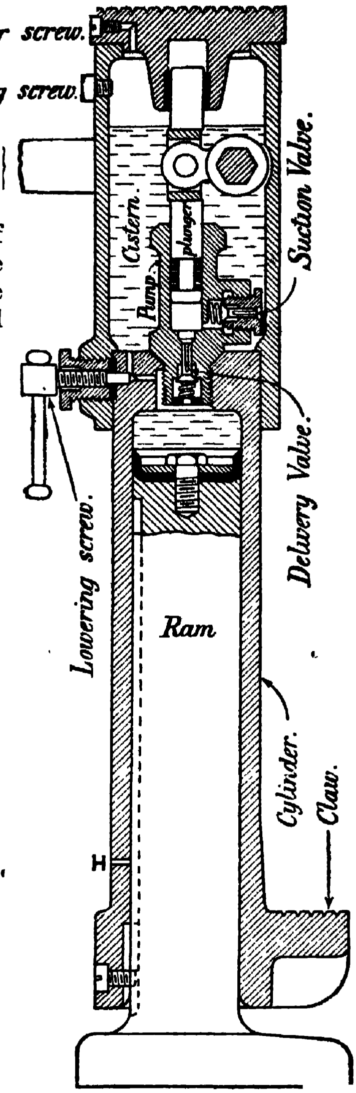


FIG. 839.

456. Hydraulic Lifting Jack.—Fig. 839 shows a hydraulic lifting jack made by Messrs. Tangyes of Birmingham. The ram is spread out at its lower end to form a base upon which the jack stands. To the top of the ram is attached a cup-leather to form a water-tight joint between the ram and the cylinder. The pump is screwed into the top of the cylinder, and is surrounded by a casing attached to the cylinder. The cover of this casing forms the bed for the load under which the jack is placed, or the load may be carried on the claw formed on the lower end

of the cylinder. The space round the pump in the casing forms a cistern for the water. A spindle, one end of which passes through the casing, carries at that end the hand lever, and inside the casing it carries a short crank, the free end of which works in a rectangular slot formed in the plunger of the pump. By this arrangement the oscillation of the lever causes the reciprocation of the plunger. The suction valve of the pump is at the side, and the delivery valve at the bottom of the pump. The valves are loaded with light spiral springs.

When the lever is worked the pump takes in water from the cistern and delivers it into the cylinder above the ram, and thus causes the cylinder and the load on it to rise. During the operation of raising the load the lowering screw must be screwed up tight. The inner end of the lowering screw is conical, and forms a valve which, when shut, closes the passage between the cylinder and cistern outside the pump. To lower the cylinder the lowering screw is unscrewed, and the water may then flow freely from the cylinder to the cistern.

When the jack is in use, the air screw at the top should be slackened. Overlifting is prevented by the hole H in the cylinder allowing the water to escape when the proper lift is exceeded. Rotation of the cylinder on the ram is prevented by a key secured in the cylinder at the lower end by a set screw; this key fits in a keyway extending nearly the whole length of the ram.

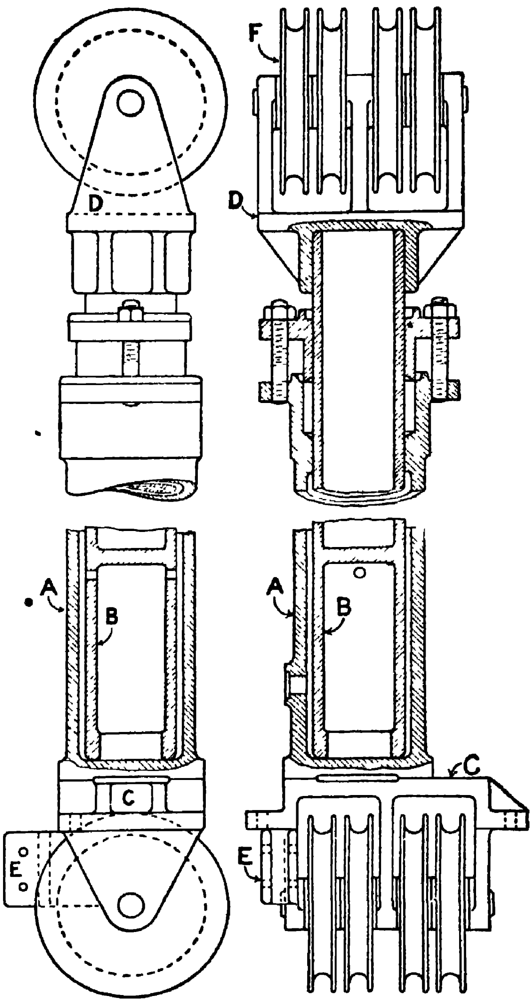


FIG. 840.

¶ 457. Hydraulic Crane.—Comparing the principal part of the mechanism of a hydraulic crane with the ordinary "block and tackle." In the block and tackle one force acting on the "fall" of the rope overcomes a greater by bringing the two blocks closer together. In the hydraulic crane the action is reversed; the two

blocks or sets of pulleys are pushed apart by a great force, which is made to overcome a smaller force at the free end of the chain. One block is attached to the bottom of a hydraulic cylinder, and the other to the ram. A chain passes over the pulleys in the two blocks in the same manner as in the ordinary block and tackle, the free end of the chain passing up the crane post and over guide pulleys to the load to be lifted.

Referring to Fig. 840, A is the stationary cylinder, and B the ram. Fixed to the bottom of the cylinder is the frame C carrying one set of pulleys, and fixed to the outer end of the ram is the frame D carrying the other set of pulleys. One end of the chain is attached to the bracket E on the fixed frame C.

From the fixed end at E the chain passes over the pulley F, and then down and up over the various pulleys, and finally passes up to and over guide pulleys in the frame of the crane to the load to be raised.

Neglecting friction, the load lifted is equal to the tension in the chain, which is equal to the total force on the ram divided by the number of straight lengths of chain which proceed from the pulleys on the ram.

Exercises XXXI.

1. If a hydraulic power company charges 15 pence per 1000 gallons of water at a pressure of 750 lbs. per square inch, what is the cost per horse-power hour to the consumer?

2. How many ft.-lbs. of work may be stored up in a hydraulic accumulator whose ram has a diameter of 12 inches, and a lift of 13 feet, when the pressure of the water is 750 lbs. per square inch?

3. What is the pressure of the water in a hydraulic accumulator having a ram 11 inches in diameter when the total load is 45 tons and friction is neglected?

4. What must be the diameter of the ram of a hydraulic accumulator which is to have a capacity of 100 horse-power minutes with a water pressure of 1120 lbs. per square inch, if the stroke is 14 times the diameter of the ram?

5. The ram of a hydraulic accumulator (Fig. 834, p. 522) is 12 inches in diameter, and the total moving load is 55 tons. If the force required to move the cylinder along the ram against the resistance of friction only is 2·5 tons, what is the pressure of the water, in lbs. per square inch, (*a*) when the load is ascending with uniform velocity, (*b*) when the load is descending with uniform velocity?

6. The ram of a hydraulic accumulator is 16 inches in diameter, the stroke is 23 feet, and the water pressure is 1120 lbs. per square inch. If the useful work given up by this accumulator during one full downward stroke is utilised in raising W tons to a height of 40 feet by means of a hydraulic crane whose efficiency is 55 per cent., find W. If this work is done in three minutes, what is the gross horse-power of the crane?

7. The ram of a hydraulic accumulator is 4 inches in diameter. The pressure of the water from the accumulator is to be 1·5 tons per square inch, and the water is used to work the hydraulic ram of a testing machine, which is 9·5 inches in diameter. (*a*) What total load must be placed on the 4-inch ram if 5 per cent. of the load is wasted in the friction of the cup-leathers, etc.? (*b*) What total load is the testing machine capable of applying, if there is a further loss of 5 per cent. in the cup-leathers of the large cylinder? [B.E.]

8. The diameters of the two parts of the ram of a differential accumulator (Fig. 835, p. 523) are 6 inches and 5 inches, and the stroke is 50 inches. If the pressure of the water is 1500 lbs. per square inch when the load is at rest or when it is moving with uniform velocity, what load is required, including the weight of the cylinder? How many ft.-lbs. of work may be stored in this accumulator? What would be the diameter of the ram of an ordinary accumulator to carry the same load with the same water pressure?

9. The total moving load on a differential hydraulic accumulator is 3 tons. The diameters of the larger and smaller parts of the ram are $4\frac{1}{2}$ inches and

4 inches respectively. Neglecting friction, what is the pressure of the water in lbs. per square inch?

10. In an intensifying accumulator (Fig. 836, p. 524) the diameter of the ram is 4 inches, and the diameter of the piston is 20 inches. If the head of water on the piston is 60 feet, what is the pressure of the water on the ram in lbs. per square inch? If the stroke of the ram and piston is 30 inches, what is the capacity of this accumulator in ft.-lbs.?

11. On board ship an accumulator is used in which the ram is 9 inches in diameter, and the pressure of the water is 800 lbs. per square inch. The load is produced by steam pressure of 60 lbs. per square inch acting on a piston connected directly to the ram and working in a separate steam cylinder. Neglecting friction, what must be the diameter of the steam piston?

12. A hydraulic intensifier is required to increase the pressure of 700 lbs. per square inch in the mains to 3000 lbs. per square inch. The stroke of the intensifier is to be 4 feet, and its capacity 3 gallons. Determine the diameters of the rams. [Inst.C.E.]

13. Referring to the hydraulic intensifier shown in Fig. 837, p. 525, if the diameters of the rams are $2\frac{1}{2}$, 4, 6, and 8 inches respectively, and if the pressure of the water introduced into the outer cylinder is 700 lbs. per square inch, what are the pressures, in lbs. per square inch, which may be obtained in the pipe leading from the interior of the smaller ram, neglecting friction?

14. What would the results of the preceding exercise be if the friction of the stuffing-boxes is allowed for, assuming that the frictional resistance of one stuffing-box, in lbs., is equal to 0·05PD, where P is the pressure of the water in lbs. per square inch, and D is the diameter of the ram in inches?

15. In a hydraulic press for squirting lead pipes (Fig. 838, p. 526) the larger ram has a diameter of 20 inches, and the smaller ram has a diameter of 5 inches. Find the intensity of the pressure on the lead when the water pressure is 1 ton per square inch, and the bore of the lead pipe is $\frac{1}{2}$ inch. If the stroke of the ram is 12 inches, and the lead pipe weighs 1·1 lbs per foot of length, find the length of pipe produced in one stroke if the lead weighs 712 lbs. per cubic foot.

16. The diameter of the ram of a hydraulic jack is 3 inches, the diameter of the plunger of the pump is $\frac{3}{4}$ inch, and the mechanical advantage of the lever is 20. If the efficiency of this jack is 75 per cent., what weight will be raised by a force of 70 lbs. at the end of the lever?

17. What is the efficiency of a hydraulic crane which uses 70 gallons of water at a pressure of 700 lbs. per square inch in raising a weight of 10 tons to a height of 27 feet?

18. In a hydraulic crane the ram is 9 inches in diameter, and the velocity ratio (or the ratio between the velocity of the lift and the velocity of the ram) is 10. The water is delivered to the crane under a pressure of 1200 lbs. per square inch, and the mechanical efficiency of the crane is 52 per cent. Find (1) what load this crane will lift; (2) the quantity of water used in gallons per 35 feet lift. Why do these cranes have such a low mechanical efficiency? [B.E.]

19. Calculate the displacement of the ram of a hydraulic crane, whose efficiency is 55 per cent., in order that, with a water pressure of 700 lbs. per square inch, it may raise a load of 5 tons to a height of 30 feet. Find the diameter of the ram if its stroke is six times its diameter.

20. A hydraulic lift with ram, load, etc., weighs 10 tons, the ram is 9 inches in diameter, and the friction in the mechanism is equal to 0·05 of the gross load. The accumulator is half a mile away, and is loaded to 800 lbs. per square inch; the diameter of the supply pipe is 3 inches. Estimate the speed of ascent of the lift, if the loss of head in the pipe is $0{\cdot}0005lv^2/d$, l being the length in feet, d the diameter in feet, v the velocity of the water in the pipe in feet per second. [B.E.]

ANSWERS

II. pp. 21–23.

1. 66; 3·75; 19; 8·2. **2.** 210; 1584; 1824; 2952·9.
3. 22·5; 35; 19; 29·83. **4.** 31·42; 28·27. **5.** 601·6. **6.** 0·0432.
7. $14\frac{2}{3}$. **8.** 0·55. **9.** 0·449. **10.** 6 seconds. **11.** 14 feet per second.
12. 93·9 feet. **13.** 22·5 seconds. **14.** 0·0675; 16·59.
15. 1343; 25·66 seconds. **16.** 224; 23·47; 1·91.
17. 26·64 feet per second at 41° 21′ to the vertical.
18. 25° 37′ north of east; 3·889 miles per hour.
19. 48° 15′, assuming that the man is approaching the line of flight of the object. **20.** 17·32 feet per second.
22. B, 5·8 feet per second; middle point of AB, 3·2 feet per second.
23. 12·42 lbs. **24.** 258·75 feet. **25.** 3821 lbs.; 4631 lbs.
26. 2 minutes 4·6 seconds; 54·72 miles per hour.
27. $2\frac{1}{27}$ feet per second per second; 10·23 tons. **28.** 8·93 feet per second.
29. 10·93 lbs. **30.** 3·22 feet per second per second.
31. (1) 844·7 lbs.; (2) 1000 lbs.; (3) 1155·3 lbs. **33.** 2020 lbs.
34. 488·13 ft.-lbs.

IIIa. pp. 28–30.

1. 40·594. **2.** 5727·6 **3.** 514·3. **4.** 366,000. **5.** 47,099. **6.** 302,467
7. $\frac{15}{16}$. **8.** 130·2 ft.-lbs. **9.** 6. **10.** 2200 lbs. **11.** 992·3. **12.** 226,286.
13. 30·55; 22,790; 170; 0·0035. **14.** 44·24; £840.
15. 10·02 lbs.; 83·3 per cent., 56·4 per cent., 74·2 per cent.
16. 85; 112·5; 0·437; 15·5 lbs.; 0·573. **17.** See Fig. 841

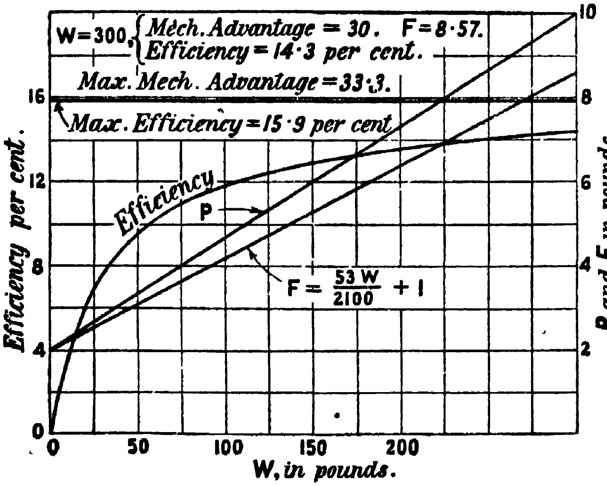


FIG. 841.

Correction.—In Fig. 841 the lower ends of e P and F lines should be one division wer, that is, when W=0, P=F=1 lb.

18. $Q=0·02W+1$. **19.** 22·45. **20.** $P=0·162W+2·8$; maximum efficiency =38·6 per cent.
21. $p=1·25$, $q=5$; efficiency=57·14 per cent.

IIIb. pp. 32–33.

1. 4489 feet; 209·5 lbs. **2.** 16·56 feet. **3.** 0·863 lb.
4. 37·51; 261·24. **5.** 4445 feet.
6. 59·86 feet; result independent of diameter of wheel.
7. 7,339,185 in lb. and foot units; 1250 ft.-lbs.
8. 54,000 ft.-lbs; 118·6 tons. **9.** 260,741 ft.-lbs.; 237.
10. 365,783 ft.-lbs.; increases at the rate of 3·087 revolutions per minute per minute. **11.** 173·5 feet. **12.** 78·22; 0·039.
13. 1,171,780 ft.-lbs. **14.** 8·7 feet per second.
15. (*a*) 6·77; (*b*) 11·4; (*c*) 2236 in lb. and inch units. **16.** 778 feet.
17. 17·3 B.Th.U. **18.** 13·6 per cent. **19.** 500·87; 171·73 tons.
20. 508·8 lbs.

IV. p. 41.

1. Magnitude, 1·99; line of action inclined at 23° 57′ to the horizontal.
2. Magnitude of resultant, 3·18; line of action inclined at 38° 56′ to the horizontal; horizontal force, 2·48; vertical force, 2.
3. Magnitude, 50; line of action, 1·2 inches from the force 20 and 0·8 inch from the force 30.
4. Magnitude, 10; line of action, 6 inches from the force 20 and 4 inches from the force 30.
5. Magnitude, 30; line of action, 3·83 inches from A and 0·67 inch from B.
6. Magnitude, 2; line of action midway between A and B.
7. P, $21\frac{1}{4}$; Q, $23\frac{3}{4}$. **8.** R_1, $3\frac{5}{32}$; R_2, $3\frac{3}{32}$.
9. Magnitude, 4·17; line of action inclined at 78° 17′ to AB, and cuts the latter at a point 1·4 inches to the right of A.
10. Magnitude of force at A, 6·32; magnitude of force at B, 4·99; lines of action inclined at 75° 8′ to AB.
11. Magnitude, 3·66; line of action inclined at 82° 42′ to AB, and cuts AB at a point 0·76 inch to the right of A.
12. P, 4·14; Q, 1·8; line of action of Q 1·41 inches above A.

Va. pp. 49–50.

1. 11·25. **1a.** 28·75. **1b.** Magnitude of P, 20 lbs.; line of action is perpendicular to BC, and cuts AC at a point $\frac{3}{4}$ inch from A.
2. Within the rectangle ABDE, 1·81 inches from AB, and 1·30 inches from AE.
3. Within the triangle ABC, 1·06 inches from AC, and 0·69 inch from BC.
3a. Within the triangle ABC, 0·96 inch from AC, and 0·61 inch from BC.
4. The string cuts CD produced 2·65 inches from C.
5. 0·89 inch below AB, and 0·39 inch to the left of AC.
6. 0·87 inch above AB, and 0·63 inch to the right of AC.
7. 9·01 feet to the right of A.
8. 0·89 inch below AB, and 1·13 inches to the right of AC.
9. 0·85 inch above AB, and 2·58 inches to the right of AC.
10. 1·23 inches from OA, and 1·31 inches from OB.
11. 1·61 inches below AB, and 0·98 inch to the right of AC.
12. 1·90 inches below AB, and 1·78 inches to the right of CE.
13. 17·6; 177,408; centre of pressure, 25 feet from ground.

Vb. pp. 61–63.

1. 265 in inch and lb. units. **2.** 2·146 in inch units.
3. (*a*) 56·30, (*b*) 9·71, both in inch units.
4. 549·1 and 364·5, both in inch units.
5. $\bar{y} = 5·34$ inches; $I = 1385$ in inch units. **6.** 3·61. **7.** 15·4.
8. 6·34. **9.** 5·96 and 5·24. **10.** 0·49. **11.** 0·568. **13.** 9·65 inches.
14. (1) 8·9 square inches; (2) 3·1 inches; (3) 54·5. These answers are the means of five solutions.

15. 424 lbs.; 10·23 inches.
16. Bending moment at centre, 27·5 foot-tons; shearing force at centre, 1·4 tons.
17. 37·5 foot-tons; 5 tons. **18.** Tension in chain, 2640 lbs.
19. Reaction at C, 12·4 tons; reaction at D, 17·6 tons; bending moment at centre, 43·6 foot-tons; shearing force at centre, 4·6 tons.

VIa. pp. 70–72.

1. (1) 22,282; (2) 0·4456; (3) 0·0007427; (4) 24·95.
2. (1) 25,133 lbs.; (2) 0·0256 inch. **3.** 460·9; 4609.
4. (1) 0·0432 inch; (2) 4800 lbs. per square inch; (3) 7248·5 lbs.
5. 9447 lbs. per square inch tensile stress in steel; 6144 lbs. per square inch compressive stress in copper; 0·2182 inch; 36,551 lbs.
7. (1) 19·9882 inches; (2) 24,083 lbs. per square inch; (3) 16,858 lbs. per square inch. **8.** 21·9875 inches; 7000 lbs. per square inch; 19·9951 inches.
9. 500 feet; increase in length, 0·18 inch. **10.** 0·1055 inch.
11. (*a*) 6972 ft.-lbs.; (*b*) 8·21 ft.-lbs.; (*a*) ÷ (*b*) 849. **12.** $\frac{5}{8}$.
13. 17,112 lbs. per square inch; 0·041 inch. **14.** 21,231 lbs. per square inch.
15. 12,937 lbs. per square inch; 0·588 inch. **16.** 1989 inch-lbs.
17. 13,650 lbs. per square inch; 0·505 inch.

VIb. pp. 78–79.

E_t = tearing efficiency per cent. E_s = shearing efficiency per cent. E_{st} = combined shearing and tearing efficiency per cent.

1. E_t, 56·25; E_s, 56·45; crushing stress, 36 tons per square inch.
2. E_t, 71·7; E_s, 71·7; crushing stress, 23·61 tons per square inch.
3. $p=3$ inches; E_t, 68·75; E_s, 67·2.
4. $d=1\frac{5}{16}$ inches; $p=4\frac{3}{4}$ inches; E_t, 72·4; E_s, 72·9.
5. E_t, 75·7; E_s, 75·3; E_{st}, 70·2.
6. $d=1\frac{7}{16}$ inches; $p=7\frac{1}{8}$ inches; E_t, 79·8; E_s, 79·8; E_{st}, 79·6.
7. $p=6\frac{5}{8}$ inches; E_t, 84·9; E_s, 85·2; E_{st}, 86·85.
8. $p=7\frac{1}{2}$ inches; E_t, 86·7; E_s, 86·0; E_{st}, 90·5.
9. E_t, 86·6; E_s, 115·2; E_{st}, 87·6.
10. $p=7\frac{7}{8}$ inches; $d=\frac{7}{8}$ inch; E_t, 88·9; E_s, 89·2; E_{st}, 88·9.
11. E_t, 68·2; E_s, 71·8; E_t, 84·1; E_s, 89·8; E_{st}, 86·1.
12. 81·35 per cent.; shearing of outer rivet and tearing between rivets in next row.
13. $d=\frac{7}{8}$ inch; $d_1=\frac{15}{16}$ inch; E_t, 82·5; E_s, 84·9; E_{st}, 82·3.
15. $\frac{7}{8}$ inch; 89·8. **16.** 140 lbs. per square inch; 2·917 tons per square inch.
17. 9600 lbs. per square inch; 5 feet 4 inches. **18.** 10,377.
19. 200. **20.** $s=1·1d$; $b=1·37d$; $t=0·38d$.
21. $d=1·60$; $b=s=1·80$; $t=0·69$.
22. $d_1=1·24d$; $b=1·55d$; $t=0·34d$; $D=1·65d$; $D_1=2·48d$.

VIc. pp. 84–86.

1. 26,260; 2·46 inches. **2.** 2941 lbs. per square inch. **3.** 3·4 inches.
4. 2885; 5557 lbs. per square inch. **5.** 6016 lbs. per square inch.
6. 311,953 inch-lbs.; 35·7; 121·2 revolutions per minute. **7.** 13,125.
8. 14·39 inches. **9.** 0·714; 50.
10. (1) 14,181 lbs. per square inch; (2) 43,507 inch-lbs.; (3) 124·25.
11. 90·8; 15·7. **12.** 1·87 inches; 5672 lbs. per square inch.
13. 8·52 inches; 1 : 1·057.
14. Total twist, 3·7 degrees; the diameters of the pulleys are not required.
15. $n=2·95$; $C=11,980,000$ lbs. per square inch.
16. 40·9 lbs.; 4·12 inches; 7·02. **17.** 4·614. **18.** 53·4 inches.
19. 0·226 inch; 22 inches.

VIIa. pp. 101–103.

8. See Fig. 842.
9. See Fig. 843.
16. See Fig. 844.
18. $w=0\cdot1$.
20. $w=0\cdot36$.
21. $w=0\cdot442$.
23. $w=0\cdot87$.
24. Bending moments—1000; 2000; 2163; 2530; 1265. Shearing forces—500; 361; 632.

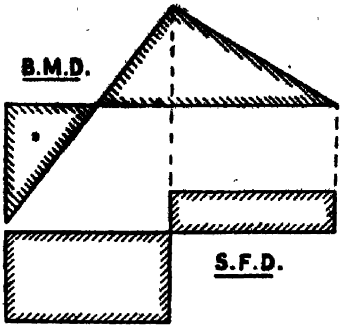


FIG. 843.

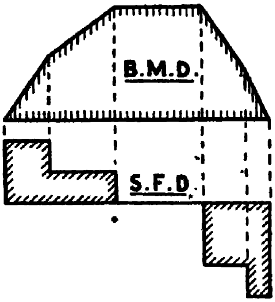


FIG. 842.

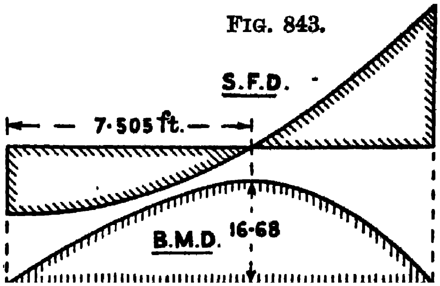


FIG. 844.

VIIb. pp. 111–112.

1. 0·682; 1·484; 2·618. 2. 0·625; 1·484; 1·25.
3. 197·34 inch-tons; 53 inch-tons. 4. 5·29 inches; 1·92.
5. (a) 127·83; (b) 118·44. 12. $d=7\cdot59$ inches; $l=13\cdot72$ inches.
13. $d=7\cdot61$ inches; $l=13\cdot69$ inches. 14. 5·02 sq. in.; 8 tons per sq. in.
16. 33·8. 17. 2·693 inches; 56. 18. 70·76. 19. 3272·5 lbs.
20. 6·71 inches; 7·89; 13,669. 21. $1\frac{2}{3}$ sq. in.; $6\frac{2}{3}$ inches.

VIII. pp. 133–137.

1. 13·44 inches; 12·48 inches; 9·62 inches; 0·853 inch; 395 feet 10 inches.
2. Depths in inches, 7·53, 6·35, 4·40; deflection, 0·163 inch; radius, 746·23 feet; maximum stresses in lbs. per square inch, 7594, 7146, 6028, 4179.
4. 6·476 inches; 0·0736 inch. 5. $\frac{1}{6}$ inch.
6. 16,394 lbs. per square inch; 0·874 inch. 7. 6667; 0·32 inch.
8. 9600. 9. 13,989 lbs.; 14,961 lbs. per square inch.
10. (a) 1223 lbs. per square inch; (b) 2·64 inches.
11. (b) 1008 lbs.; (c) 12,150,000; (d) 388; (e) 43,200 lbs. per square inch.
12. (a) 2·52; (b) 2·10. 14. 0·367 inch. 15. $\frac{wL^4}{4EI}$.
16. 2636 lbs. per square inch; 0·47 inch. 17. 0·03 radian = 1·72 degrees.
18. Each side beam load 60 lbs., stress $166\frac{2}{3}$ lbs. per square inch; centre beam load 480 lbs., stress $333\frac{1}{3}$ lbs. per square inch.
19. 18,000 lbs. per square inch; 68·9.
21. Taking $E=13,000$ tons per square inch, maximum deflection $=\frac{4600}{I}$ inches, where I is the moment of inertia of the section of each girder in inch-units; maximum deflection at 19·4 feet from left-hand support.
22. 11·232 tons; 0·27 inch.
23. Thrust $=32\frac{1}{16}$ tons; bending moment zero at 2·725 feet, 10·275 feet, and 16 feet from fixed end.
24. 45 foot-tons at ends; 15 foot-tons at centre; zero bending moment at 4·5 feet from ends.

26. $-312\cdot5$ foot-tons; $+175\cdot8$ foot-tons; $62\cdot5$ tons at centre; $18\cdot75$ tons at ends.

27. $-\frac{wL^2}{96}$ inch-tons; $-\frac{wL^2}{96}\pm\frac{3EIx}{L^2}$ inch-tons, where L is the length of each span in inches, E is in tons per square inch, and I is in inch-units.

28. $10\cdot51$ tons, or $0\cdot263$ tons per foot. 29. $159\frac{3}{8}$ tons; $444\frac{19}{24}$ tons; $95\frac{5}{8}$ tons.

30. $66\cdot46$ tons; $234\cdot84$ tons; $73\cdot7$ tons; -2104 foot-tons.

31. $M_B = 2707$ foot-tons; $M_C = -1382$ foot-tons. 35. $11\frac{1}{3}$; 100.

IXa. pp. 151–153.

2. 1; $0\cdot866$. 4. $f=\sqrt{pq}$; $\tan\theta=\sqrt{\left(\frac{q}{p}\right)}$.

5. $p=f\cot\theta$; $q=f\tan\theta$; $s=f(\tan\theta-\cot\theta)$. 6. 49,672; 12,418.

7. 19,763. 8. $2\cdot876$ inches. 9. 3119. 10. $338\cdot7$. 11. 4 inches; $2\cdot98$ inches.

12. Bending moment, 432 inch-tons; twisting moment, 288 inch-tons; stress, $0\cdot5855$ ton per square inch.

13. 36,000 inch-lbs. at A; 3487 inch-lbs. at C.

14. 36,000 inch-lbs. at A; 26,358 inch-lbs. at C.

15. (*a*) B, $4\cdot47$ inches; C, $5\cdot03$ inches; D, $4\cdot75$ inches;
(*b*) B, $4\cdot68$ inches; C, $5\cdot05$ inches; D, $4\cdot77$ inches.

16. 345 lbs. per square inch.

17. Maximum shear stress $\frac{5}{8}$ ton per square inch on each section; maximum direct stresses $\frac{5}{8}$ and $1\frac{1}{8}$ ton per square inch on the respective sections.

19. $4\cdot324$ tons per square inch; $1\cdot035$.

20. (*a*) 4500; (*b*) 3122; (*c*) 6000; (*d*) 4500; (*e*) 4153; (*f*) 2280; (*g*) 2068. 21. $0\cdot81$.

IXb. pp. 159–161.

1. $12\cdot108$; $4\cdot629$ tons per square inch. 2. 20 tons; $\frac{1}{2}$-inch. 3. $\frac{1}{3}d$.

5. $1\cdot72$; compressive; $3\cdot298$ tons per square inch.

6. $1\cdot674\sqrt{W}$; compressive; $4\cdot091$ tons per square inch.

7. (*a*) 2958; (*b*) 3471. 8. $0\cdot336\sqrt{W}$.

9. $3\cdot32$ tons per square inch; $3\cdot53$ tons per square inch.

10. $56\cdot55$ feet.

12. $0\cdot0736$ cubic inch, or $0\cdot0077$ per cent.

13. $3\frac{1}{3}$ tons per square inch; $\frac{29}{4200}$ inch.

14. (*a*) $0\cdot00033$; (*b*) $0\cdot00029$.

15. $E=29{,}878{,}000$ lbs. per square inch; $C=11{,}672{,}000$ lbs. per square inch; $K=22{,}625{,}000$ lbs. per square inch; Poisson's ratio $=0\cdot2799$.

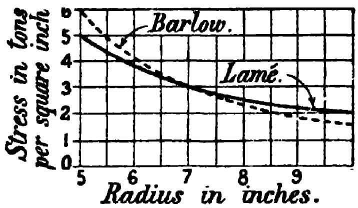


FIG. 845.

16. 5 inches; 1000 lbs. per square inch. 17. 973. 18. $1\cdot32$ inches.

19. 1585 lbs. per square inch; 2300; 1877; 1603; 1115. 21. See Fig. 845.

X. pp. 168–169.

1. $14\cdot9$ lbs. 2. $324\cdot5$ tons. 3. $24\cdot13$ tons. 4. $42\cdot4$. 6. $4\cdot10$ inches.

7. $4\cdot11$ inches; $37\cdot4$. 8. $12\cdot65$ inches; $10\cdot12$ inches. 9. $6\cdot2$ inches.

10. 62,112 lbs. 11. $\sqrt{7\cdot66}$ inches; $220\cdot6$ tons.

12. $106\cdot3$ tons, with factor of safety 5. 13. $1\cdot045$ inches.

15. $4\cdot7$ inches; $0\cdot47$ inch.

XI. pp. 187–190.

3. (*b*) 29,900,000; (*c*) $4\cdot4$; (*d*) $4\cdot6$; (*f*) 5040.

5. $c=77\cdot3$; $b=19\cdot2$; $e=26$. 6. $1\cdot52:1$. 7. $a=67\cdot5$; $b=40$.

8. $c=11\cdot5$. 9. $25\cdot21$ tons per square inch; $\mu=0\cdot29$.

10. $21\cdot18$ tons per square inch: $\mu=0\cdot445$. 11. $3\cdot4$ square inches.

XII. pp. 198–200.

1.	BH EL	CJ DK	GH GL	HJ LK	JK
	$\underline{1677}$	$\underline{1118}$	1500	$\underline{559}$	500

2.	Dead Load.	Wind on Left.	Wind on Right.	Maximum.
BK	$\underline{3606}$	$\underline{4250}$	$\underline{3250}$	$\underline{7856}$
CL	$\underline{2884}$	$\underline{3417}$	$\underline{3250}$	$\underline{6301}$
DM	$\underline{2884}$	$\underline{4750}$	$\underline{3250}$	$\underline{7634}$
EO	$\underline{2884}$	$\underline{3250}$	$\underline{4750}$	$\underline{7634}$
FP	$\underline{2884}$	$\underline{3250}$	$\underline{3417}$	$\underline{6301}$
GQ	$\underline{3606}$	$\underline{3250}$	$\underline{4250}$	$\underline{7856}$
QP	$\underline{721}$	0	$\underline{2167}$	$\underline{2888}$
PO	$\underline{800}$	0	$\underline{2404}$	$\underline{3204}$
ON	1342	0	4031	5373
NM	1342	4031	0	5373
ML	$\underline{800}$	$\underline{2404}$	0	$\underline{3204}$
LK	$\underline{721}$	$\underline{2167}$	0	$\underline{2888}$
KJ	3000	5108	1502	8108
JN	1800	1502	1502	3302
JQ	3000	1502	5108	8108

3.	
BK and GT	$\underline{5590}$
CM ,, FR	$\underline{4472}$
DO ,, EP	$\underline{3354}$
JN ,, JQ	4000
JL ,, JS	5000
JK ,, JT	5000
KL ,, TS	0
LM ,, SR	$\underline{1118}$
MN ,, RQ	500
NO ,, QP	$\underline{1414}$
OP	2000

Thick underline denotes compression.

4.	
AD	$\underline{2814}$
DB	$\underline{2151}$
DC	$\underline{2946}$
BC	1771

5.	
AG	$\underline{3500}$
BK	$\underline{2000}$
CK	2236
DH	2981
EF	5590
FG	$\underline{2121}$
GH	1863
HK	$\underline{943}$

	6 (*a*).	6 (*b*).	6 (*c*).
AG	$\underline{6·71}$	$\underline{6·71}$	$\underline{13·42}$
AH	$\underline{6}$	$\underline{6}$	$\underline{12}$
AK	$\underline{12}$	$\underline{12}$	$\underline{18}$
AM	$\underline{8}$	$\underline{10}$	$\underline{18}$
AO	$\underline{4}$	$\underline{8}$	$\underline{12}$
AP	$\underline{4·47}$	$\underline{8·94}$	$\underline{13·42}$
BP	2	4	6
CN	6	9	15
DL	10	11	18
EJ	9	9	15
FG	3	3	6
GH	6·71	6·71	13·42
HJ	$\underline{6·71}$	$\underline{6·71}$	$\underline{6·71}$
JK	6·71	6·71	6·71
KL	4·47	2·24	0
LM	$\underline{4·47}$	$\underline{2·24}$	0
MN	4·47	2·24	6·71
NO	$\underline{4·47}$	$\underline{2·24}$	$\underline{6·71}$
OP	4·47	8·94	13·42

7.	
BM and GV	$\underline{15}$
CO ,, FT	$\underline{24}$
DQ ,, ER	$\underline{27}$
KP ,, KS	24
KN ,, KU	15
KL ,, KW	0
AL ,, HW	$\underline{18}$
LM ,, WV	21·21
MN ,, VU	$\underline{15}$
NO ,, UT	12·73
OP ,, TS	$\underline{9}$
PQ ,, SR	4·24
QR	$\underline{6}$

	8 (*a*).	8 (*b*).
BM and JY	$\underline{5218}$	$\underline{3975}$
CN ,, HX	$\underline{4472}$	$\underline{3230}$
DQ ,, GU	$\underline{4100}$	$\underline{2857}$
ER ,, FT	$\underline{3354}$	$\underline{2112}$
SR ,, ST	1118	497
SP ,, SV	373	$\underline{248}$
LO ,, LW	2236	$\underline{248}$
LM ,, LY	2609	124
MN ,, YX	$\underline{471}$	$\underline{471}$
NO ,, XW	500	500
OP ,, WV	943	157
PQ ,, VU	1000	1000
QR ,, UT	$\underline{943}$	$\underline{943}$
LS	2500	...

8 (*b*) *continued.*—Reaction at top hinge horizontal and $=1667$ lbs. Reaction at each bottom hinge $=4333$ lbs. and inclined at angle ϕ to vertical, $\tan\phi=\frac{5}{12}$.

9. X = 11·8 outwards; Y = 8·0 inwards; Z = 6·7 outwards; force in AB, 3·5, compression.

12. $\frac{\sqrt{11}}{20}$.

14. Total thrust in top member = 71·67; thrust in BJ, DL, EM, and FN = 10; thrust in CK = 20; tension in AJ and HN = 11·78; tension in AK = 29·81; tension in AL and HL = 15·81; tension in HK = 27·49; tension in AM = 13·74; tension in HM = 14·91; tension in AN and HJ = 8·50.

16. Total thrust in top member = 67·2; thrust in BF and DK = 8; thrust in CH = 16; thrust in EL = 32; tension in AF, CF, CK, and EK = 5·12; tension in AH and EH = 15·09; tension in AL = 53·64.

XVI. pp. 278–282.

2. $\theta = \phi$. **5.** 1 in 7·11. **7.** 37,600; 5600. **8.** 396; 454·9.
9. 24·6. **10.** 20·48, neglecting difference between length and base of plane.
12. (1) 0·30; (2) 0·26; (3) 0·24.

13. *d*	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$
Eff. %	25·7	20·3	18·2	18·3	16·6	15·9	14·8

14. 32·3 lbs.; 36·8 per cent.; 8·2 lbs. **15.** (1) 22,620; (2) 3343.
16. 0·193. **17.** 71·2; 1·58. **18.** (1) 51·5 lbs.; (2) 48·5 lbs.
19. To raise W, (*a*) 207 lbs.; (*b*) 203 lbs.; to lower W. (*a*) 193·1 lbs.; (*b*) 197 lbs. **20.** (*a*) 203·6 lbs.; (*b*) 196·5 lbs.
21. (1) P = 550 lbs.; (2) P = 598 lbs.; (1) R = 1260 lbs.; (2) R = 1291 lbs.
23. (*a*) 4885 lbs.; (*b*) 4603 lbs. **24.** 515 lbs. **25.** 0·34.
26. 135·7. **27.** 352·3. **28.** 7·1. **29.** 128 lbs.
30. 0·00066 lb.; 0·0264; 37·7 : 1.
31. Horse-power = 1866 K, if V is in feet per second.
32. 257 lbs.; 39 lbs. **33.** 1174 lbs.; 8·5 lbs. **34.** 0·22.
35. $3\frac{1}{4}$. **36.** (*a*) 215 lbs.; (*b*) 497 lbs.

XVIIa. pp. 294–296.

1. 98; 63·91 lbs. **2.** 128 ft.-lbs.; 10·15 feet per second; 11 feet per second.
3. 14·11. **4.** Mean pressures, (*a*) 99; (*b*) 99·2; (*c*) 95·4 for forward stroke, and 103 for return stroke.
5. 60·5 feet per second. **6.** 2·48. **7.** 2·8. **8.** 28·09 feet per second.
9. 1·21. **10.** Velocities: 19; 30; 36; 39. Accelerations: 3; 1·4; 0·9; 0·6.
11. When $t = 0·075$ second, angular velocity = 8·3, and angular acceleration = 17·4; 8679 ft.-lbs.
12. I.: (*a*) $6\frac{2}{3}$; (*b*) 2·32; (*c*) $3\frac{1}{3}$; (*d*) 0·644. II.: (*a*) 10; (*b*) 4·01; (*c*) 5; (*d*) 1·93. III.: (*a*) $13\frac{1}{3}$; (*b*) 6·55; (*c*) $6\frac{2}{3}$; (*d*) 4·51.
13. (*a*) 10·72; (*b*) 4·23; (*c*) 4; (*d*) 1·07.
14. $x = 10$, $v = 25·43$; $x = 30$, $v = 42·15$; $x = 50$, $v = 51·5$; $x = 70$, $v = 57·54$; from $x = 45$ to $x = 55$, time = 0·194 second; from $x = 0$ to $x = 75$, time = 2·24 seconds.
15. Stops at 18·2 feet; time from start to stop, 1·92 seconds
16. Time average of force = 262·4 lbs.; space average of force = 267·9 lbs.; average velocity = 12·73 miles per hour; distance = 653·5 feet.
17. Length of stroke, 9·46 feet; time for up stroke, 1·12 seconds.

XVIIb. p. 299.

1. 1226 lbs. **2.** 0·495 second. **3.** 3·25 lbs. **4.** 1·107 seconds.
5. 6·12 inches. **6.** 2·06 seconds. **7.** 289. **8.** 2·972 in lb. and foot units.

XVIII. pp. 309–312.

1. (1) 7·07 ; (2) 8·20 ; (3) 9·74.

2.	x	1·34	..	1·62	18·94	..	...	1·97	19·30	...	...
	v	5·0	6·0	5·97	4·03	6·55	5·49	7·24	2·76	7·28	4·93

4. 400 ; 641. **5.** 104 and 42 feet per second.
6. (1) 104 ; (2) 117 ; (3) 136. See Fig. 846.
7. (1) 190 ; (2) 182 ; (3) 176. See Fig. 847. **8.** 218·8.
11. (1) 2·32 ; (2) 5·36 ; (3) 12.
12. Piston, 1148 ; shaft, 759 ; cross-head pin, 126 ; crank pin, 989.

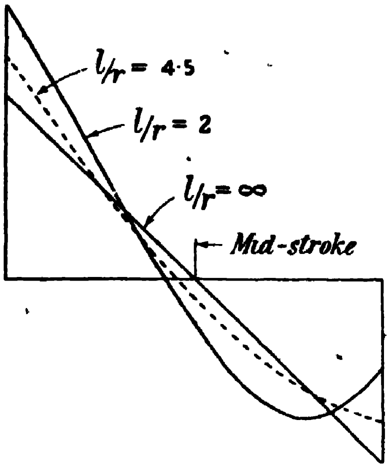


FIG. 846.

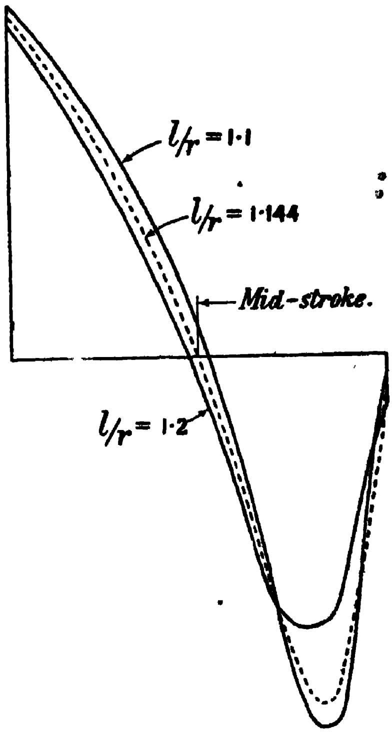


FIG. 847.

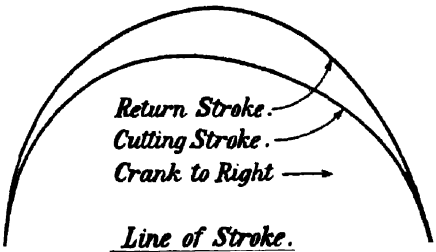


FIG. 848.

13. (1) 31·5 ; (2) 77·7 ; (3) 0. **14.** Length of stroke, 16·16 inches.
15. Radius of crank, 2·806 inches ; (*a*) 28·6 by construction ; (*b*) 36·1 by construction ; 1·17 : 1 ; for curves, see Fig. 848.
16. (*a*) 1·47 ; (*b*) 10·88 ; (*c*) 420. **17.** 5⅝ inches ; 1·51 : 1.
18. 1·38 : 1. **19.** Velocities of sliding in feet per minute.—At C, 125 ; at E, 109 or 275 ; time ratio of cutting and return strokes, 2·73.
20. 132 feet per minute.

XIX. pp. 322-328.

1. 30·5, assuming exhaust pressure to be atmospheric; 342. **2.** 578. **3.** (*a*) 2554 lbs.; (*b*) 1277 lbs. **4.** 1100 lbs.

6. 15·1 and 8·1 lbs. per square inch of piston, both downwards; 127 revolutions per minute; 28·5 lbs. per square inch of piston, downwards.

7. (*a*) 0·105; (*b*) 0·129.

8. Coefficient of fluctuation of energy, (*a*) 0·16; (*b*) 0·18.

$$\text{Ratio } \frac{\text{maximum torque}}{\text{mean torque}}, \ (a)\ 2{\cdot}42;\ (b)\ 2{\cdot}58.$$

9. 42·02 lbs.; 9·45 lbs.; 55·43 lbs.

10. Forward stroke: 38·4 lbs.; 7·4 lbs.; 59·2 lbs. Return stroke: 46·3 lbs.; 11·7 lbs.; 53·1 lbs.

11. 109,495 lbs.; 133,390 ft.-lbs.

12. 0·049. **13.** 0·044.

14.

16. (i.) mean of nine solutions, 0·076.

17. (6)

18. 15,258. **19.** 284,462 ft.-lbs. **20.** 1392. **21.** 574·2 lbs. **22.** 101·6.

23. 84·879 and 85·121 revolutions per minute; 2760·5.

24. 27·54. **25.** 0·00137. **26.** 1118·5; 50,994 ft.-lbs.

27. 404; 606 revolutions per minute.

28. $M=\frac{5}{2}\frac{6}{1}$; $I=15{,}377$ in lb. and foot units. **29.** 30,000 ft.-lbs.

30. 1·2. **31.** 41,671 in lb. and foot units; 207.

XX. pp. 341–343.

2. 24° 35′; 50° 50′; 51·2; 56·7. **3.** 0·92 inch; 1·05 inches.

5. 6·33; 10·98; 3·18.

6. 6·33; 11·07; 3·06.

7. See Fig. 849.

8. 4·93 lbs.; 196 to 221 revolutions per minute.

9. 0·82 inch.

10. (1) 3·18; (2) 212, 222, 234; (3) 221, 232, 245; (4) 223 212, 202.

11. 1·45.

12. 198 and 212; 198 to 224 revolutions per minute.

13. 6·73 inches; 2·23 lbs. at sleeve; 13·3 revolutions per minute, or 5·54 per cent.

14. 411 revolutions per minute; 100·8 lbs.

15. T=16·84 lbs.; Q=695·2 lbs. **16.** (*a*) 290; (*b*) 111·7 lbs.

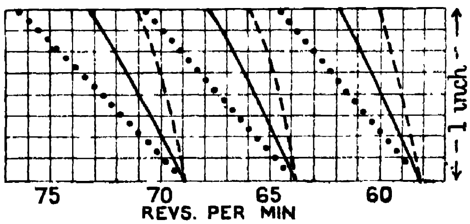


FIG. 849.

XXI. pp. 361–362.

1. 284·8. **2.** 48·4 lbs. **3.** 4·61. **4.** 60·12 feet.

5. Emergency stop from speed of 60 miles per hour equals 2 + 16 = 18 seconds, and the distance covered in that time is 880 feet; retardation compared with gravity=0·171; resisting force=383 lbs. per ton. **6.** (*a*) 120·96; (*b*) 89·6.

7. 4·27. **8.** 34·86. **9.** 5·51. **10.** 1·14. **12.** 2742.

XXII pp. 372-374.

1. (*a*) 3141·6; (*b*) 3233·2; 2·83. **2.** 29·2 inches; 7·3 inches.

3. (*a*) 1458·3; (*b*) 1423·4; 2·45. **4.** 7·79 inches.

5. (*a*) 196·46; (*b*) 196·01. **6.** (*a*) 182·52; (*b*) 182·51.

7. Open belt: (*a*) 193·30; (*b*) 193·14; 0·08. Crossed belt: (*a*) 205·10; (*b*) 204·25; 0·41. **8.** 24·95; 9·45; 37·80; 174·40.

9. $d_1=10$; $D_2=24{\cdot}64$; $d_2=16{\cdot}43$; $D_3=13{\cdot}57$; $d_3=27{\cdot}14$; $l=164{\cdot}8$; all in inches.

10. $d_1=10$; $D_2=24$; $d_2=16$; $D_3=13\frac{1}{3}$; $d_3=26\frac{2}{3}$; $l=170\cdot8$; all in inches.
11. 10·25; 12·34; 14·25; all in inches.
12. B and d, 14·20; C and c, 12·17; D and b, 10·04; all in inches.
13. 183·82 lbs.; 91·91 lbs. **14.** 123·75 lbs.; 68·75 lbs.
15. (1) H=14·91; (2) H=13·37. **16.** (1) H=58·47; (2) H=53·95.
17. (1) $b=5\cdot13$ inches; (2) $b=5\cdot43$ inches.
18. (1) $b=16\cdot76$ inches; (2) $b=18\cdot68$ inches.
19. (1) $f=302$ lbs. per square inch; (2) $f=344$ lbs. per square inch.
20. (1) V=2514 ft. per minute; (2) V=2710 ft. per minute. **21.** 80·9.
22. Maximum horse-power=29·56, when $v=96\cdot91$ feet per second.
25. 19·15; 42·61 lbs.; 4·29. **26.** 125·5. **27.** 14 feet 6 inches.
28. Velocity, 170; horse-power, 167.
29. Maximum horse-power=200, when $v=176$. **30.** 1·5; 7·3.

XXIII. p. 387.

1. 23·75 inches; 0·7854 inch. **2.** 0·9425p'; 1·2566p'.
3. 4 · 5; (1) 7 and 9; (2) 29 and 35.

XXIV. pp. 393–396.

1. 14·8 inches, 59·2 inches. **2.** 20, 90, 39·39. **3.** 450; 1003·5 lbs. **4.** 16.
5. 60. **6.** (*a*) $142\frac{6}{7}$; (*b*) 19·8 seconds; (*c*) 1465 lbs. **7.** 1·573.
8. (1) 0; (2) +49; (3) −51.
9. n revolutions per minute anti clockwise; straight line through centre of A; ellipse, centre at centre of A, major axis equal 3 times distance between centres of A and C, minor axis equal $\frac{1}{3}$ major axis. **10.** 50 in same direction.
11. (1) +9; (2) −25·2. **12.** 62·95 inches. **13.** 44.
14. $7\frac{1}{2}$ in same direction. **15.** −176. **16.** 30 to 1.
17. 62 and $29\frac{1}{4}$ revolutions per minute in same direction as arm; 206·45 lbs.
18. +70. **19.** (1) 0; (2) +80; (3) −48.
20. 240 revolutions per minute in same direction as E.
21. (1) +21; (2) −22·8.

XXV. pp. 411–413.

6. (*a*) 23·56; (*b*) 35·34.

XXVIa. pp. 417–419.

1. 75 lbs; 126° 52′. **2.** 861·5 lbs.; 108° 26′ to first radius; 25·3 lbs.
3. 36·1 lbs., 15 lbs. **4.** 9·386, 4·614; 102·8.
5. 572·2 lbs., 381·4 lbs.; 20 lbs. **6.** 24·64 lbs. on A, 20·36 lbs. on B.
7. 6415 lbs. on the bearing nearest to the crank, and 1480 lbs. on the other; 541 lbs. in the plane nearest to the crank, and 283·4 lbs. in the other.
8. 201 lbs. each; left-hand balance weight 157° 37′ in advance of left-hand crank; right-hand balance weight 202° 23′ in advance of right-hand crank.
9. Forces on bearings.—Left-hand, 50·2 lbs.; right-hand, 120·5 lbs. Balance weights.—Left-hand, 12·6 lbs.; right-hand, 38·1 lbs. If the 10 lbs. mass leads, then the 12·6 lbs. mass is 159° 58′ ahead, and the 38·1 lbs. mass is 127° 20′ ahead of the 10 lbs. mass
10. 37·8 lbs. in plane P at 227° 5′; 34·5 lbs. in plane Q at 34° 1′.
11. $x=83\cdot67$ lbs.; angle between D and A, 37° 51′.

XXVIb. pp. 428–430.

1. $\frac{1}{2}$ inch; 4087 lbs. **2.** 0·004 inch. **3.** 382 lbs.; $255\frac{1}{3}$ lbs.; $126\frac{1}{3}$ lbs.
4. 787 lbs.; $\theta_1=\theta_2=23°\ 12'$ (see Fig. 692, p. 423).
5. 311·5 lbs.; $\theta_1=\theta_2=22°\ 23'$ (see Fig. 692, p. 423).
6. 333 lbs.; $\theta_1=\theta_2=5°\ 43'$ (see Fig. 693, p. 424).
7. 162 lbs. in each driving wheel, and 68 lbs. in each trailing wheel.
8. 366 lbs. in each driving wheel, and 108 lbs. in each trailing wheel.
9. 2·46 tons. **10.** 5·6 tons. **11.** 360 lbs. and 240 lbs.

12. 385 lbs. and 175 lbs.; all cranks in same plane, intermediate crank on opposite side of shaft to outside cranks.

13. X, 271 lbs., 19° 24′ in front of C, A leading; Y, 550 lbs., 70° 27′ in front of B, A leading.

14. C, 5411 lbs., 103° 58′ behind B; D, 3530 lbs., 52° 28′ behind A.

XXVII. pp. 436–438.

1. 9600 lbs. **2.** 9·45; 0·51 lb. **3.** 69·2; 416. **4.** 22,670. **5.** 8392 lbs.
6. $253\frac{1}{2}$ tons; $10\frac{1}{6}$ ft. **7.** 7·30 tons; 13·04 tons.
8. 3·245 lbs. in each bottom screw, 0·649 lb. in each top screw.
9. 604·5 lbs., 641·5 lbs.; in horizontal line 0·4 inch below, and equally distant from, centre of door; 623 lbs. **10.** 6·59 feet. **11.** 5586 lbs.
12. 444·3 tons; 8·3 feet. **13.** (i.) $6\frac{1}{15}$ lbs.; (ii.) $11\frac{4}{15}$ lbs.
14. 4·2 inches from bottom; 10·21. **15.** 1·798 feet; 1·752 feet.
16. 4300 tons at beginning of voyage. **18.** 2°.

XXVIIIa. pp. 454–456.

1. 19·3 lbs. per square inch; 1·92 inches. **2.** 1·22 lbs. per square inch.
3. 31·0 lbs. per square inch. **4.** 48·11 cubic feet per second. **5.** 1924.
6. 2520. **7.** 3·48 inches. **8.** (1) 650; (2) 1300.
9. 5·67 radians per second. **10.** 78·2 feet per second. **11.** 8·7 feet per second.
12. 40·4 feet per second. **13.** (i.) 25·38; (ii.) 25·39.
15. 0·613; 0·957; 0·641. **16.** ·622.
17. 1481 with coefficient of discharge, $=0·61$.
18. 3 hours 6 minutes 49 seconds with coefficient of discharge $=0·62$.
19. $6\frac{1}{2}$ seconds with coefficient of discharge $=0·62$.
20. 8 seconds with coefficient of discharge $=0·62$.
21. 37 minutes 19 seconds. **22.** 0·619. **23.** 89·7. **24.** 91·9.
25. 8·88 feet. **26.** 22·1 cubic feet per second with $k=0·6$.
27. 10·3 inches with $k=0·6$. **28.** 4·87.
29. $2·64h^{\frac{3}{2}}$ cubic feet per second; 672.
30. Total discharge about 6480 cubic feet. **31.** 13·7 inches. **32.** 130,489.

XXVIIIb. pp. 472–475.

1. 32·8. **2.** 61·8. **3.** 0·655 inch; 1·99 inches; when $k=0·66$.
4. 30,805; 1419. **5.** 126,367,280; 5470. **6.** (1) 2·47 feet; (2) 1·17 feet.
7. 4·92. **8.** 51·9 feet. **9.** 169. **10.** 13·5 cubic feet per second. **11.** 0·73 foot.
12. Discharge, 2·33 cubic feet per second; hydraulic gradient, 1 in 62·1; pressure head, 15·5 feet. **13.** $1 : \sqrt{2}$.
14. 46,100; 8390 gallons, or 18·2 per cent. **15.** 158·5 feet.
16. 3·55, assuming that the total loss of head is equal to the difference in the levels of the two ends of the pipe.
17. 2·12 and 1·52 cubic feet per second.
18. 59·8 cubic feet to B, and 69·3 cubic feet to C. **19.** 70·1 feet.
20. 2·75 inches. **22.** 527.
23. The curves are shown in Fig. 850. Total loss of horse-power asked for $=6·94$.
24. 5·50 inches.
25. 4·15 inches; 840 lbs. per square inch; 303·2.
26. $s=6·5$ feet; $d=3·72$ feet; $b=12·84$ feet.
28. 1·062 : 1.
29. 3·17 feet per second; 976 cubic feet per second.

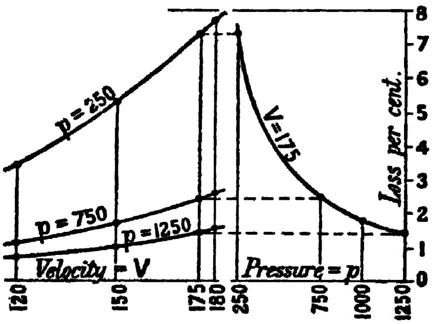


FIG. 850.

30. 1·49 feet. **31.** 6 feet; 1 in 3517. **32.** 476. **35.** 84,790.

XXVIIIc. pp. 483–484.

1. (*a*) 38 lbs.; (*b*) 19 lbs. **2.** (*a*) 152 lbs.; (*b*) 59·4 lbs. **3.** 0·863.
4. Maximum horse-power = 9·59 at 25 feet per second.
5. 1831; 29·2 per cent. **6.** 1393; 22·2 per cent.
7. 2973; 41·1 per cent. **8.** 220 lbs. **9.** (*a*) 126 lbs.; (*b*) 56 lbs.
10. 1260; 93·3 per cent. **11.** 974 lbs.; 67·3.
12. (*a*) 72·6 lbs.; (*b*) 18·1 lbs.

13. Acute angles between AB and the tangents to the vane at entrance and exit, 53° 48′ and 36° 12′ respectively. Pressure on vane, 0·645 lb.

14. Acute angles between horizontal and tangents to vane at entrance and exit, 73° 41′ and 47° 16′ respectively. Pressure in direction of motion 68·4 lbs.

15. Direction of total pressure inclined at 7° 56′ to direction of motion of bucket or vane. Magnitude of total pressure, 193·7 lbs. Pressure in direction motion, 191·8 lbs. Maximum efficiency $\frac{1}{2}$, when velocity of buckets is 82·8 feet per second.

16. 9317 ft.-lbs. **17.** 862·6 lbs.
18. 34·7 feet per second; 25·7; 2051 lbs.; 21·9 miles per hour.

XXIX. pp. 496–498.

1. 6600. **2.** 63·7. **3.** 4·03.
5. (*a*) 75·7 per cent.; (*b*) 181 feet; (*c*) 0·459 : 1.
6. Maximum B.H.P. = 1·61; maximum efficiency, 79·3 per cent.
7. Taking efficiency at 80 per cent., horse-power = 235·5.

8. Mean diameter of wheel, 2·28 feet; diameter of nozzle, 0·461 inch; horse-power, 16·3.

10. $v_1 = 152·5$; $c_1 = 75·9$; $c_2 = 87·3$; $u_1 = 84·5$; $u_2 = 94·9$; $v_2 = 24·9$; $\theta_2 = 99°$; revolutions per minute = 181·2; horse-power = 318·5.

11. $c_1 = u_1 = 42·57$; $c_2 = u_2 = 53·21$; $v_2 = 13·9$; $\theta_2 = 82°\ 30'$; revolutions per minute = 203·3; horse-power = 273.

12. 5·4; 81·5 per cent. **13.** 68·7 per cent.; 11·8.

14. (Referring to Fig. 795, p. 494) $\theta_1 = 41°\ 13'$; angle $U_1B_1C_1 = 78°\ 54'$; $\phi_2 = 21°\ 52'$.

15. 9° 28′; 747·6 lbs. **16.** Inlet, 48° 54′; outlet, 11° 19′.

17. At inlet, 65° 9′ to the tangent to the periphery; at outlet, 21° to the tangent to the periphery; 89 per cent.

19. Velocity of outer periphery of wheel, 35·8 feet per second; guide angle, 6° 23′; vane angle at exit, 12° 37′; outer diameter, 1·95 feet; inner diameter, 0·975 feet; widths at inlet and outlet, 0·34 feet and 0·68 feet respectively, neglecting thickness of vanes.

XXX. pp. 518–519.

1. 22·09; 3·34. **2.** 2851 lbs.; (*a*) 2·25 cubic feet; (*b*) 0·102 cubic feet.
3. (*a*) 489·3 lbs.; (*b*) 873·8 lbs.; 1·65.
4. (i.) 319·7; (ii.) 38·07; (iii.) 76·1 per cent.; (iv.) 39,560; (v.) 39·27.
5. 1638 lbs.; 1583 lbs.; 14,496. **6.** 141·5; 196·7; 10 lbs.
11. (1) 11° 34′; (2) 104·6 feet; (3) 47·6 feet. **12.** 23·7 feet per second.
13. 60 per cent.; 250; 198 revolutions per minute.

XXXI. pp. 528–529.

1. 1·71 pence. **2.** 1,102,700. **3.** 1061 lbs. per square inch. **4.** 14·76 inches.
5. (*a*) 1139; (*b*) 1040. **6.** 40·24; 66·22.
7. (*a*) 19·84 tons; (*b*) 101·01 tons. **8** 12,959 lbs.; 53,996; 3·32 inches.
9. 2013. **10.** 649; 20,388. **11.** 32·9 inches. **12.** 4·70 inches; 9·73 inches.
13. 1792; 4032; 7168. **14.** 1720; 3890; 6934.
15. 16·16 tons per square inch; 87·4 feet. **16.** $7\frac{1}{2}$ tons. **17.** 53·4 per cent.
18. (1) 3970 lbs.; (2) 9·63. **19.** 6·06 cubic feet; 13 inches.
20. 1·53 feet per second.

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